Leiden
University

Foundations of Computer Science 1

Sets 1

What is a Set?

humour

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

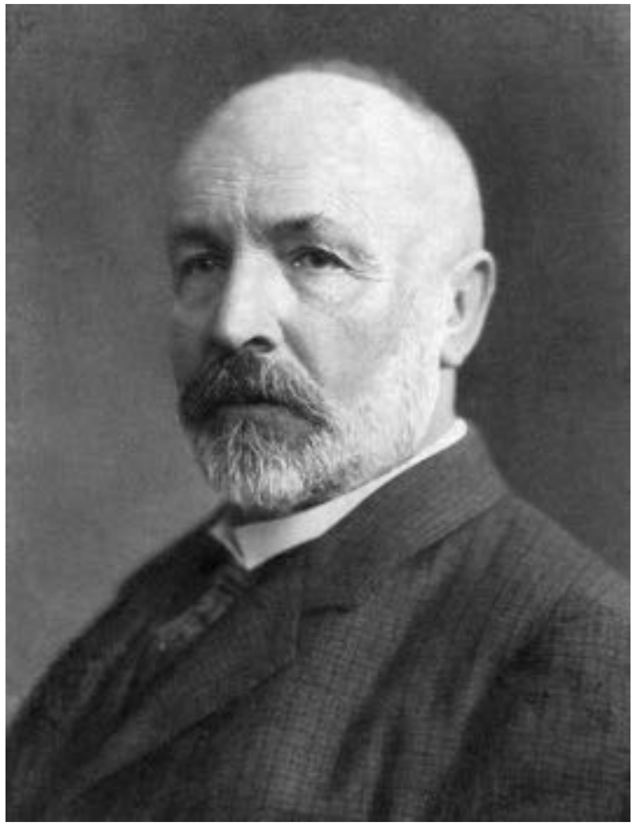
humour

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx$$

"well-defined collection of objects"…which is itself an object

Set

- **fundamental mathematical object**
- **one of the hardest objects to define**
  (*how do you learn the first word?*)
- **Georg Cantor: naive set theory**
- **abstract, yet pervasive**
- **Build basic (somewhat precise) vocabulary by examples**



source: wikipedia

- **Working definition: a set is a well-defined collection of objects**
- **"object x is an element of the set A…" "or object *x belongs to set A*…"**

$x \in A$; **otherwise** $x \notin A$

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We write: $x \in A$; otherwise $x \notin A$

<u>How are sets specified</u>

- **<u>extensional</u>; $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ …**
- $A = \{x_1, x_2, x_3 \dots\}$
- **example: $A = \{1,3,5,7\}$**

- **How are sets specified**

- **<u>extensional</u>;** $x_1 \in A$ **and** $x_2 \in A$ **and** $x_3 \in A$ **…**
- $A = \{x_1, x_2, x_3 \ldots\}$
- **example:** $A = \{1,3,5,7\}$

- **<u>intensional;</u>** "*A contains all odd numbers smaller than 9*"

  read: all numbers **satisfying the property** that they are odd and smaller than 9.

- Let *P(x)* = "true" if x is odd and smaller than 9

- $A = \{x \,|\, x \text{ is a number and } P(x) = true\} = \{x \,|\, P(x)\}$

- **Basic properties**
  - An object appears only once in the set (no duplicates)

  - **Definition**: Two sets A and B are equal if they have the same elements;
    We write $A = B$.

    Note: $A = B$ *if and only if* $[x \in A$ *if and only if* $x \in B]$

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Let's work this out

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    - {a,b,c} = {a,a,a,a,b,c} = {b,c,a}

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$a \in A$ and $b \in A$ and $c \in A$	$a \in A$ and $a \in A$ and $a \in A$ and $a \in A$ and $b \in A$ and $c \in A$	$b \in A$ and $c \in A$ and $a \in A$

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Set + multiple copies? "Multiset"
Set + ordering? Ordered list

- **Definition**: Two sets A and B are equal if they have the same elements; We write $A = B$.

- **Observation**: If two sets A & B are specified via properties *P and Q* *(so* $A = \{x \mid P(x)\}$ *and* $B = \{x \mid Q(x)\}$ *)*

then $\quad A = B \qquad$ ***if and only if*** $\qquad P = Q \qquad [P(x) = Q(x)$ ***for all*** $x]$

- **Careful! Non-trivial examples**

P(x) **is true if** *x* **is positive integer divisible by 3**
Q(x) **is true** *x* **is a positive integer and if the sum of its digits is divisible by 3**

$A = \{x \mid P(x)\}; \quad B = \{x \mid Q(x)\}$

- *A*=*B*?

- **Careful! Non-trivial examples**

Another example:

$P(x)$ **is true if** x **is positive integer and** x^2 **is divisible by 4**
$Q(x)$ **is true** x **is a positive integer and it is divisible by 2**

$A = \{x \mid P(x)\}; \quad B = \{x \mid Q(x)\}$

- $A=B$?

Let's work this out

$a = 2 \times k$

$a^2 = (2 \times k)^2 = 2^2 \times k^2 = 4 \times k$

$\sqrt{x^2} = \sqrt{4 \times k}$

$x = \sqrt{4} \times \sqrt{k} = 2\sqrt{k}$

- **Careful! Non-trivial examples**

P(x) **is true if** x **is an integer and** x^2 **is a positive integer divisible by 4**

Q(x) **is true if** x **is an integer and** $\frac{x}{2}$ **is a positive integer**

$A = \{x \mid P(x)\}; \quad B = \{x \mid Q(x)\}$

- $A=B$?

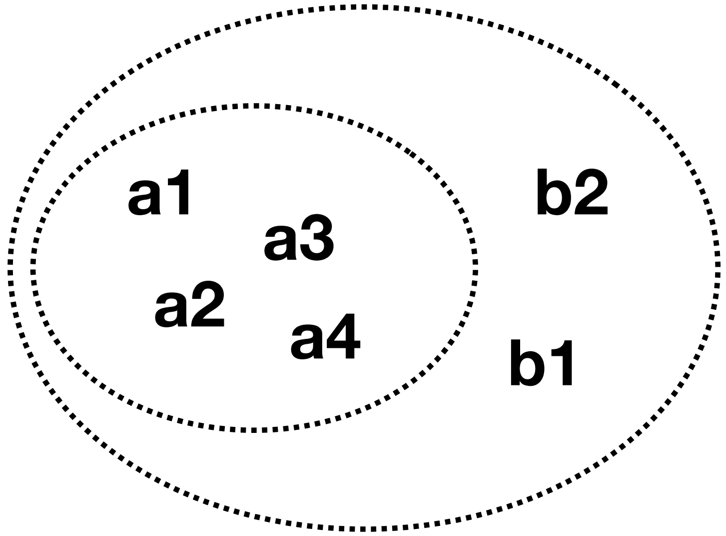
Let's work this out

Definition. *A* **is a subset of** *B* **if**

If $x \in A$, then $x \in B$

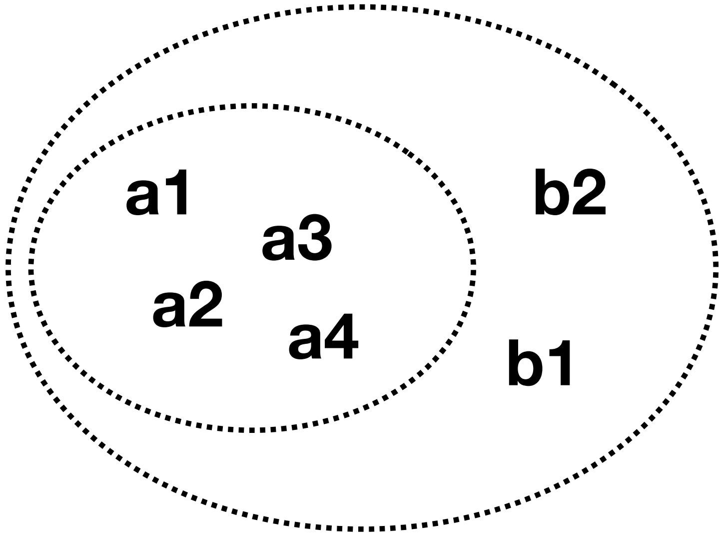
We write:

$$A \subseteq B$$



If $A \subseteq B$ and $B \subseteq A$ then

$$A = B$$



Let's work this out

$A = B$ $\quad x \in A \Rightarrow x \in B$
$\quad\quad\quad\quad x \in B \Rightarrow x \in A$

$A \subseteq B$ $\quad x \in A \Rightarrow x \in B$

$B \subseteq A$ $\quad x \in B \Rightarrow x \in A$

If $A \subseteq B$ and $B \subseteq A$ then

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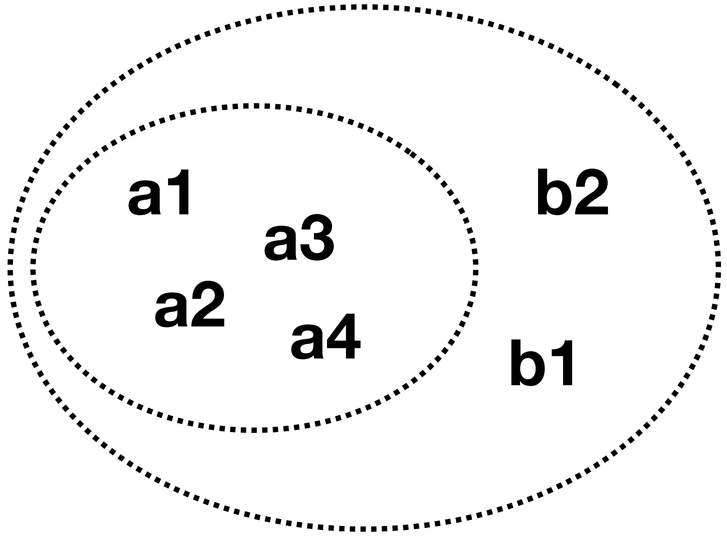


Definition. A **is a** **proper subset** **of** B **if**

$A \subseteq B$ **and** $A \neq B$

We write: $A \subset B$ *or* $A \subsetneqq B$

Conventions!

C.f. $a \leq b, a < b, a \lneqq b$

Quiz 1.

$$E = \{x \mid x^2 - 3x + 2 = 0\}$$

$$F = \{2, 1\}$$

$$G = \{1, 2, 2, 1\}$$

a) $E = F$?

b) $F = G$?

c) $E = G$?

$$x^2 - 3x + 2 = 0$$

$$x^2 - 2x - x + 2 = 0$$

$$x(x-2) - (x-2) = 0$$

$$(x-1)(x-2) = 0$$

$\Rightarrow$ $x = 1$

or

$x = 2$

- Useful special sets:

  - $\varnothing$ – the empty set; also written {}; also called *null set*
  - U - the universal set

- The "universal set" is tricky. Contextual.

$\mathbb{N}$ $\mathbb{N}$ $\mathbb{Z}$

- **Special sets: sets of numbers**
  - $\mathbb{N}$:= **{0,1,2,3…} - natural numbers; (**$\mathbb{N}^+ = \mathbb{N}_{>0}$:= **{1,2,3…})**
  - $\mathbb{Z}$:= **{0,1,-1,2,-2,3,-3…} - integer (whole) numbers;**
  - $\mathbb{Q} := \{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N}^+\}$ **- rational numbers (fractions);**
  - $\mathbb{R} := \{a + 0.d_1d_2d_3\ldots \mid a \in \mathbb{Z}, d_k \in \{0,\ldots,9\}\}$ $[almost]$ **- real numbers;**
  - $\mathbb{C} := \{x + iy \mid i = \sqrt{2}, x, y \in \mathbb{R}\}$ **- complex numbers**
  - transcendental numbers, quaternions **(**$\mathbb{H}$**)…**

Theorem. For all sets A *(it holds that)* $\emptyset \subseteq A$

Let's work this out

$B \subseteq A$ $\qquad x \in B \Rightarrow x \in A$ $\qquad A \Rightarrow B$

$\emptyset \subseteq A$ ..... $x \in \emptyset \Rightarrow x \in A$

Vacuous truths!

