Logica (I&E)

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http://liacs.leidenuniv.nl/~vlietrvan1/logica/

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1.4 Semantics of propositional logic

Voetbal speel je met het hoofd, want de bal is vlugger dan de benen.

A slide from lecture 2:

1.4.3. Soundness of propositional logic

Definition 1.34.
If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to T, ψ evaluates to T as well, we say that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds and $\vDash$ the *semantic entailment* relation.

Theorem 1.35. (Soundness)
Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas.
If

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is valid, then

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds.

Proof: By mathematical induction (course-of-values) on the length of the proof of

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

$M(k)$:

For all sequents

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

($n \geq 0$) which have a proof of length k, it is the case that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds.

$M(k)$:

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$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds.

Base case. . .

$M(k)$:

For all sequents

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

($n \geq 0$) which have a proof of length k, it is the case that

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

holds.

Suppose that $M(k)$ is valid for all $k \leq k_0$ (induction hypothesis)

Now, consider a sequent with a proof of length $k_0 + 1$.

Induction step

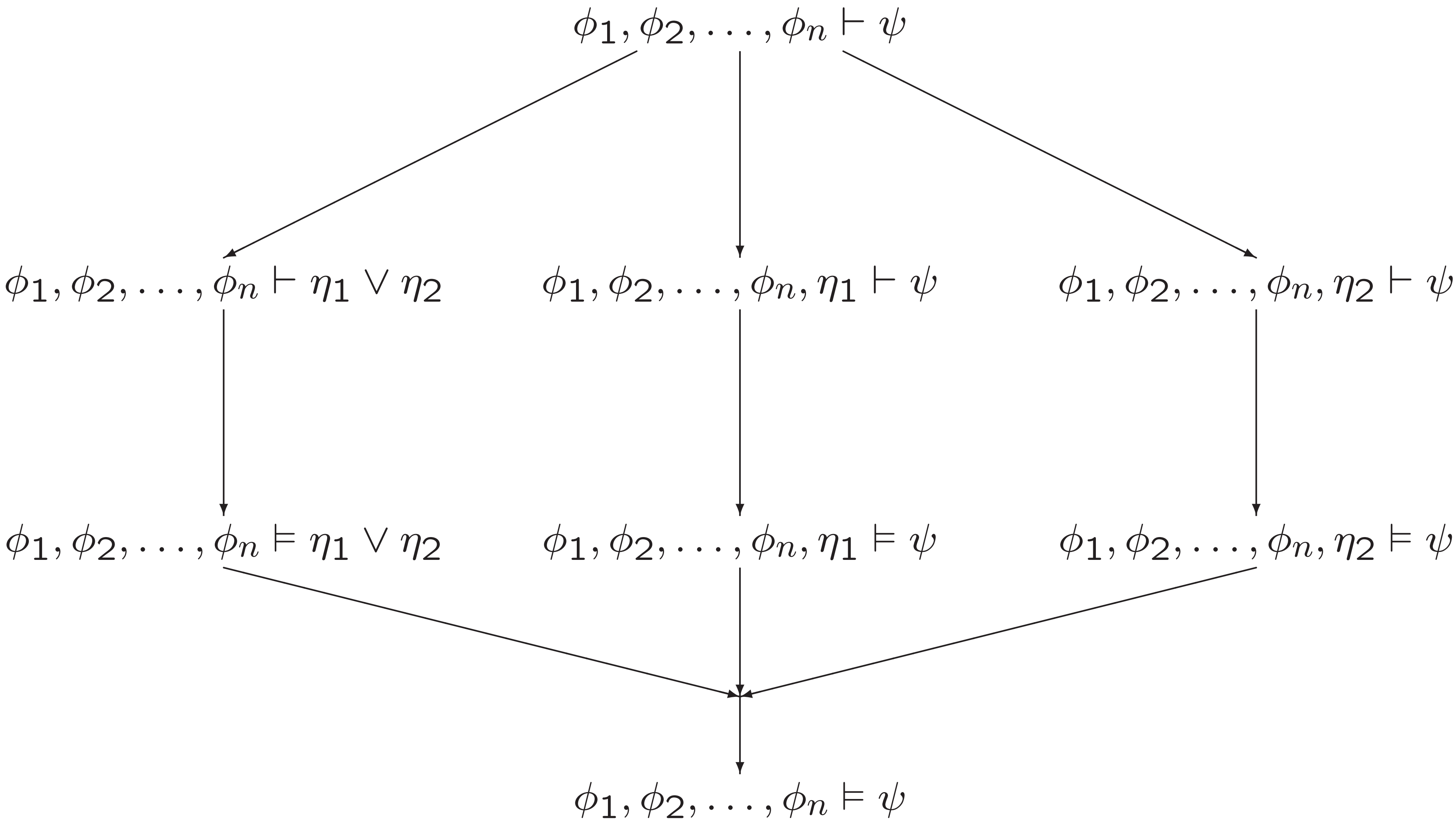
Complication:

1	$p \wedge q \to r$	premise
2	p	assumption
3	q	assumption
4	$p \wedge q$	$\wedge$i 2,3
5	r	$\to$e 1,4
6	$q \to r$	$\to$i 3–5
7	$p \to (q \to r)$	$\to$i 2–6

Induction step

Solution:

1	$p \wedge q \to r$	premise
2	p	premise
3	q	assumption
4	$p \wedge q$	$\wedge$i 2,3
5	r	$\to$ e 1,4
6	$q \to r$	$\to$ i 3–5


