Logica (I&E)

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`http://liacs.leidenuniv.nl/~vlietrvan1/logica/`

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2. Predicate logic
2.4. Semantics of predicate logic
Semantic tableaux for predicate logic

Wat is snelheid? Vaak verwisselt de sportpers snelheid met inzicht. Kijk, als ik iets eerder begin te lopen dan een ander, dan lijk ik sneller.

A slide from lecture 12:

Definition 2.14.
Let $\mathcal{F}$ be a set of function symbols and $\mathcal{P}$ a set of predicate symbols, each symbol with a fixed arity.
A model $\mathcal{M}$ of the pair $(\mathcal{F}, \mathcal{P})$ consists of the following set of data:

1. A non-empty set A, the universe of concrete values (one set);

2. for each nullary symbol $f \in \mathcal{F}$, a concrete element $f^{\mathcal{M}}$ of A;

3. for each $f \in \mathcal{F}$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$ from A^n, the set of n-tuples over A, to A;

4. for each $P \in \mathcal{P}$ with arity $n > 0$, a subset $P^{\mathcal{M}} \subseteq A^n$ of n-tuples over A;

5. $=^{\mathcal{M}}$ is equality on A

A slide from lecture 12:

Definition 2.17.
A look-up table or environment for a universe A of concrete values is a function l : **var** $\to A$ from the set of variables **var** to A.
For such an l, we denote by $l[x \mapsto a]$ the look-up table which maps x to a and any other variable y to $l(y)$.

A slide from lecture 12:

Definition 2.18.
Given a model $\mathcal{M}$ for a pair $(\mathcal{F}, \mathcal{P})$ and given a look-up table l, we define the satisfaction relation $\mathcal{M} \vDash_l \phi$ for each logical formula ϕ over the pair $(\mathcal{F}, \mathcal{P})$ and look-up table l by structural induction on ϕ.

If $\mathcal{M} \vDash_l \phi$ holds, we say that ϕ computes to T in the model $\mathcal{M}$ with respect to the look-up table l.

A slide from lecture 12:

Definition 2.18. (continued)

P: If ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, then we interpret the terms $t_1, t_2, \ldots, t_n$ in our set A by replacing all variables with their values according to l. In this way we compute concrete values $a_1, a_2, \ldots, a_n$ from A for each of these terms, where we interpret any function symbol $f \in \mathcal{F}$ by $f^{\mathcal{M}}$.

Now $\mathcal{M} \vDash_l P(t_1, t_2, \ldots, t_n)$ holds, iff $(a_1, a_2, \ldots, a_n)$ is in the set $P^{\mathcal{M}}$.

A slide from lecture 12:

Definition 2.18. (continued)

$\forall x$: The relation $\mathcal{M} \vDash_l \forall x \psi$ holds, iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for all $a \in A$.

$\exists x$: The relation $\mathcal{M} \vDash_l \exists x \psi$ holds, iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for some $a \in A$.

A slide from lecture 12:

Definition 2.18. (continued)

$\neg$: The relation $\mathcal{M} \vDash_l \neg\psi$ holds, iff $\mathcal{M} \vDash_l \psi$ does not hold.

$\vee$: The relation $\mathcal{M} \vDash_l \psi_1 \vee \psi_2$ holds, iff $\mathcal{M} \vDash_l \psi_1$ or $\mathcal{M} \vDash_l \psi_2$ holds.

$\wedge$: The relation $\mathcal{M} \vDash_l \psi_1 \wedge \psi_2$ holds, iff $\mathcal{M} \vDash_l \psi_1$ and $\mathcal{M} \vDash_l \psi_2$ holds.

$\rightarrow$: The relation $\mathcal{M} \vDash_l \psi_1 \rightarrow \psi_2$ holds, iff $\mathcal{M} \vDash_l \psi_2$ holds whenever $\mathcal{M} \vDash_l \psi_1$ holds.

Example 2.19.
$\mathcal{F} \stackrel{\text{def}}{=} \{\textbf{alma}\}$ (constant)
$\mathcal{P} \stackrel{\text{def}}{=} \{\textbf{loves}\}$ (binary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$
$\textbf{alma}^{\mathcal{M}} \stackrel{\text{def}}{=} a$
$\textbf{loves}^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (b, a), (c, a)\}$

None of Alma's lovers' lovers love her.

In predicate logic: $\phi = \ldots$

Is $M \models \phi$?

Example 2.19. (continued)
$\mathcal{F} \stackrel{\text{def}}{=} \{$**alma**$\}$ (constant)
$\mathcal{P} \stackrel{\text{def}}{=} \{$**loves**$\}$ (binary)

Model $\mathcal{M}'$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$
alma$^{\mathcal{M}'} \stackrel{\text{def}}{=} a$
loves$^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(b, a), (c, b)\}$

None of Alma's lovers' lovers love her.

In predicate logic: $\phi = \ldots$

Is $M' \ \vDash \ \phi$?

2.4.2. Semantic entailment

Definition 2.20.
Let Γ be a (possibly infinite) set of formulas in predicate logic and ψ a formula of predicate logic.

1. Semantic entailment $\Gamma \vDash \psi$, iff for all models $\mathcal{M}$ and look-up tables l, whenever $\mathcal{M} \vDash_l \phi$ holds for all $\phi \in \Gamma$, then $\mathcal{M} \vDash_l \psi$ holds as well.

3. Formula ψ is valid, iff $\mathcal{M} \vDash_l \psi$ holds for all models $\mathcal{M}$ and look-up tables l in which we can check ψ, i.e., iff $\vDash \psi$.