Theorem 1.1. The following claims hold:

- $A \subseteq A$
- $\textit{If } A \subseteq B \textit{ and } B \subseteq A \textit{ then } A = B$
- $A \subseteq B \textit{ and } B \subseteq C \textit{ imply } A \subseteq C$

Alternative phrasing/notation:

$$\begin{cases} A \subseteq B \textit{ and } B \subseteq A \textit{ imply } A = B \\ (A \subseteq B \;\&\; B \subseteq A) \;\Rightarrow\; A = B \end{cases}$$

Let's work this out

$A \subseteq A$ ✓

$B \subseteq A \mid x \in B \Rightarrow x \in A$

$x \in A \Rightarrow x \in A$ ✓

Theorem 1.1. The following claims hold:

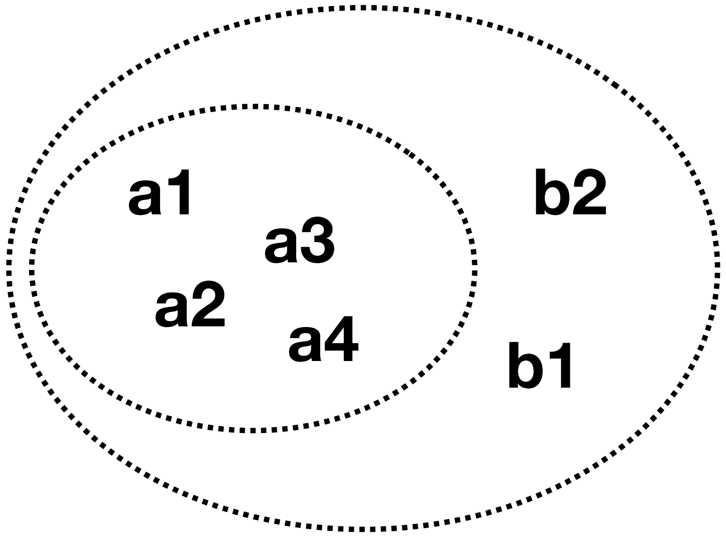
- $A \subseteq A$ — reflexive (reflexivity)
- *If* $A \subseteq B$ *and* $B \subseteq A$ *then* $A = B$ — antisymmetric (antisymmetricity)
- $A \subseteq B$ *and* $B \subseteq C$ *imply* $A \subseteq C$ — transitive (transitivity)

Partial order (discussed later)

