Logica (I&E)

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`http://liacs.leidenuniv.nl/~vlietrvan1/logica/`

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1.4 Semantics of propositional logic
Semantische tableaus

Voetbal hoort gespeeld te worden in een 4-3-3 opstelling. Je kunt niet met twee spitsen spelen, want dan krijg je oneven getallen en kan een elftal nooit functioneren.

Part of a slide from lecture 5:

Theorem 1.35. (Soundness)
Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas.
If

$$\phi_1, \phi_2, \ldots, \phi_n \;\vdash\; \psi$$

is valid, then

$$\phi_1, \phi_2, \ldots, \phi_n \;\vDash\; \psi$$

holds.

A slide from lecture 5:

Step 2:

If

$$\phi_1, \phi_2, \ldots, \phi_n \ \vDash \ \psi$$

is valid, then

$$\begin{array}{ll} \text{Step 1:} & \vDash \ \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))) \\ \text{Step 2:} & \vdash \ \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))) \\ \text{Step 3:} & \phi_1, \phi_2, \ldots, \phi_n \ \vdash \ \psi \end{array}$$

Theorem 1.37.
If $\vDash \ \eta$ holds, then $\vdash \ \eta$ is valid.
In other words, if η is a tautology, then η is a theorem.

A slide from lecture 5:

'Encode' each line in the truth table of η as a sequent.

Proposition 1.38.
Let ϕ be a formula such that $p_1, p_2, \ldots, p_m$ are its only propositional atoms.
Let l be any line in ϕ's truth table.
For all $1 \leq i \leq m$, let $\hat{p}_i$ be p_i if the entry in line l of p_i is T, otherwise $\hat{p}_i$ is $\neg p_i$.
Then we have

1. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m \vdash \phi$ is provable if the entry for ϕ in line l is T

2. $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m \vdash \neg\phi$ is provable if the entry for ϕ in line l is F

Proof: by structural induction on formula ϕ...

A slide from lecture 5:

Example.

$m = 7$

p_1	p_2	p_3	p_4	p_5	p_6	p_7	ϕ	provable sequent
T	T	T	T	T	T	T	T	$p_1, p_2, p_3, p_4, p_5, p_6, p_7 \ \vdash \ \phi$
T	T	F	T	F	F	T	T	$p_1, p_2, \neg p_3, p_4, \neg p_5, \neg p_6, p_7 \ \vdash \ \phi$
T	F	F	F	T	T	F	T	$p_1, \neg p_2, \neg p_3, \neg p_4, p_5, p_6, \neg p_7 \ \vdash \ \phi$
F	F	F	F	F	F	F	T	$\neg p_1, \neg p_2, \neg p_3, \neg p_4, \neg p_5, \neg p_6, \neg p_7 \ \vdash \ \phi$
...	...	...	...	...	...	...	...	...
T	T	T	F	T	F	F	F	$p_1, p_2, p_3, \neg p_4, p_5, \neg p_6, \neg p_7 \ \vdash \ \neg\phi$
F	T	T	F	T	T	T	F	$\neg p_1, p_2, p_3, \neg p_4, p_5, p_6, p_7 \ \vdash \ \neg\phi$

A slide from lecture 5:

Proposition 1.38.
Let ϕ be a formula $(\ldots)$
Then we have

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Inductive step:
Suppose that Proposition 1.38 is valid for all formulas ϕ with height at most k_0 (induction hypothesis).

Now, consider a formula ϕ with height $k_0 + 1$.
If $\phi = \neg\phi_1 \ldots$

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Then we have

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Inductive step:
Suppose that Proposition 1.38 is valid for all formulas ϕ with height at most k_0 (induction hypothesis).

Now, consider a formula ϕ with height $k_0 + 1$.
If $\phi = \phi_1 \circ \phi_2$

Step 2:

If

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

is valid, then

Step 1: $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots)))$

Step 2: $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots (\phi_n \rightarrow \psi) \ldots)))$

Step 3: $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

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Consider

$$\phi = \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$$

Example.

$$p \wedge q \rightarrow p$$

Corollary 1.39. (Soundness and Completeness)
Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be formulas of propositional logic. Then

$$\phi_1, \phi_2, \ldots, \phi_n \ \vDash \ \psi$$

holds, iff the sequent

$$\phi_1, \phi_2, \ldots, \phi_n \ \vdash \ \psi$$

is valid.

3.2. Semantische Tableaus

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From [Van Benthem et al., 2003]:

Gevolgtrekking:

$$\phi_1, \ldots, \phi_n \ / \ \psi$$

'ψ volgt logisch uit $\phi_1, \phi_2, \ldots, \phi_n$'

Model:

Definitie 2.5. Een waardering V heet een *model* van een formule ϕ als geldt: $V(\phi) = 1$. (1 = T, 0 = F)

Model van formuleverzameling:

Definitie 2.6. Een waardering V heet een *model* van een formuleverzameling $\Sigma = \{\phi_1, \ldots, \phi_n\}$ als $V(\phi_i) = 1$ voor elke $\phi_i \in \Sigma$.

3.2. Semantische Tableaus

From [Van Benthem et al., 2003]:

Sequent:

$$\phi_1, \ldots, \phi_n \circ \psi_1, \ldots, \psi m$$

Met $m, n \geq 0$

Tegenvoorbeeld:
Een waardering V heet een *tegenvoorbeeld* van een sequent

$$\phi_1, \ldots, \phi_n \circ \psi_1, \ldots, \psi_m$$

als $V(\phi_1) = \ldots = V(\phi_n) = 1$ en $V(\psi_1) = \ldots = V(\psi_m) = 0$.

Voorbeeld 3.4.

$$\neg q, p \rightarrow q \;/\; \neg p$$

Voorbeeld 3.4.

$$\neg p, p \to q \; / \; \neg q$$

Voorbeeld 3.5.

$$p \vee q, p \to r, q \to s \;/\; r \vee s$$

Full notation

Voorbeeld 3.6.

$$p \vee q, p \to r, q \to s \ / \ r \vee s$$

Simplified notation

Voorbeeld 3.7.

$$(p \to q) \to (p \to r) \ / \ q \to r$$