Logica (I&E)

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`http://liacs.leidenuniv.nl/~vlietrvan1/logica/`

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Semantic tableaux for predicate logic
2.5. Undecidability of predicate logic
2.6. Expressiveness of predicate logic
Als Italianen één kans krijgen, maken ze er twee.

A slide from lecture 13:

Voorbeeld 9.6.

$$\forall y \exists x R(x,y) \ / \ \exists x \forall y R(x,y)$$

Valid or not?

Infinite branch,
which yields counter example with infinite domain.

E.g. $D \stackrel{\text{def}}{=} \mathbb{N}$, $\qquad R^{\mathcal{M}} \stackrel{\text{def}}{=} ' >'$

9.3. Een verfijning van de methode

Voorbeeld 9.7.

$$\forall y \exists x R(x,y) \;/\; \exists x \forall y R(x,y)$$

9.4. Samenvatting en opmerkingen

Possible situations:

1. Tableau closes (and is finite), hence *gevolgtrekking* is valid

2. There is a non-closing branch

   2.1 finite

   2.2 infinite

   describing counter example

Adequacy

A *gevolgtrekking* is valid, if and only if there is a closed tableau.

Undecidability

How to decide that we are on an infinite branch?

2.5. Undecidability of predicate logic

Deciding $\vDash \phi$ in propositional logic...

Deciding $\vDash \phi$ in predicate logic...

Decision problem: problem for which the answer is ‘yes’ or ‘no’:

Given . . . , is it true that . . . ?

Given an undirected graph $G = (V, E)$,
does G contain a Hamiltonian path?

Given a list of integers $x_1, x_2, \ldots, x_n$,
is the list sorted?

Given a state in a chess game,
will the white player win (assuming both players play optimally) ?

Solution to a decision problem. . .

Definition. Validity in predicate logic.

Given a logical formula ϕ in predicate logic,
does $\models \phi$ hold ?

Post correspondence problem = PCP

Instance:

1	10	011
101	00	11

Solution. . .

Instance:

1	10	011
101	00	11

Solution:

1	011	10	011
101	11	00	11
1	3	2	3

Instance:

0	01
10	1

No solution

Definition. The Post correspondence problem.
Given a finite sequence of pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ such that all s_i and t_i are binary strings of positive length,
is there a sequence of indices $i_1, i_2, \ldots, i_n$ with $n \geq 1$ such that the concatenation of strings $s_{i_1} s_{i_2} \ldots s_{i_n}$ equals $t_{i_1} t_{i_2} \ldots t_{i_n}$?

$i_1, i_2, \ldots, i_n$ need not all be distinct.

Exercise.

In each case below, either find a match for the instance of PCP or show that none exists.

a.

100	101	110
10	01	1010

b.

1	01	0	001
10	101	101	0

001	01	01	10
0	011	101	001

`http://jamesvanboxtel.com/projects/pcp-solver`

Problem reduction

Given: PCP is undecidable

Theorem 2.22. (Church, 1936)
The decision problem of validity in predicate logic is undecidable: no program exists which, given any ϕ, decides whether $\models \phi$.

Proof:

Assume that Validity *is* decidable.
Then an algorithm for PCP would be:

- Given an instance $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ of PCP, construct formula ϕ (such that . . .)

- Decide whether or not $\models \phi$

ϕ contains:

- constant e
  ('empty string')

- unary function symbols f_0 and f_1
  ('append 0/1 to string')

- binary predicate symbol P
  ('$P(s,t)$: there is sequence of indices $i_1, i_2, \ldots, i_m$ with $m \geq 1$, such that $s = s_{i_1} s_{i_2} \ldots s_{i_m}$ and $t = t_{i_1} t_{i_2} \ldots t_{i_m}$')