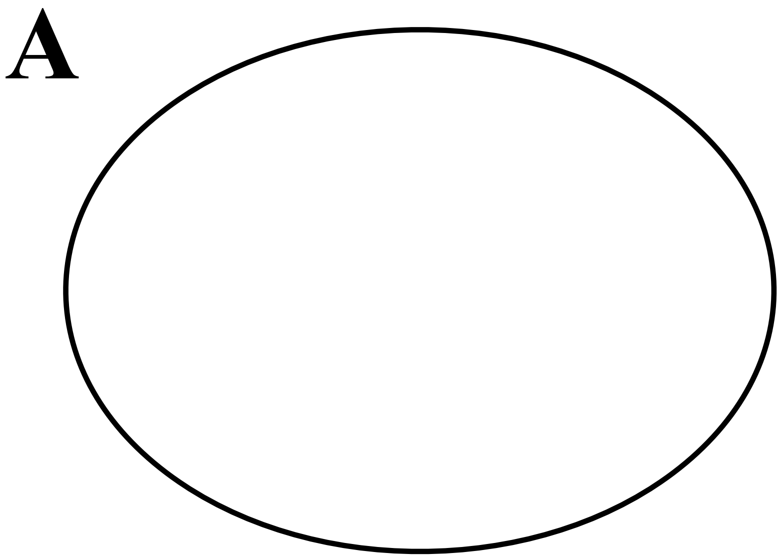
compare to $\leq$

Venn diagrams:
visualization of set relationships

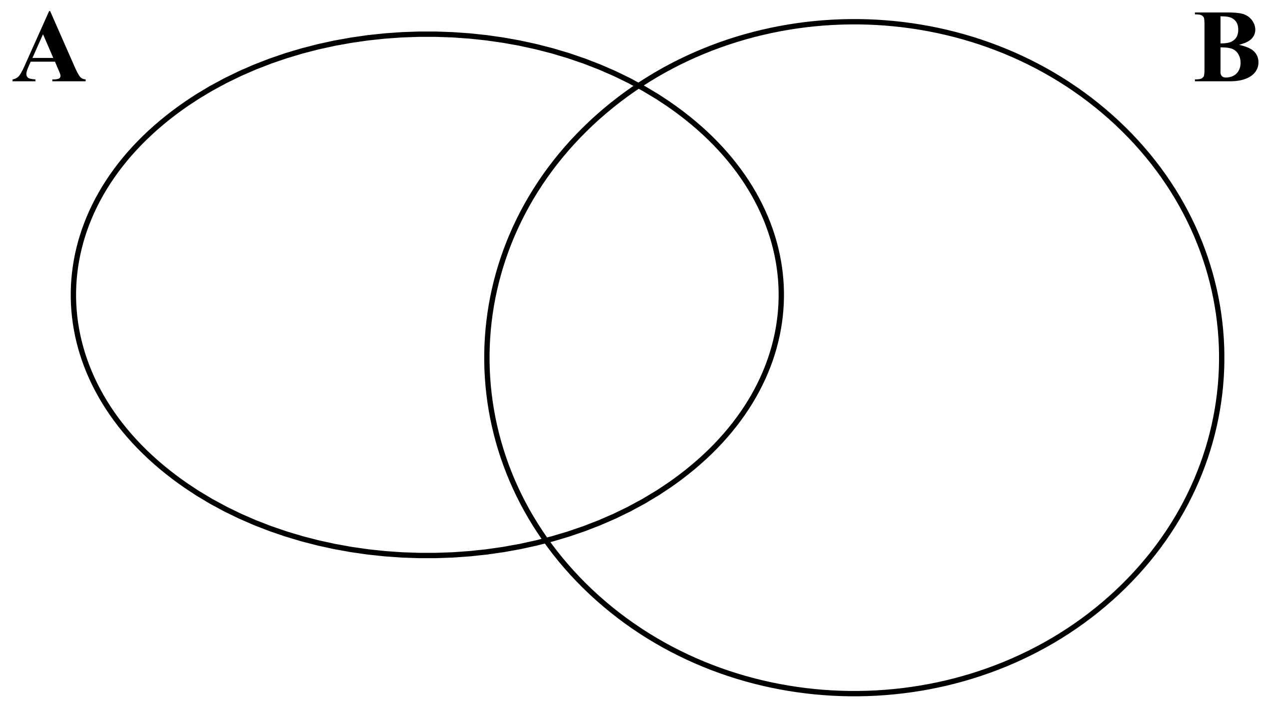
Venn diagrams



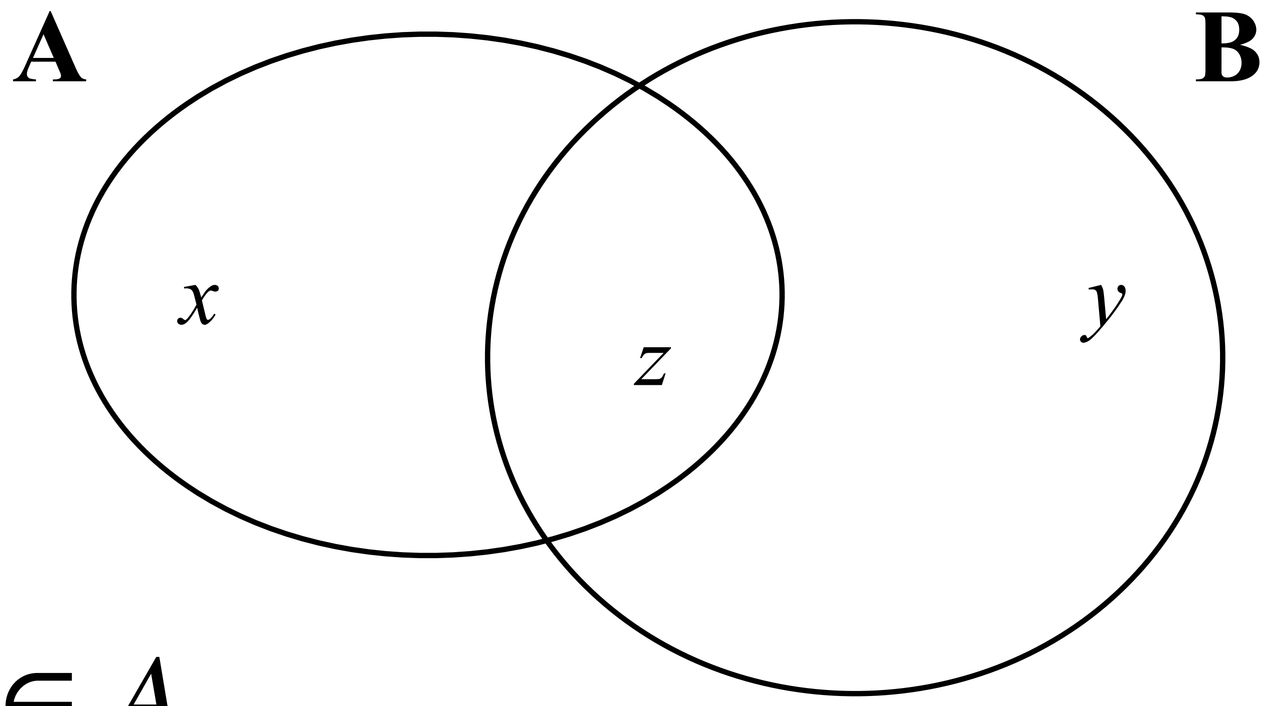
Venn diagrams



Venn diagrams



Venn diagrams

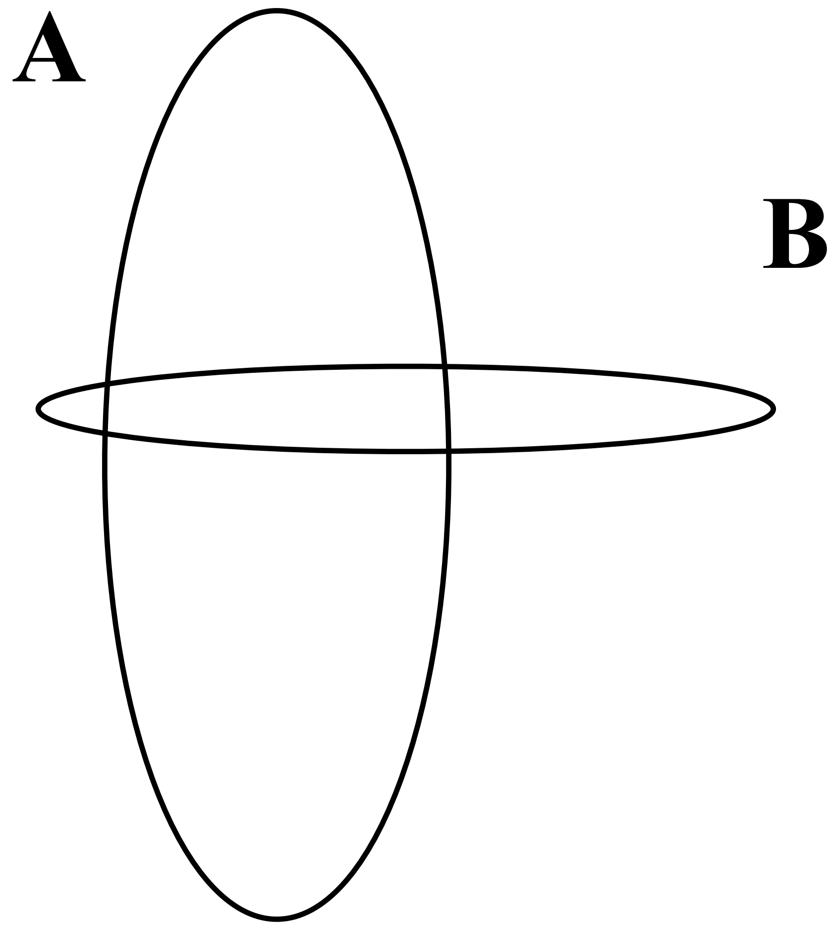


$$x \in A$$

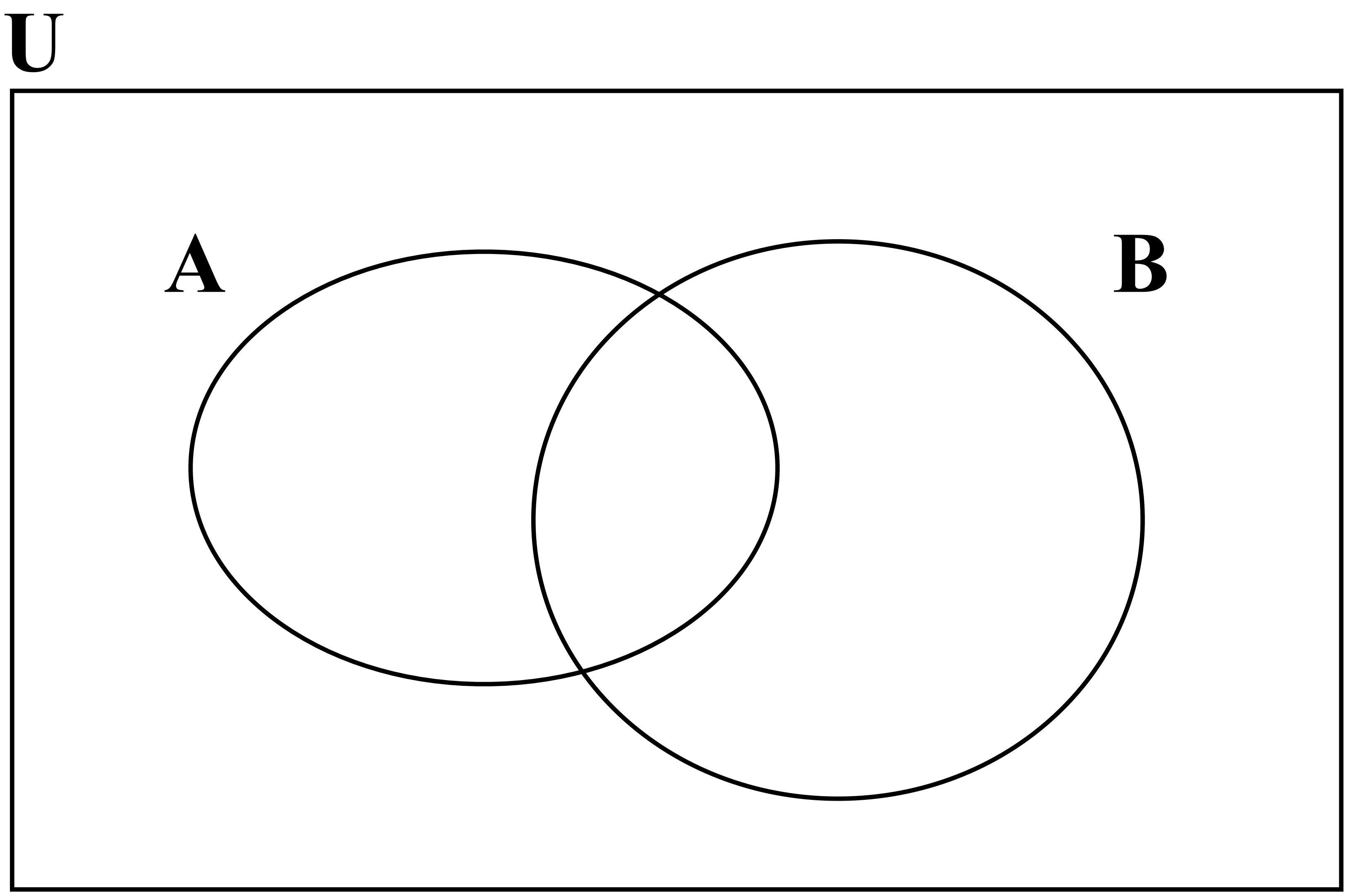
$$y \in B$$

$$z \in A \;\&\; z \in B$$

Venn diagrams



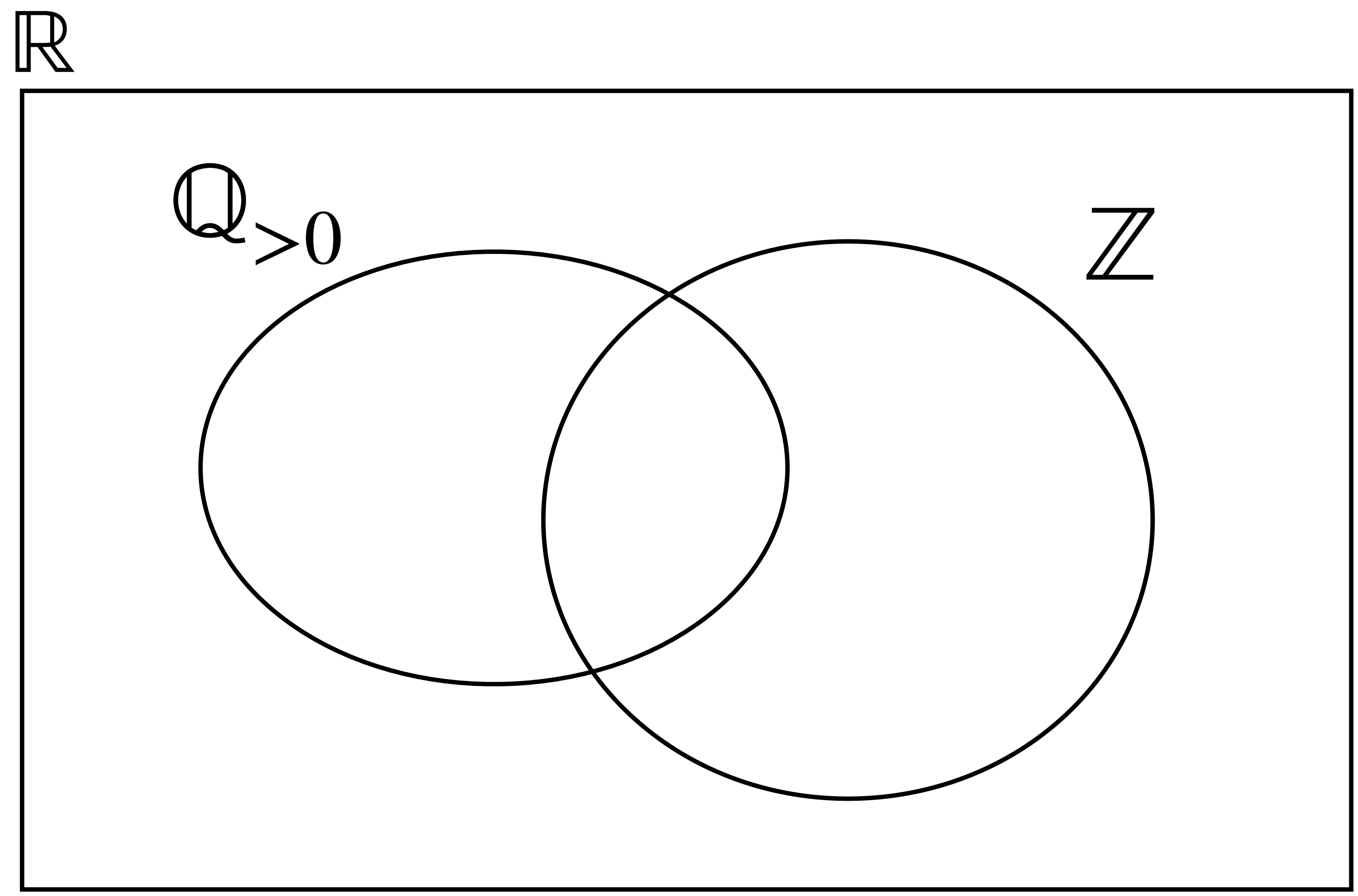
Venn diagrams



Context: universe **U**