$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$
$\phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \vee \eta_2$
$\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi$
$\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi$
$\phi_1, \phi_2, \ldots, \phi_n \vDash \eta_1 \vee \eta_2$
$\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vDash \psi$
$\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vDash \psi$
$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$

A slide from lecture 4:

Basic rules of natural induction

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \qquad \psi}{\phi\wedge\psi}\ \wedge\mathsf{i}$	$\dfrac{\phi\wedge\psi}{\phi}\ \wedge\mathsf{e}_R \qquad \dfrac{\phi\wedge\psi}{\psi}\ \wedge\mathsf{e}_L$
$\vee$	$\dfrac{\phi}{\phi\vee\psi}\ \vee\mathsf{i}_R \qquad \dfrac{\psi}{\phi\vee\psi}\ \vee\mathsf{i}_L$	$\dfrac{\phi\vee\psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathsf{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi\rightarrow\psi}\ \rightarrow\mathsf{i}$	$\dfrac{\phi \qquad \phi\rightarrow\psi}{\psi}\ \rightarrow\mathsf{e}$

A slide from lecture 4:

Basic rules of natural induction

	introduction	*elimination*
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \qquad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$		$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

A slide from lecture 4:

Some useful derived rules

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}\ \text{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{PBC} \qquad\qquad \frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

Exercise 1.4.11.
For the soundness proof of Theorem 1.35 on page 46,

(a) explain why we could not use mathematical induction,
but had to resort to course-of-values induction

(b) give justifications for all inferences that were annotated with 'why?'

(c) complete the case analysis ranging over the final proof rule applied;
inspect the summary of natural deduction rules in the foregoing slides to see which cases are still missing.
Do you need to include derived rules?
What about the copy rule?

1.4.4. Completeness of propositional logic

If

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

is valid, then

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

holds.

1.4.4. Completeness of propositional logic

If

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

is valid, then

Step 1: $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$

Step 2: $\vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$

Step 3: $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

$$\vDash \phi$$

Step 1:

Definition 1.36.
A formula of propositional logic ϕ is called a *tautology* iff it evaluates to T under all its valuations, i.e., iff $\models \phi$.

Step 1:

If

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

is valid, then

$$\begin{array}{ll} \text{Step 1:} & \vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))) \\ \text{Step 2:} & \vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots))) \\ \text{Step 3:} & \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \end{array}$$

Step 3:

If

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

is valid, then

Step 1: $\vDash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots)))$

Step 2: $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\phi_3 \rightarrow (\ldots(\phi_n \rightarrow \psi)\ldots)))$

Step 3: $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

Step 2:

If

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

is valid, then

Step 1: $\vDash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$

Step 2: $\vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$

Step 3: $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

Theorem 1.37.
If $\vDash \eta$ holds, then $\vdash \eta$ is valid.
In other words, if η is a tautology, then η is a theorem.

‘Encode’ each line in the truth table of η as a sequent.

Proposition 1.38.
Let ϕ be a formula such that $p_1, p_2, \ldots, p_m$ are its only propositional atoms.
Let l be any line in ϕ's truth table.
For all $1 \leq i \leq m$, let $\widehat{p}_i$ be p_i if the entry in line l of p_i is T, otherwise $\widehat{p}_i$ is $\neg p_i$.
Then we have

1. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \phi$ is provable if the entry for ϕ in line l is T

2. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \neg\phi$ is provable if the entry for ϕ in line l is F

Example.

$m = 7$

p_1	p_2	p_3	p_4	p_5	p_6	p_7	ϕ	provable sequent
T	T	T	T	T	T	T	T	$p_1, p_2, p_3, p_4, p_5, p_6, p_7 \vdash \phi$
T	T	F	T	F	F	T	T	$p_1, p_2, \neg p_3, p_4, \neg p_5, \neg p_6, p_7 \vdash \phi$
T	F	F	F	T	T	F	T	$p_1, \neg p_2, \neg p_3, \neg p_4, p_5, p_6, \neg p_7 \vdash \phi$
F	F	F	F	F	F	F	T	$\neg p_1, \neg p_2, \neg p_3, \neg p_4, \neg p_5, \neg p_6, \neg p_7 \vdash \phi$
...	...	...	...	...	...	...	...	...
T	T	T	F	T	F	F	F	$p_1, p_2, p_3, \neg p_4, p_5, \neg p_6, \neg p_7 \vdash \neg \phi$
F	T	T	F	T	T	T	F	$\neg p_1, p_2, p_3, \neg p_4, p_5, p_6, p_7 \vdash \neg \phi$

Proposition 1.38.
Let ϕ be a formula such that $p_1, p_2, \ldots, p_m$ are its only propositional atoms.
Let l be any line in ϕ's truth table.
For all $1 \leq i \leq m$, let $\widehat{p}_i$ be p_i if the entry in line l of p_i is T, otherwise $\widehat{p}_i$ is $\neg p_i$.
Then we have

1. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \phi$ is provable if the entry for ϕ in line l is T

2. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \neg\phi$ is provable if the entry for ϕ in line l is F

Proof: by structural induction on formula ϕ

Base case...

Proposition 1.38.
Let ϕ be a formula (...)
Then we have

1. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \phi$ is provable if the entry for ϕ in line l is T

2. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \neg\phi$ is provable if the entry for ϕ in line l is F

Inductive step:
Suppose that Proposition 1.38 is valid for all formulas ϕ with height at most k_0 (induction hypothesis).

Now, consider a formula ϕ with height $k_0 + 1$.
If $\phi = \neg\phi_1 \ldots$