2. Formula ψ is satisfiable, iff there is some model $\mathcal{M}$ and some look-up table l such that $\mathcal{M} \vDash_l \psi$ holds.

4. The set Γ is consistent or satisfiable, iff there is some model $\mathcal{M}$ and and some look-up table l such that $\mathcal{M} \vDash_l \phi$ holds for all $\phi \in \Gamma$.

$\mathcal{M} \models \phi$ vs. $\phi_1, \phi_2, \ldots, \phi_n \models \psi$

Computational . . .

In propositional logic. . .

Example 2.21.

Is

$$\forall x(P(x) \to Q(x)) \models \forall x P(x) \to \forall x Q(x)$$

valid?

Is

$$\forall x P(x) \to \forall x Q(x) \models \forall x(P(x) \to Q(x))$$

valid?

2.4.3. The semantics of equality

Mild requirements on model. . .

$\phi_1, \phi_2, \ldots, \phi_n \ \vDash \ \psi$

Special predicate $=$: $t_1 = t_2$

Semantically, $=^{\mathcal{M}} \ = \ \ldots$

9. Predikaatlogica: semantische tableaus

[Van Benthem et al]

To find counter example of a *gevolgtrekking*

$$\phi_1, \ldots, \phi_n \;/\; \psi$$

in predicate logic

Predicate $P(x) = Px$ $\qquad R(x, y) = Rxy$

Substitution: $\phi[t/x] = [t/x]\phi$

Definition 2.14.
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1. A non-empty set A, the universe of concrete values;
2. for each nullary symbol $f \in \mathcal{F}$, a concrete element $f^{\mathcal{M}}$ of A;
3. for each $f \in \mathcal{F}$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$ from A^n, the set of n-tuples over A, to A;
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5. $=^{\mathcal{M}}$ is equality on A

1. = domein D 2–4 = interpretatiefunctie I
look-up table l = bedeling b

Extending semantic tableaux from propositional logic

- reduction rules for $\forall$ and $\exists$
- building up domain D
- building up *interpretatiefunctie* I (and *bedeling* b)

We ignore function symbols (including constants) and free variables.

Voorbeeld 9.1.

$$\forall x(A(x) \to B(x)),\ \forall x(B(x) \to C(x))\ /\ \forall x(A(x) \to C(x))$$

Valid or not?

Extra reduction rules

Suppose we already have $D = \{d_1, d_2, \ldots, d_k\}$

$\forall_L$:

$$\Phi, \forall x \phi \circ \Psi$$
$$\Phi, \phi[d/x] \circ \Psi$$

$\forall_R$:

$$\Phi \circ \forall x \phi, \Psi$$
$$\Phi \circ \phi[d_{k+1}/x], \Psi$$

where d is any existing d_i, and d_{k+1} is new

Voorbeeld 9.2.

$$\forall x(A(x) \to \forall y B(y)) \ / \ \forall x \forall y(A(x) \to B(y))$$

Valid or not?

Voorbeeld 9.3.

Alle kaaimannen zijn reptielen. Geen reptiel kan fluiten. Dus geen kaaiman kan fluiten.

$$\forall x(K(x) \to R(x)),\ \neg\exists x(R(x) \land F(x))\ /\ \neg\exists x(K(x) \land F(x))$$

Valid or not?

Study this example yourself

Voorbeeld 9.4.

Geen A is B. Geen B is C.
Dus geen A is C.

Geen professor is student. Geen student is gepromoveerd.
Dus geen professor is gepromoveerd.

$$\neg\exists x(A(x) \wedge B(x)),\ \neg\exists x(B(x) \wedge C(x))\ /\ \neg\exists x(A(x) \wedge C(x))$$

Valid or not?

Extra reduction rules

Suppose we already have $D = \{d_1, d_2, \ldots, d_k\}$

$\forall_{\mathsf{L}}$: $\begin{array}{c} \Phi, \forall x \phi \circ \Psi \\ \mid \\ \Phi, \phi[d/x] \circ \Psi \end{array}$	$\forall_{\mathsf{R}}$: $\begin{array}{c} \Phi \circ \forall x \phi, \Psi \\ \mid \\ \Phi \circ \phi[d_{k+1}/x], \Psi \end{array}$
$\exists_{\mathsf{L}}$: $\begin{array}{c} \Phi, \exists x \phi \circ \Psi \\ \mid \\ \Phi, \phi[d_{k+1}/x] \circ \Psi \end{array}$	$\exists_{\mathsf{R}}$: $\begin{array}{c} \Phi \circ \exists x \phi, \Psi \\ \mid \\ \Phi \circ \phi[d/x], \Psi \end{array}$

where d is any existing d_i, and d_{k+1} is new

Voorbeeld 9.5.

$$\exists x \forall y R(x, y) \ / \ \forall y \exists x R(x, y)$$

Valid or not?

Study this example yourself

Voorbeeld 9.6.

$$\forall y \exists x R(x,y) \;/\; \exists x \forall y R(x,y)$$

Valid or not?

Voorbeeld 9.6.

$$\forall y \exists x R(x,y) \ / \ \exists x \forall y R(x,y)$$

Valid or not?

Infinite branch,
which yields counter example with infinite domain.

E.g. $D \stackrel{\text{def}}{=} \mathbb{N}$, $\quad R^{\mathcal{M}} \stackrel{\text{def}}{=} ' >'$

9.4. Samenvatting en opmerkingen

Possible situations:

1. Tableau closes (and is finite), hence *gevolgtrekking* is valid

2. There is a non-closing branch

   2.1 finite

   2.2 infinite

   describing counter example

Undecidability

How to decide that we are on an infinite branch?

Adequacy

A *gevolgtrekking* is valid, if and only if there is a closed tableau.