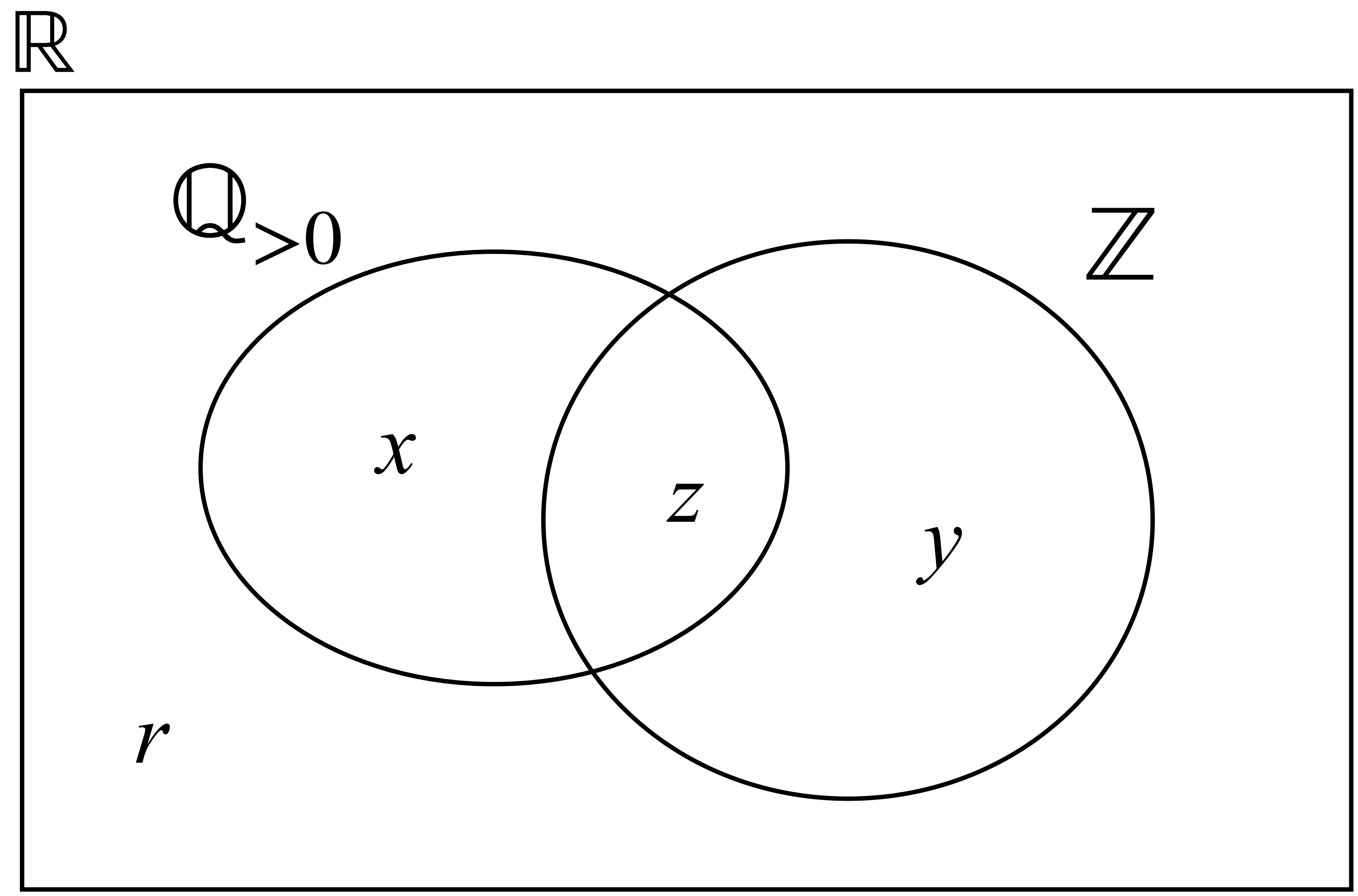
Venn diagrams

Example



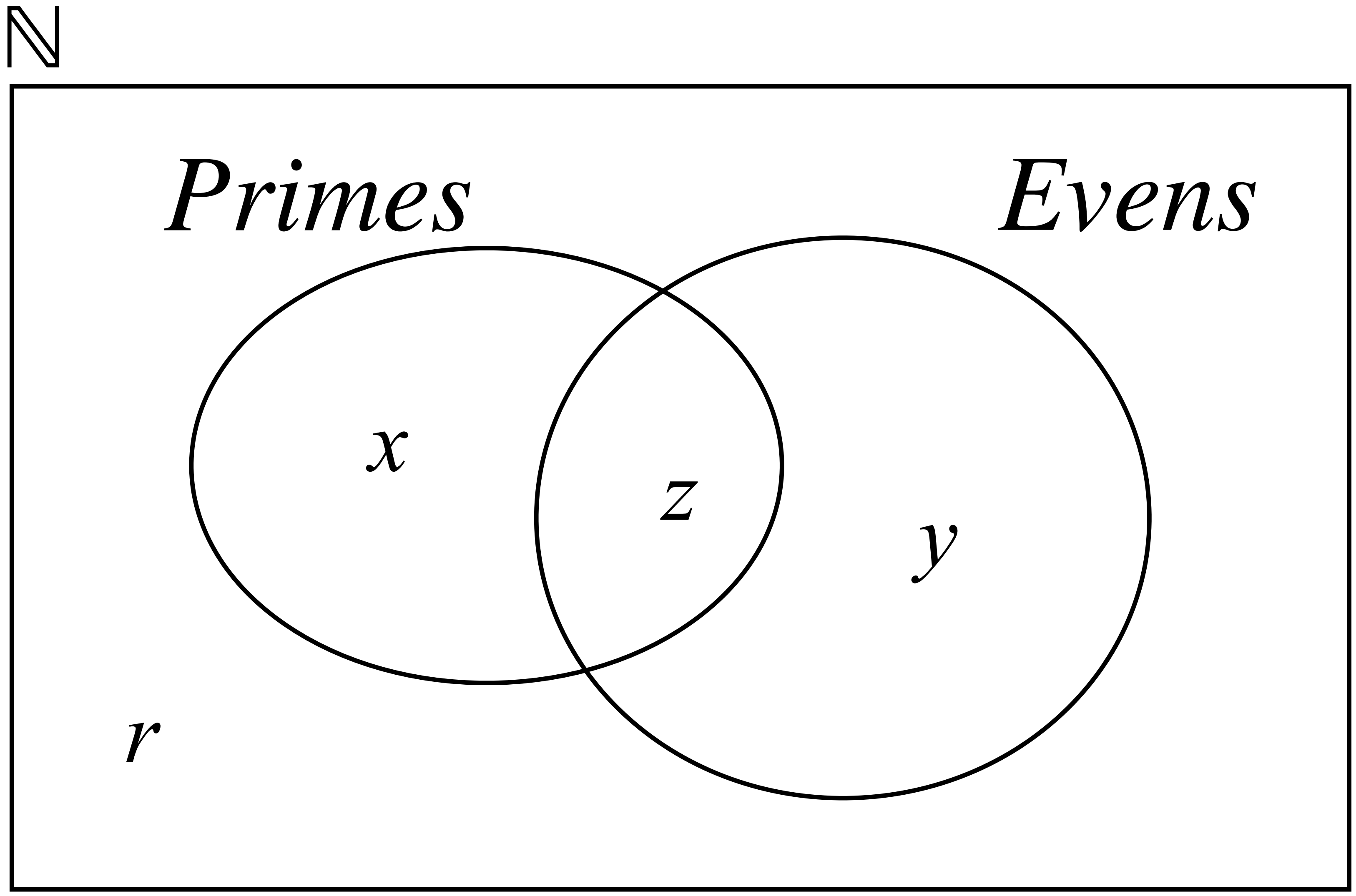
Venn diagrams

Example



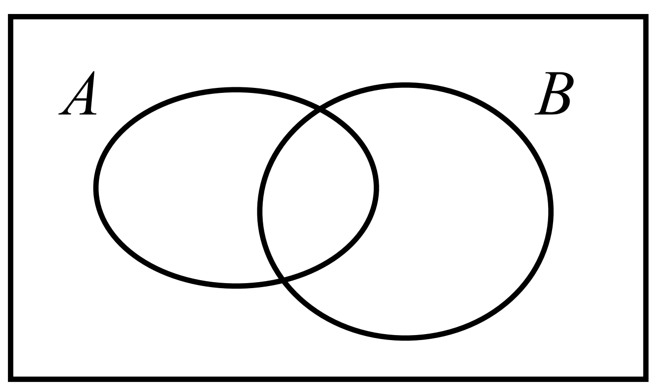
Venn diagrams

Example

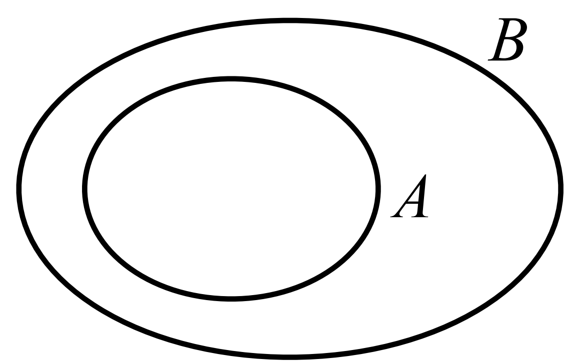


Venn diagrams

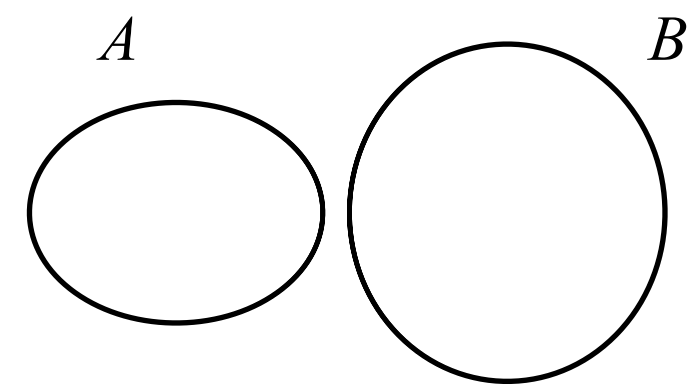
Terminology



General position



$A \subset B$ **(subset)**

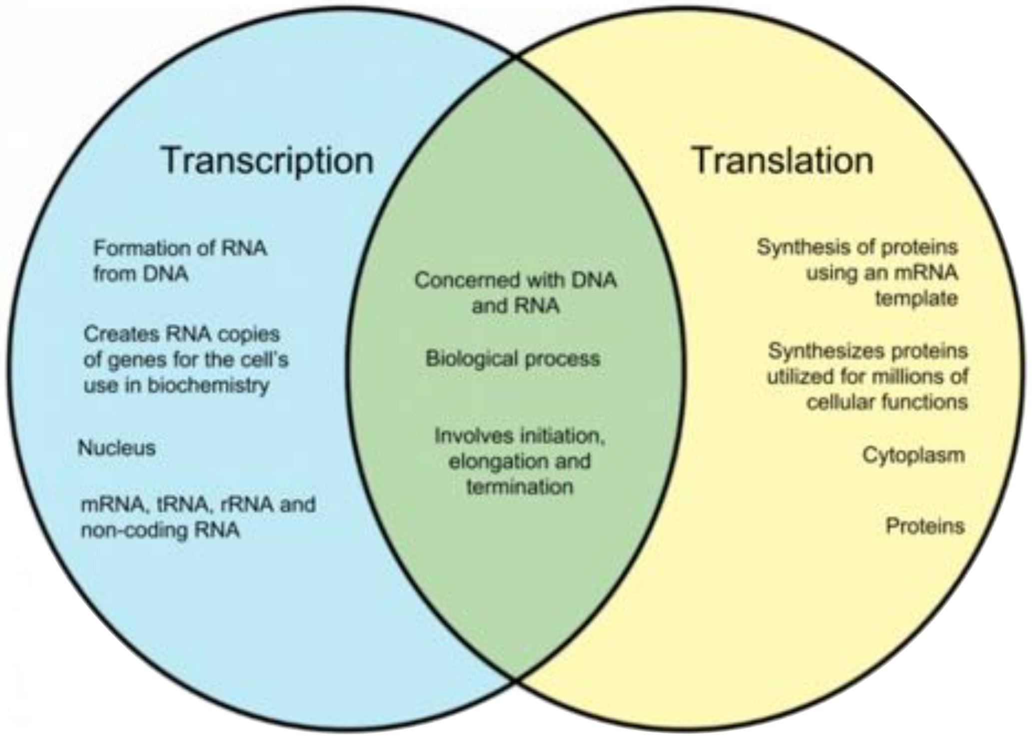


Disjoint

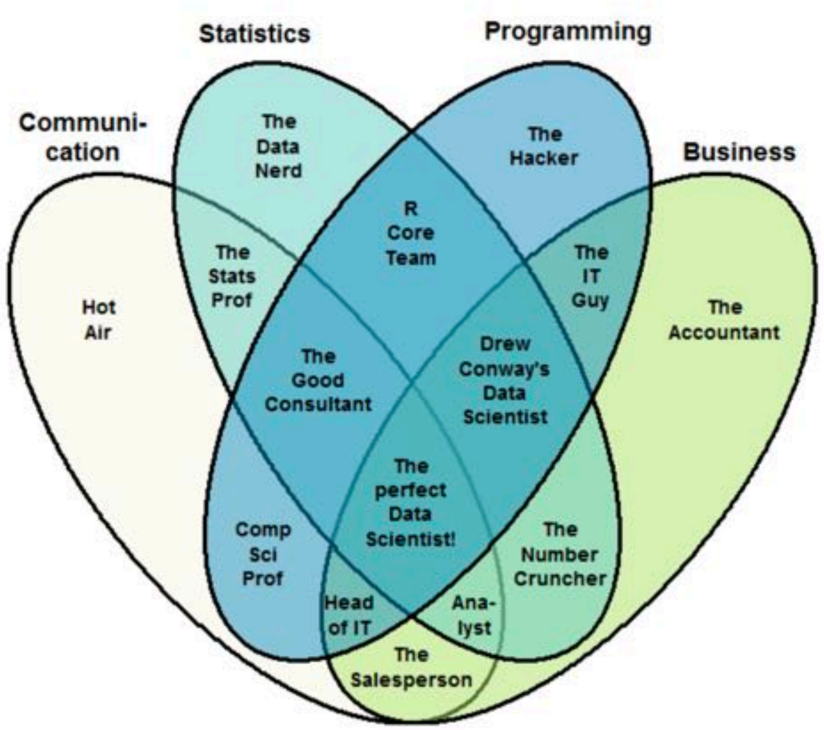
John Venn (1834-1923)

