Logica (I&E)

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http://liacs.leidenuniv.nl/~vlietrvan1/logica/

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1.2 Natural deduction

Als je één goal meer maakt dan de ander, dan win je.

A slide from lecture 2:

1.2. Natural deduction

Proof rules

Premises $\phi_1, \phi_2, \ldots, \phi_n$

Conclusion ψ

Sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

A slide from lecture 1:

Propositional logic

Example 1.1. If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late.
Therefore, there were taxis at the station.

Example 1.2. If it is raining and Jane does not have her umbrella with her, then she will get wet. Jane is not wet. It is raining.
Therefore, Jane has her umbrella with her.

General structure:
If p and not q, then r. Not r. p. *Therefore*, q.

Propositional logic

Example 1.1. If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late.
Therefore, there were taxis at the station.

Example 1.2. If it is raining and Jane does not have her umbrella with her, then she will get wet. Jane is not wet. It is raining.
Therefore, Jane has her umbrella with her.

General structure:
$p \wedge \neg q \rightarrow r, \neg r, p \;\vdash\; q$

A slide from lecture 2:

The rules for conjunction

And-introduction:

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \wedge\mathsf{i}$$

And-elimination:

$$\frac{\phi \wedge \psi}{\phi} \wedge\mathsf{e}_1 \qquad\qquad \frac{\phi \wedge \psi}{\psi} \wedge\mathsf{e}_2$$

Alternative notation

And-elimination:

$$\frac{\phi \wedge \psi}{\phi}\ \wedge\mathsf{e}_R \qquad \frac{\phi \wedge \psi}{\psi}\ \wedge\mathsf{e}_L$$

A slide from lecture 2:

Example 1.4. Proof of: $p \wedge q, r \ \vdash \ q \wedge r$

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$q \wedge r$	$\wedge i$ 3, 2

In tree-like form. . .

Example 1.6. Proof of: $(p \wedge q) \wedge r, s \wedge t \ \vdash \ q \wedge s$

The rules of double negation

It is not true that it does not rain.

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\mathsf{e} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathsf{i}$$

Example 1.5.

$$p, \neg\neg(q \wedge r) \;\vdash\; \neg\neg p \wedge r$$

Proof. . .

The rule for eliminating implication

= Modus ponens

$$\frac{\phi \qquad \phi \to \psi}{\psi} \to \mathsf{e}$$

Je maakt één goal meer dan de ander.
Als je één goal meer maakt dan de ander, dan win je.

Example.

$$p \to (q \to r), p \to q, p \ \vdash \ r$$

Proof. . .

Another rule for eliminating implication

= Modus tollens

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\ \mathsf{MT}$$

Als je één goal meer maakt dan de ander, dan win je.
Je wint niet.

Example 1.7.

$$p \to (q \to r), p, \neg r \;\vdash\; \neg q$$

Proof. . .

Examples 1.8.

$$\neg p \rightarrow q, \neg q \ \vdash \ p$$

Proof. . .

$$p \rightarrow \neg q, q \ \vdash \ \neg p$$

Proof. . .

The rule implies introduction

Example.

$$p \to q \ \vdash \ \neg q \to \neg p$$

Proof. . .

The rule implies introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi} \to \mathrm{i}$$

We can only use a formula ϕ in a proof at a given point, if . . .

Example 1.9.

$$\neg q \to \neg p \ \vdash \ p \to \neg\neg q$$

Proof. . .

One-line argument

1	p	premise

Definition 1.10.
Logical formulas ϕ with valid sequent $\vdash \phi$ are *theorems*.

Example 1.11.

$$\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$$

Proof. . .

Boxproof

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	$\neg\neg$i 3
5	$\neg\neg q$	MT 2,4
6	q	$\neg\neg$e 5
7	r	$\to$ e 1,6
8	$p \to r$	$\to$ i 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	$\to$ i 2–8
10	$(q \to r) \to ((\neg q \to \neg p) \to (p \to r))$	$\to$ i 1–9

Structure of possible proof

Structure of formula (tree)

Remark 1.12.
This way, we may transform any proof of

$$\phi_1, \phi_2, \phi_3, \ldots, \phi_n \ \vdash \ \psi$$

into a proof of

$$\vdash \ \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$$

Example.

$$p \to (q \to r), p \to q, p \ \vdash \ r$$

Proof:

1	$p \to (q \to r)$	premise
2	$p \to q$	premise
3	p	premise
4	$q \to r$	$\to$ e 1,3
5	q	$\to$ e 2,3
6	r	$\to$ e 4,5

Example 1.13.

$$p \wedge q \to r \ \vdash \ p \to (q \to r)$$

Proof. . .

Example 1.14.

$$p \to (q \to r) \;\vdash\; p \wedge q \to r$$

Proof. . .

Hence, equivalent formulas:

$$p \wedge q \to r \;\dashv\vdash\; p \to (q \to r)$$

The rules for disjunction

Or-introduction:

$$\frac{\phi}{\phi \lor \psi} \lor\mathsf{i}_1 \qquad \frac{\psi}{\phi \lor \psi} \lor\mathsf{i}_2$$

Regardless of ψ / ϕ . . .

Alternative notation

Or-introduction:

$$\frac{\phi}{\phi \vee \psi}\ \vee\mathsf{i}_R \qquad\qquad \frac{\psi}{\phi \vee \psi}\ \vee\mathsf{i}_L$$

Or-elimination

How to infer χ from $\phi \vee \psi$?

Or-elimination

$$\frac{\phi \vee \psi \qquad \boxed{\begin{matrix}\phi\\ \vdots\\ \chi\end{matrix}} \qquad \boxed{\begin{matrix}\psi\\ \vdots\\ \chi\end{matrix}}}{\chi} \; \vee\mathrm{e}$$

Example.

$$p \vee q \ \vdash \ q \vee p$$

Proof. . .