0100110

$$\phi \stackrel{\text{def}}{=} \phi_1 \wedge \phi_2 \rightarrow \phi_3$$

with

$$\begin{aligned}
\phi_1 &\stackrel{\text{def}}{=} \bigwedge_{i=1}^{k} P(f_{s_i}(e), f_{t_i}(e)) \\
\phi_2 &\stackrel{\text{def}}{=} \ldots \\
\phi_3 &\stackrel{\text{def}}{=} \ldots
\end{aligned}$$

$$\phi \stackrel{\text{def}}{=} \phi_1 \wedge \phi_2 \to \phi_3$$

with

$$\begin{aligned}
\phi_1 &\stackrel{\text{def}}{=} \ldots \\
\phi_2 &\stackrel{\text{def}}{=} \forall v \forall w \left(P(v,w) \to \bigwedge_{i=1}^{k} P(f_{s_i}(v), f_{t_i}(w)) \right) \\
\phi_3 &\stackrel{\text{def}}{=} \ldots
\end{aligned}$$

$$\phi \stackrel{\text{def}}{=} \phi_1 \wedge \phi_2 \to \phi_3$$

with

$$\begin{aligned}
\phi_1 &\stackrel{\text{def}}{=} \ldots \\
\phi_2 &\stackrel{\text{def}}{=} \ldots \\
\phi_3 &\stackrel{\text{def}}{=} \exists z P(z, z)
\end{aligned}$$

Suppose that $\models \ \phi$ . . .

Suppose that $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$ has some solution $(i_1, i_2, \ldots, i_n)$ . . .

Corollary 1.
Satisfiability for predicate logic

Corollary 2.

Provability: $\vdash \phi$

2.6. Expressiveness of predicate logic

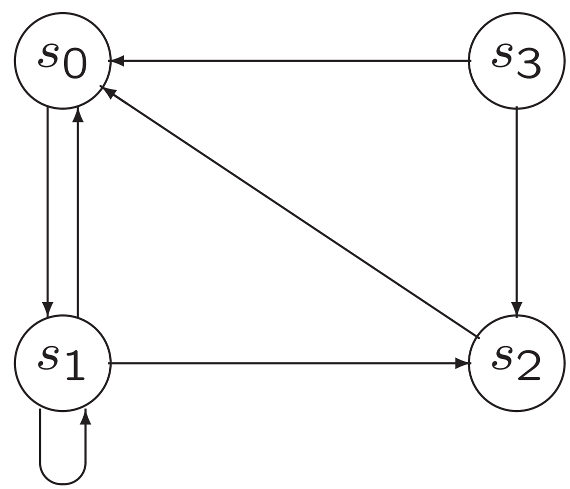
Reachability

```
int A[10];

int main ()
{ ...
  A[x*(y-1)] = 42;
  ...
  return 0;
}
```

Good state vs bad state

Reachability: Given nodes n and n' in a directed graph, is there a finite path of transitions from n to n' ?



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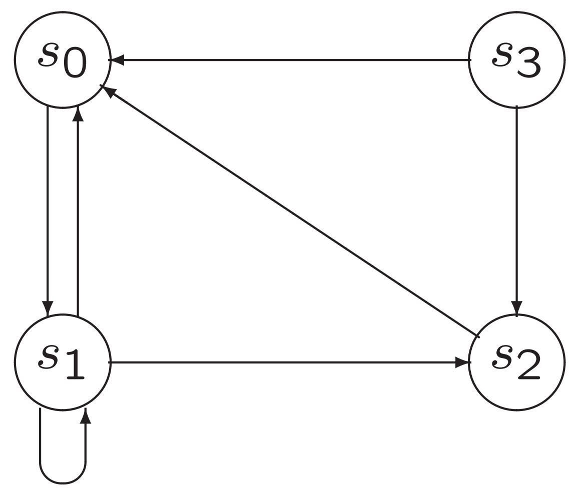

Example 2.23.
Take $R^{\mathcal{M}} = \{(s_0, s_1), (s_1, s_0), (s_1, s_1), (s_1, s_2), (s_2, s_0), (s_3, s_0), (s_3, s_2)\}$

Theorem 2.26.

Reachability is not expressible in predicate logic:
there is no predicate-logic formula ϕ with u and v as its only free variables and R as its only predicate symbol (of arity 2), such that ϕ holds in directed graphs iff there is a path in that graph from the node associated to u to the node associated to v.