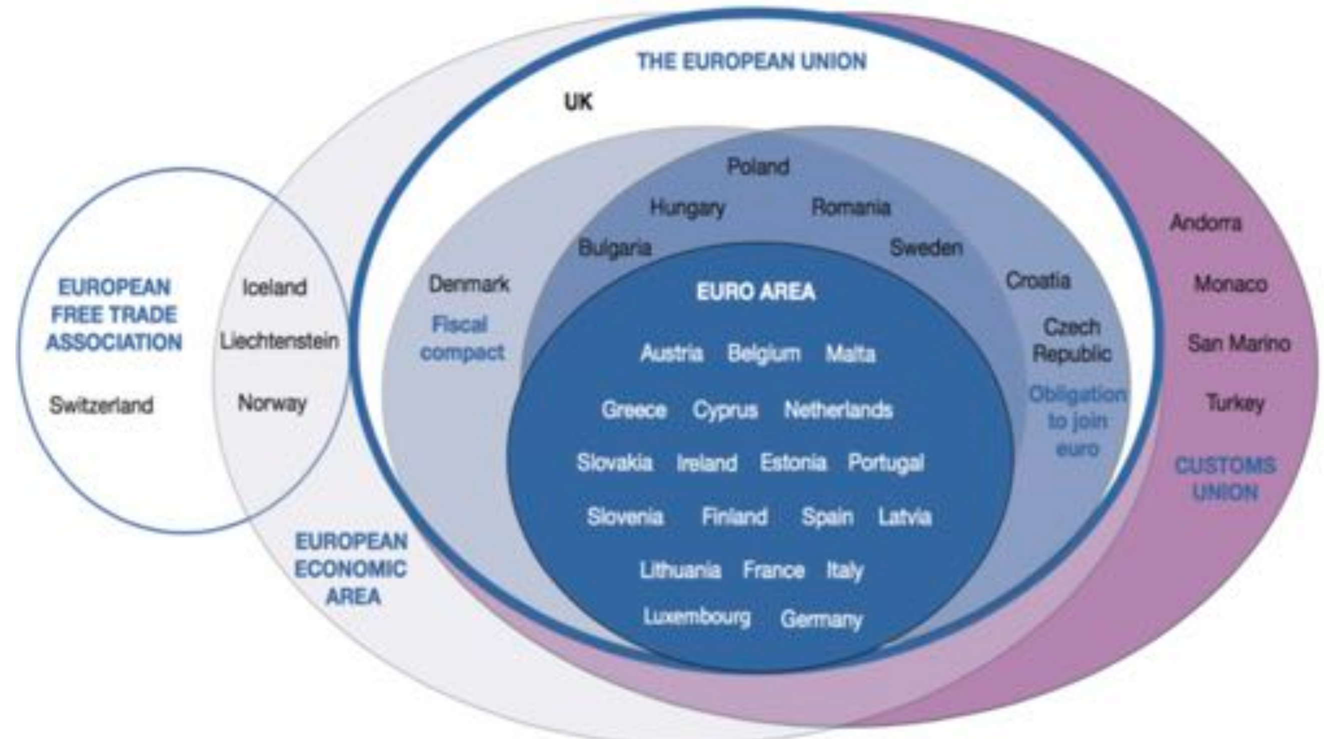
- **English mathematician, logician and philosopher**
- **1866 *The Logic of Chance***
- Helped standardize eponymous diagrams (also Leibniz)

source: wikipedia



The Data Scientist Venn Diagram





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https://whatsthebigdata.com/2016/07/08/the-new-data-scientist-venn-diagram/

Quiz 2.

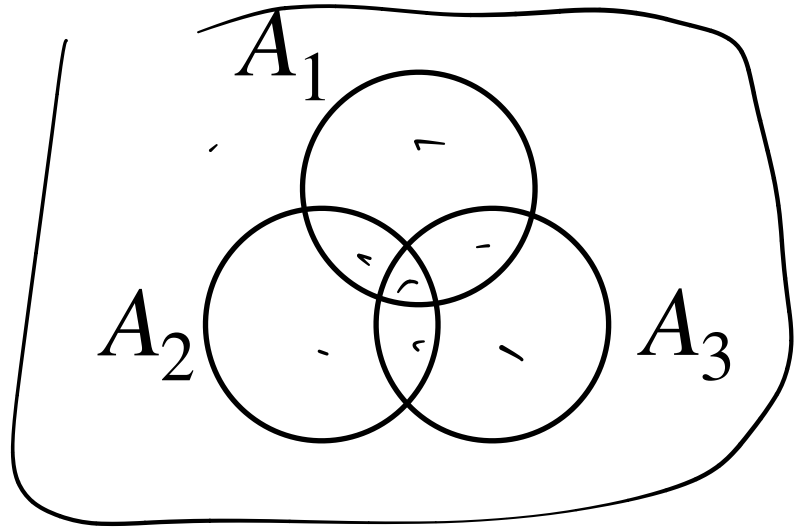
S_0 : ALL MY FRIENDS ARE MUSICIANS

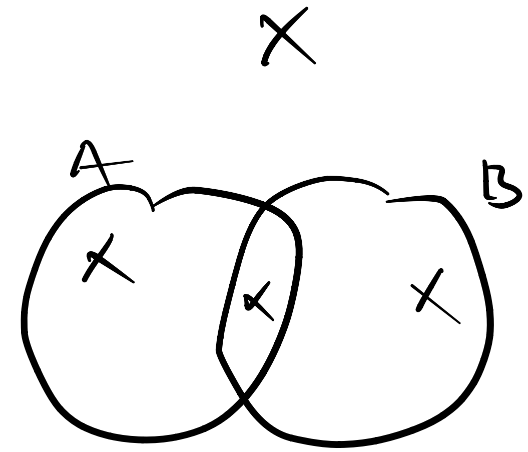
S_1 : JOHN IS MY FRIEND

S_3 = NONE OF MY NEIGHBOURS ARE MUSICIANS

S : JOHN IS NOT MY NEIGHBOUR

One step further: more than 2 sets?





How many distinct areas for *n sets* in a general position?

$$2^n = \underbrace{2 \times 2 \times \ldots \times 2}_{n}$$

One step further: more than 2 sets?

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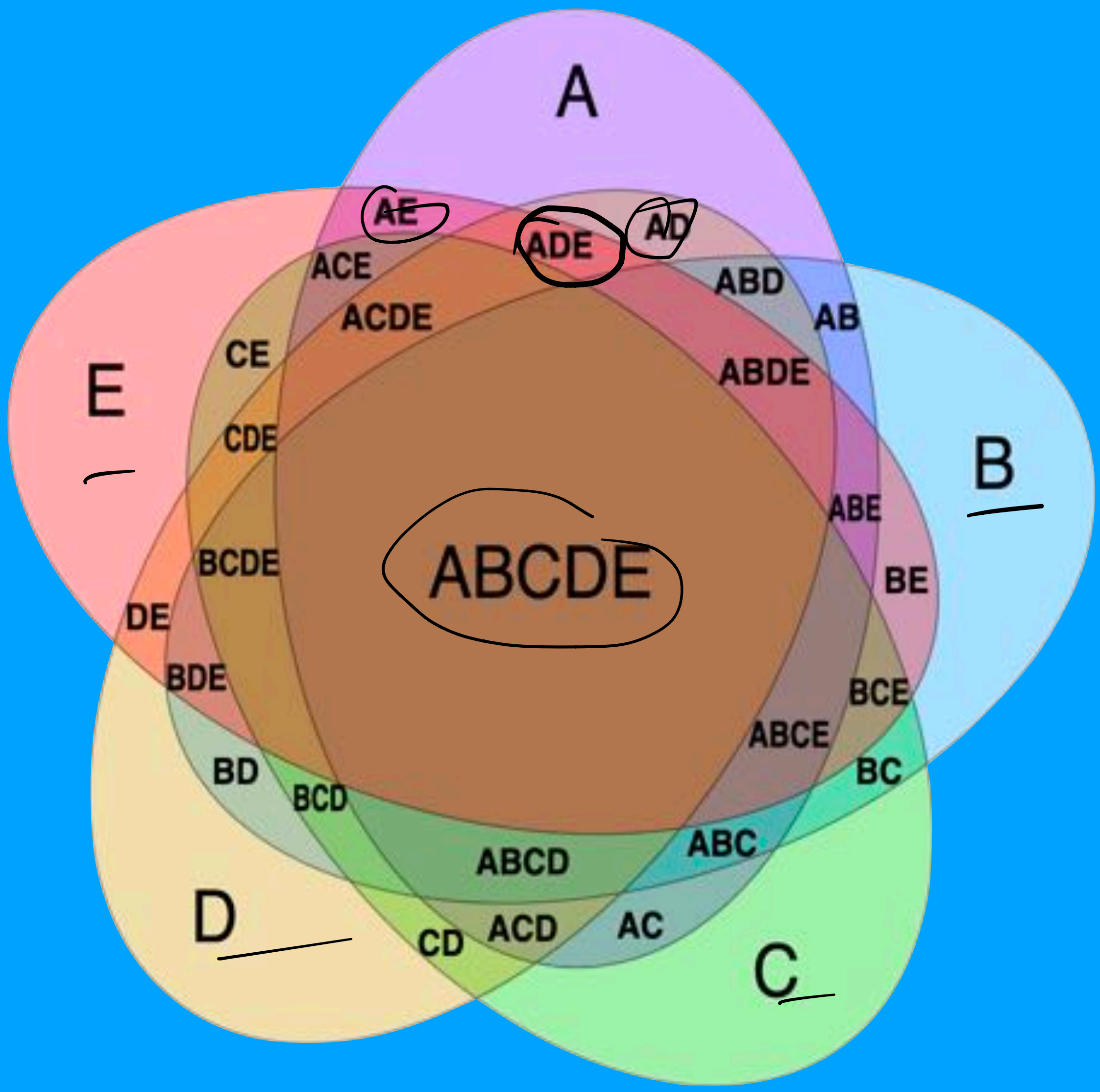


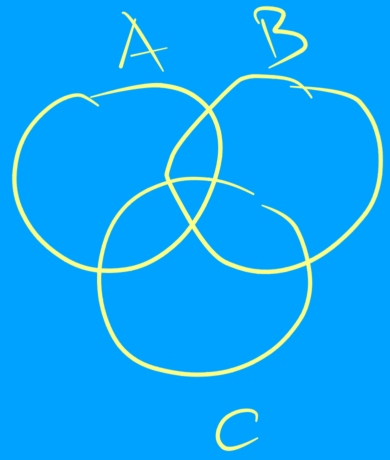
$\binom{n}{k}$

$\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

$= 2^n$

One step further: more than 2 sets?

How many distinct areas for *n sets* in a general position?



AB C D

$4 \times 2 \times 2 \cdots$

$\Rightarrow 2^n$

EACH NEW SET DOUBLES

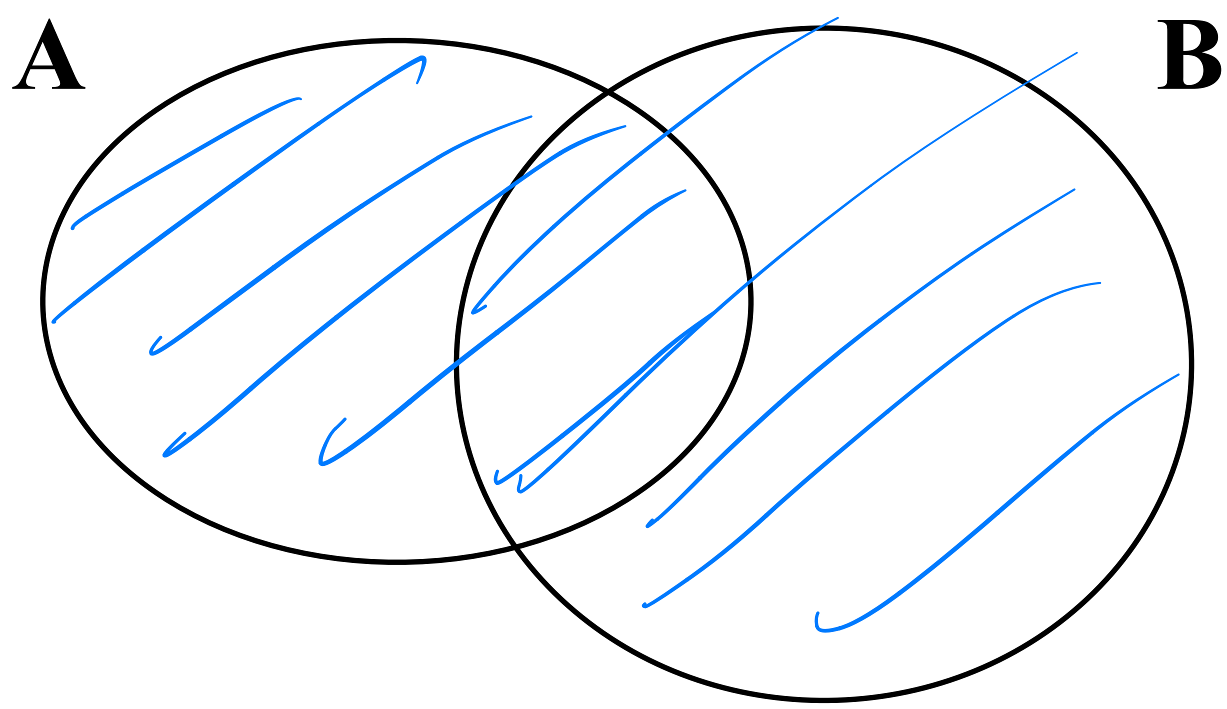
Operations on sets

Venn diagrams

UNION

- **New notation**
  - $\boxed{A \cup B} := \{x | x \in A \ or \ x \in B\}$ - **<u>union</u> of A and B**
  - $\boxed{A \cap B} := \{x | x \in A \ and \ x \in B\}$ - **<u>intersection</u> of A and B**

Intersection



Venn diagrams

- **New notation**
  - $A \cup B := \{x \mid x \in A \ or \ x \in B\}$ **- <u>union</u> of A and B**
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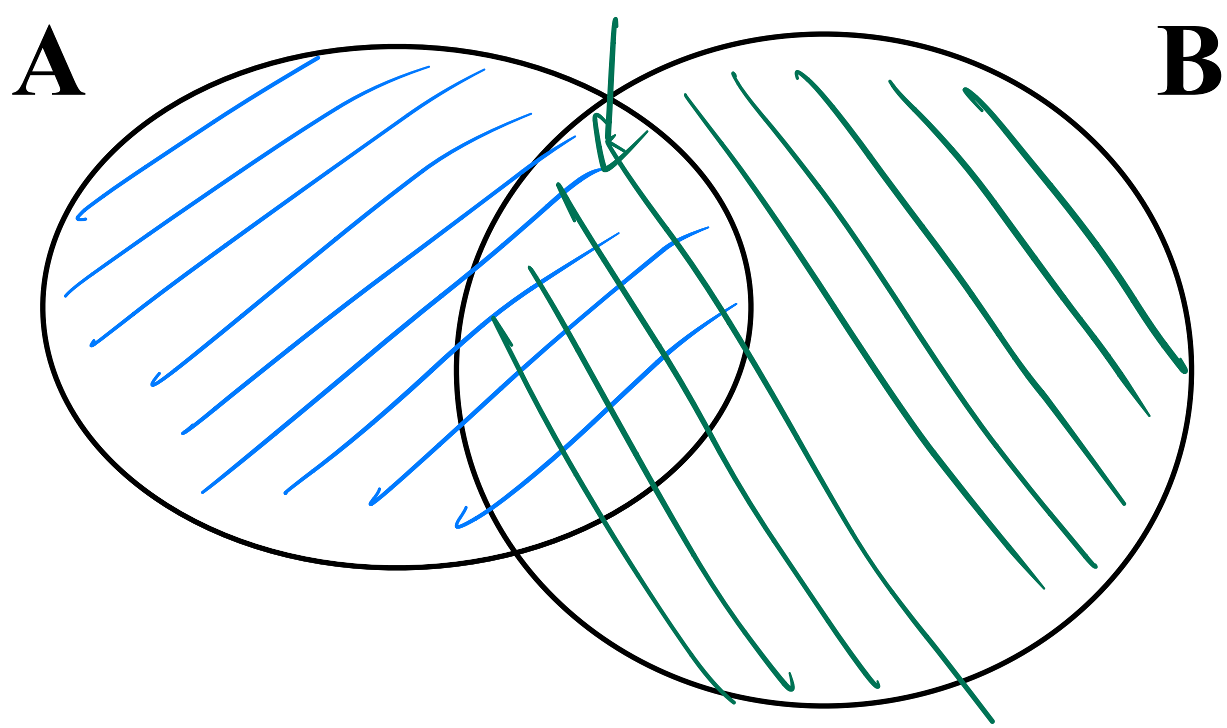

Venn diagrams

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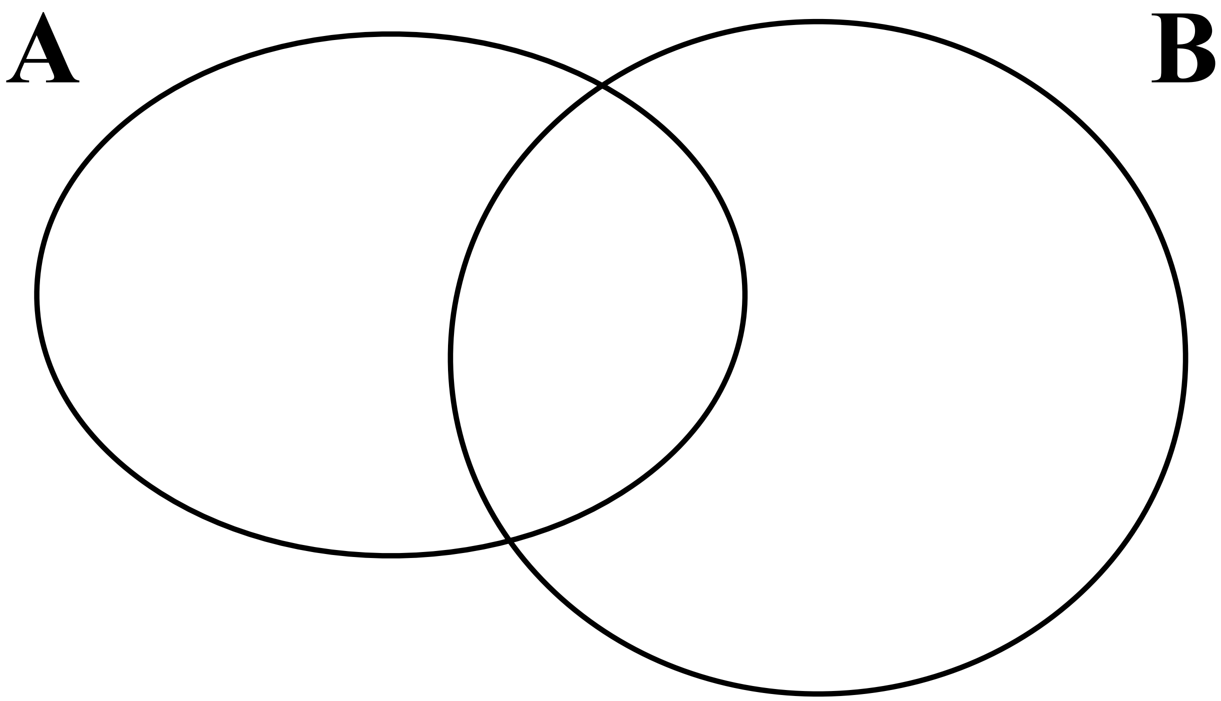
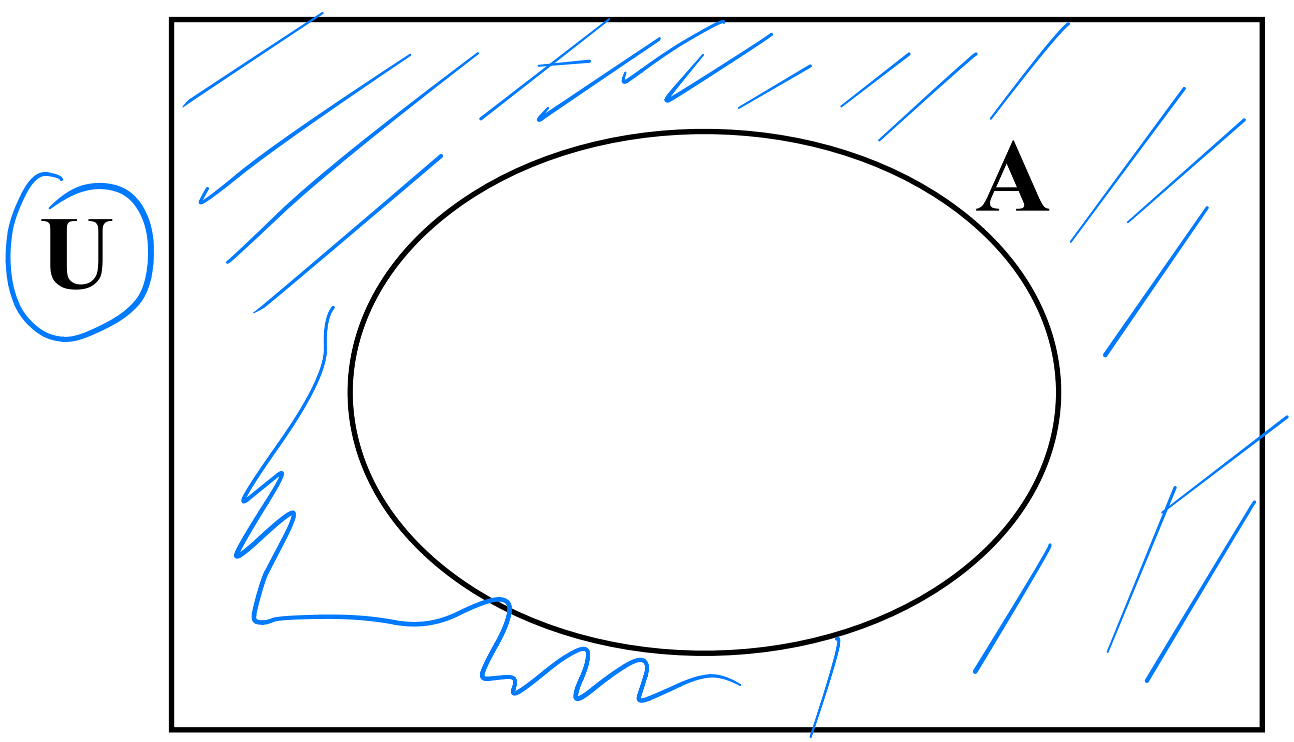


Venn diagrams

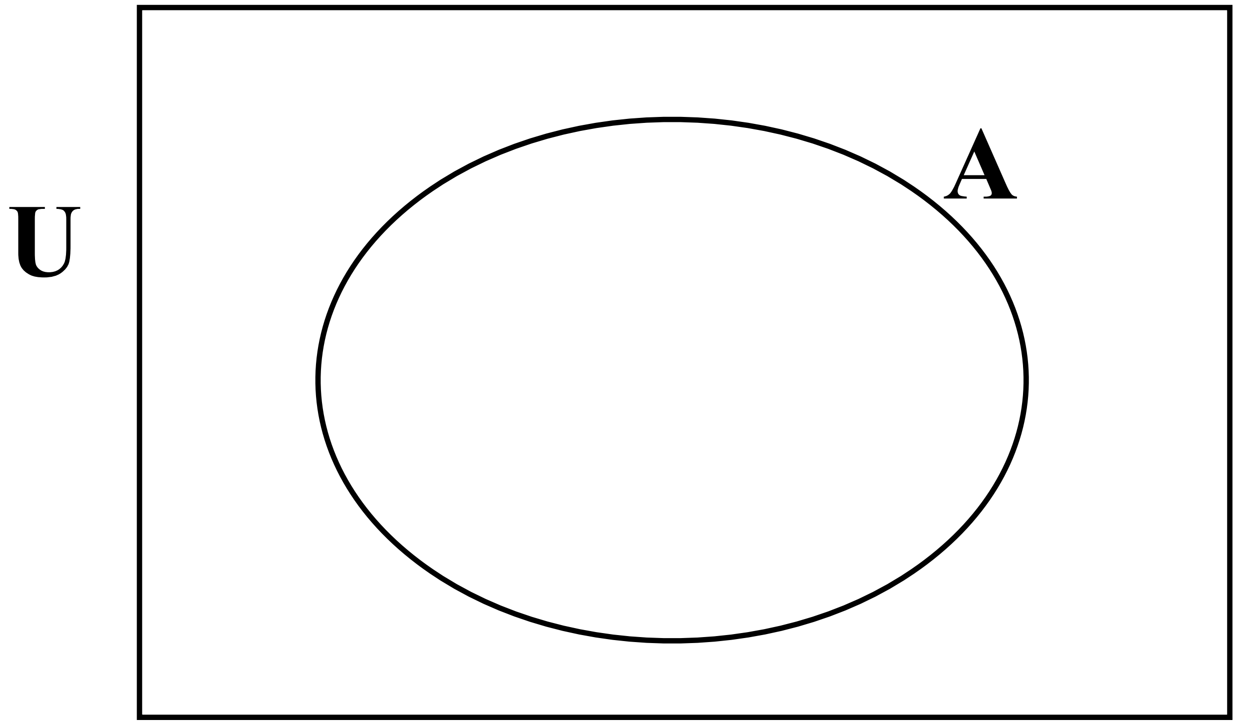
- **New notation**
  - A^c - **<u>complement</u> of A (in U)**

$A^c = \{x \mid x \notin A\}$



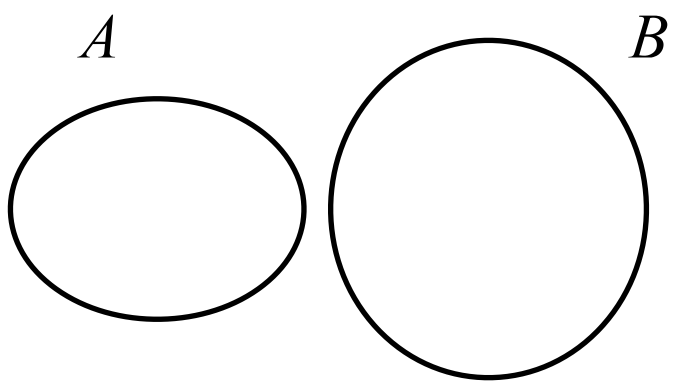
Venn diagrams

- **New notation**
  - A^c - **complement of A (in U)**



- **Definition. Two sets** A **and** B ***<u>are disjoint</u> if***

$$A \cap B = \varnothing$$



There is no element defined in the set defined by the intersection of A and B.
That is the *definition of the empty set.*

- **Theorem 1.4.**
  The following claims are equivalent:
  (1) $A \subseteq B$
  (2) $A \cap B = A$
  (3) $A \cup B = B$

Math lingo:
is it clear what this means?

Let's work this out

(1) => (2)

(1) => $x \in A \Rightarrow x \in B$

(2) $x \in A \Leftrightarrow \underbrace{x \in A \cap B}_{x \in A \ \& \ x \in B}$

(2) => $x \in A$ & (1) => $x \in B$ ✓

<= ✓

Venn diagrams

- **Theorem 1.4.**
  The following claims are equivalent:
  (1) $A \subseteq B$
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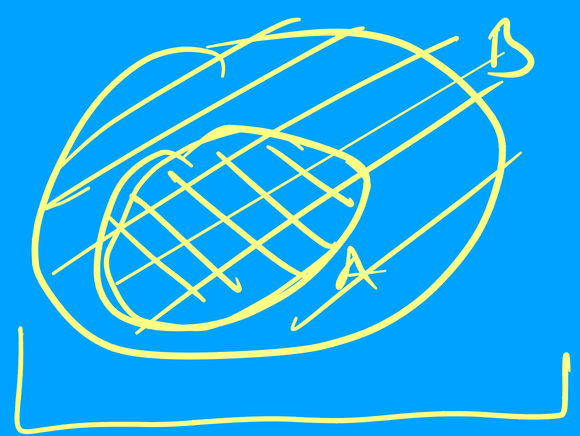
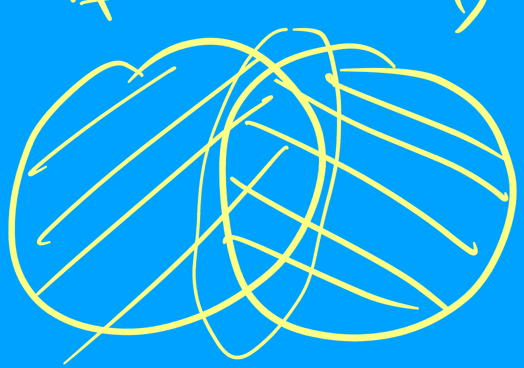
Let's work this out

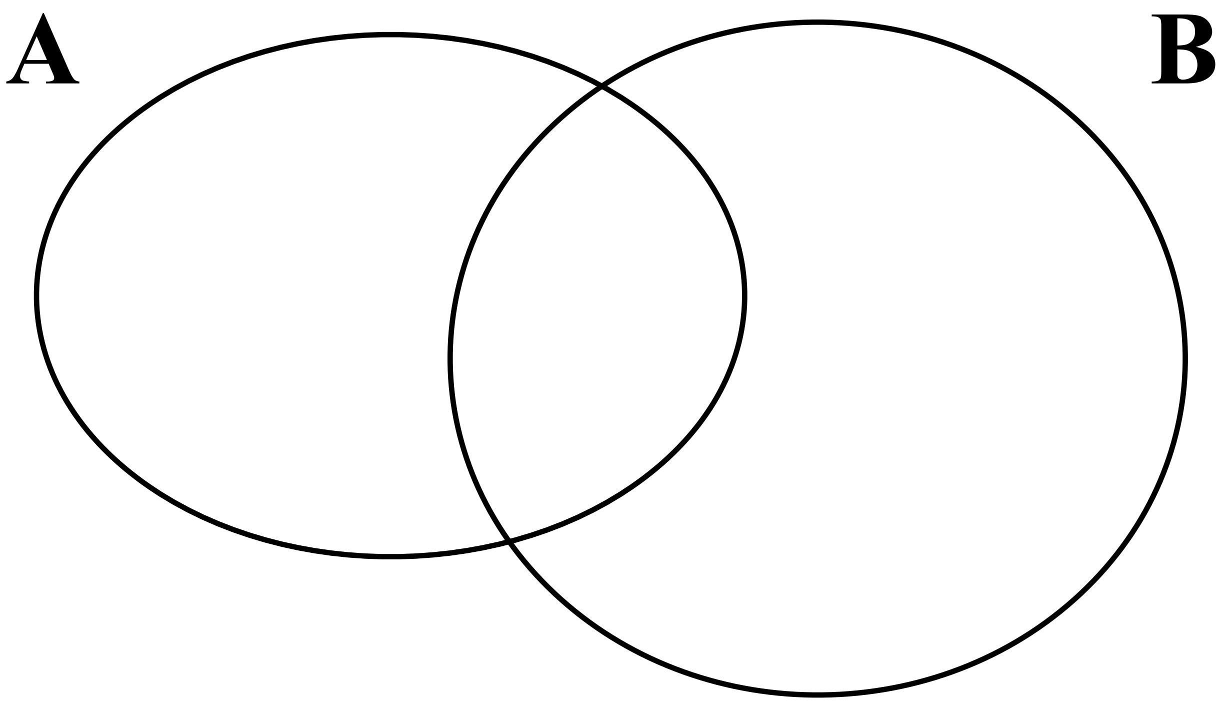
$(1) \Rightarrow (2)$





Operations continued

- **New notation**
  - $A - B$ **(or** $A \backslash B$**)** **-** **<u>set difference</u> of A and B**



Operations continued

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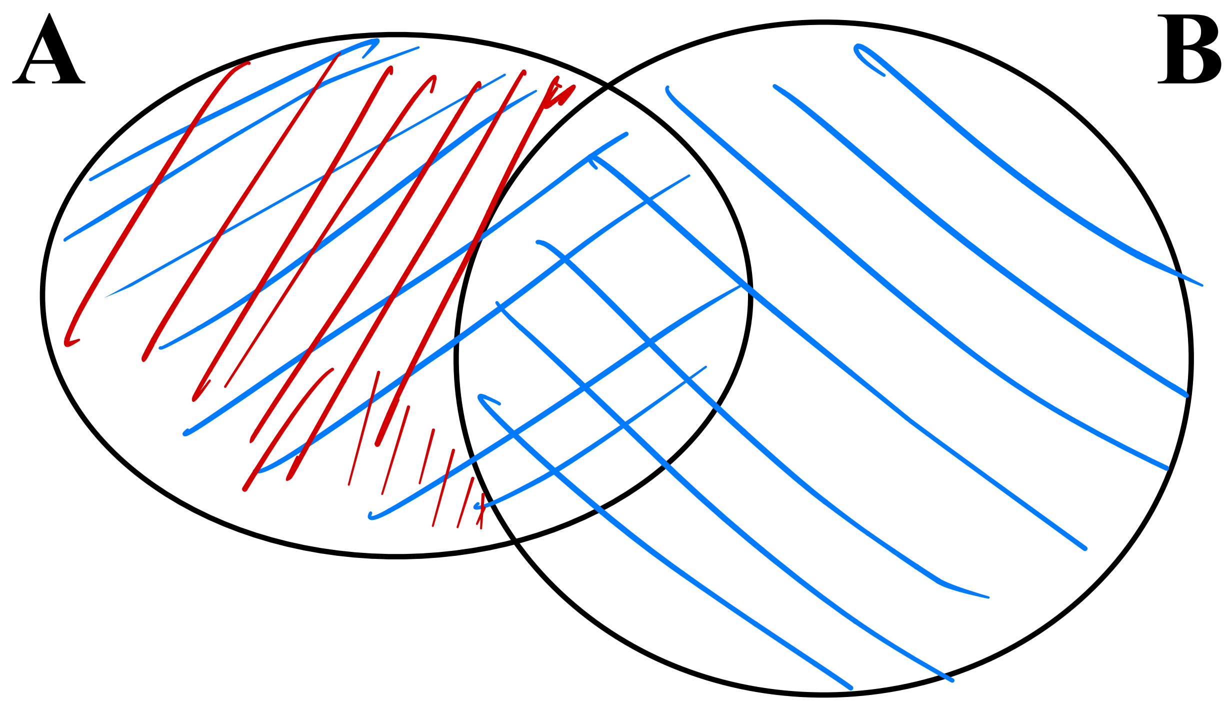
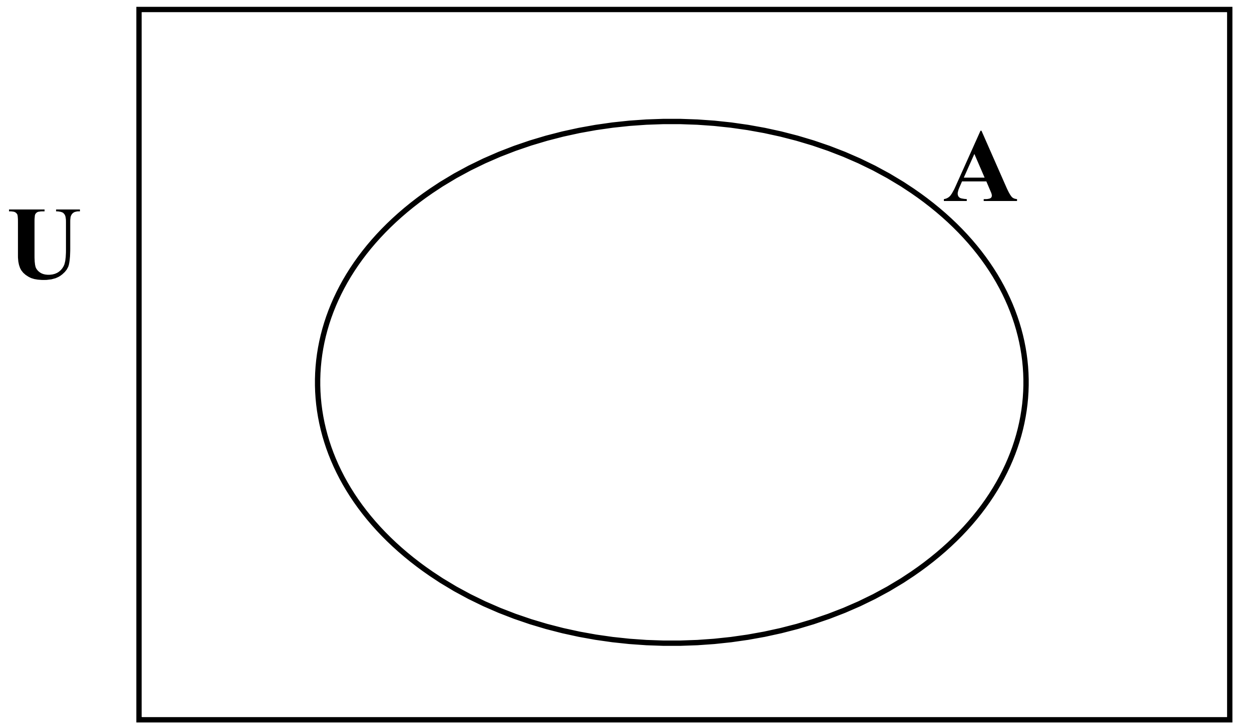


$A - B = \{ x \mid x \in A \;\&\; x \notin B \}$

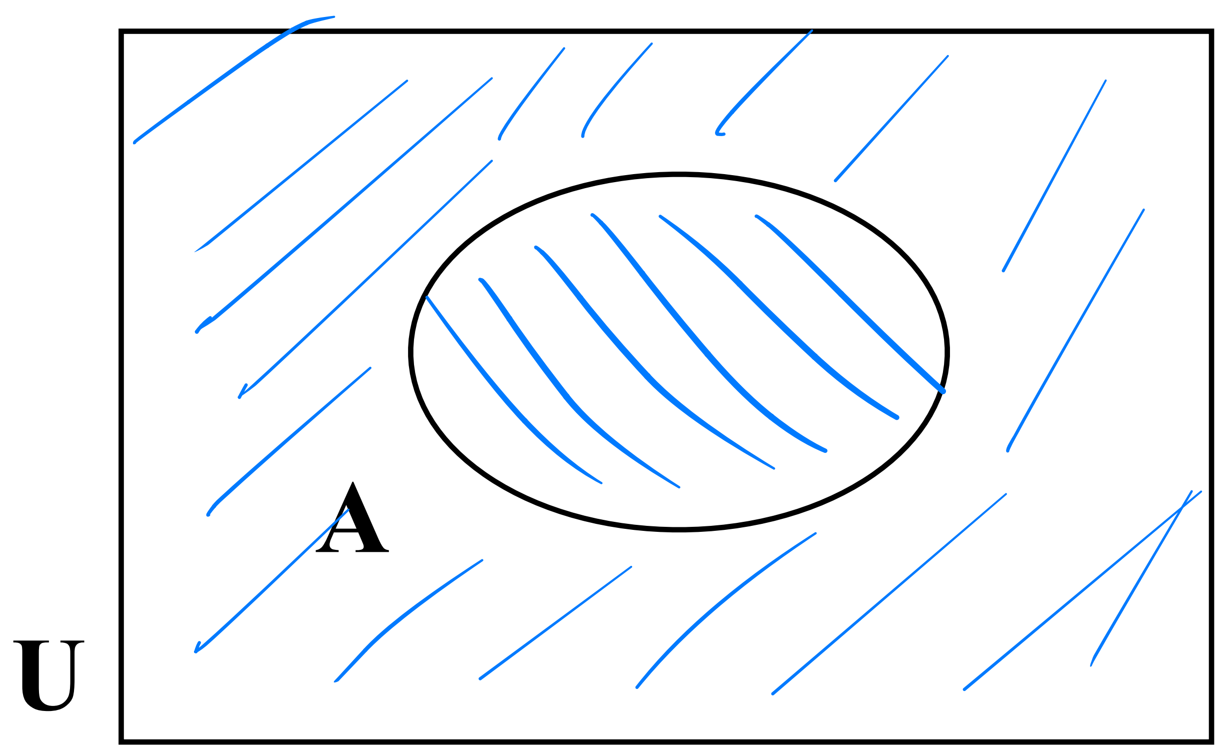
Venn diagrams

- **Note:** $A^c = U - A$
- **set difference A - B = "relative complement of B with respect to A "**



Venn diagrams

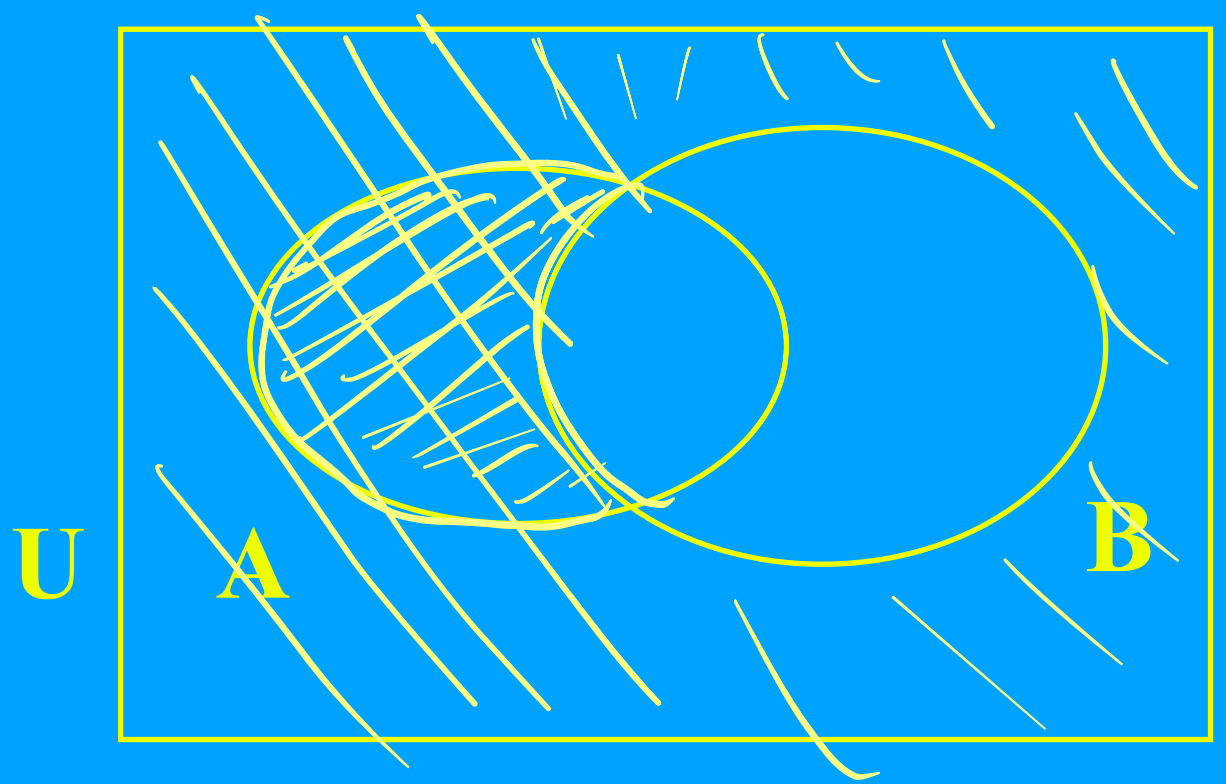
- **Question: what is $(A^c)^c$**



Venn diagrams

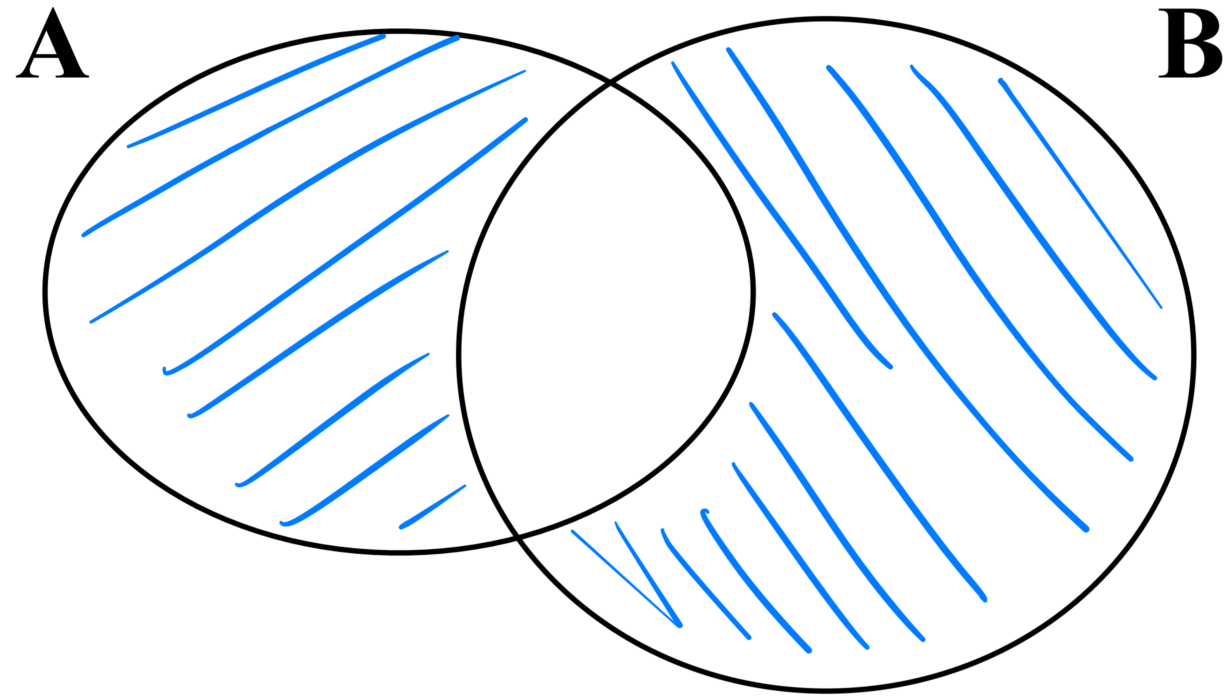
- **Note:** $A - B = A \cap B^c$

Let's work this out



Venn diagrams

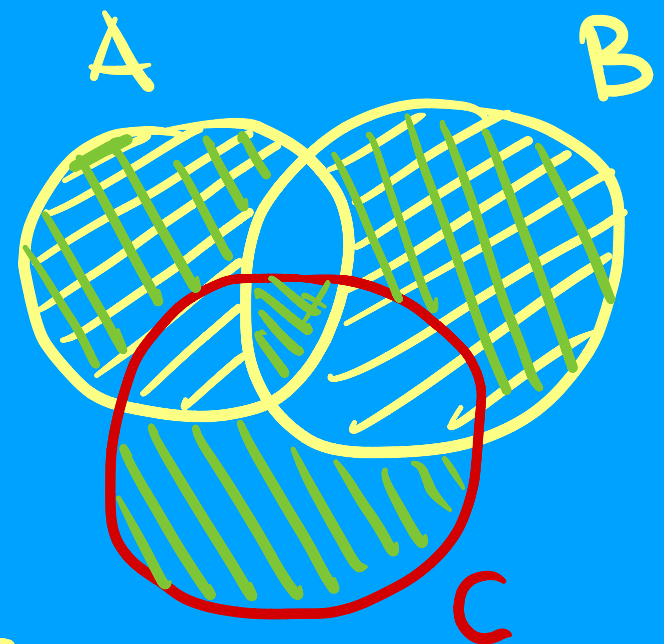
- **New notation**
  - $A \oplus B := (A \backslash B) \cup (B \backslash A)$ - **<u>symmetric difference</u> of A and B**
  - **also** $A \bigtriangleup B$
  - **"XOR" (exclusive OR)**



Venn diagrams

- $A \oplus B := (A \backslash B) \cup (B \backslash A)$

Quiz: $A \oplus B \oplus C$



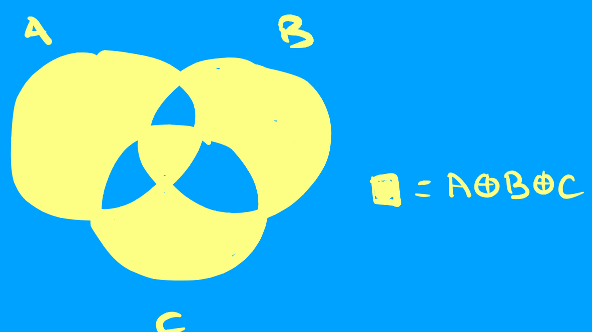
$A' = A \oplus B =$

$\underbrace{A' \oplus C}$

$\hookrightarrow A' \backslash C =$

$C \backslash A' =$

$\Rightarrow$



Example: show $A - B = A \cap B^c$ with $U = \mathbb{N}$

- **More properties**
  - **We know** $A - B = A \cap B^c$
  - **Also**
    - $(A \cup B)^c = A^c \cap B^c$
    - $(A \cap B^c) = A^c \cup B$

Let’s work this out: Venn, then algebra

- **Theorem 1.4.**
  The following claims are equivalent:

(1) $A \subseteq B$ (4) $B^c \subseteq A^c$
(2) $A \cap B = A$ (5) $A \cap B^c = \varnothing$
(3) $A \cup B = B$ (6) $A^c \cup B = U$

Let's work one out

See Schaum, Problem 1.8 and 1.31