LIACS
Leiden University

Examination Logic
19 December 2016, 10:00 - 13:00

1. **[0,5 point]** Show that $p \vee q, \neg p \vee \neg q \models q \to \neg p$ using a truth table.

p	q	$p \vee q$	$\neg p \vee \neg q$	$q \to \neg p$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	T

2. **[1,5 points]** Give a proof by means of natural deduction of the following sequents:

a) $p \to q \vdash \neg p \vee q$

1	$p \to q$	premise		
2	$p \vee \neg p$	LEM		
3	p	assumption	$\neg p$	assumption
4	q	$\to$e 3,1	$\neg p \vee q$	$\vee$i 3
5	$\neg p \vee q$	$\vee$i 4		
6	$\neg p \vee q$	$\vee$e 2 3-5,3-4		

b) $(p \to r) \vee (q \to r) \vdash (p \wedge q) \to r$

1	$p \wedge q$	assumption		
2	$(p \to r) \vee (q \to r)$	premise		
3	$p \to r$	assumption	$q \to r$	assumption
4	p	$\wedge$e 2	q	$\wedge$e 2
5	r	$\to$e 4, 3	r	$\to$e 4, 3
6	r	$\vee$e 1, 3-5		
7	$(p \wedge q) \to r$	$\to$i 2, 6		

c) $p \to \neg p, \neg p \to p \vdash \bot$

1	$p \to \neg p$	premise		
2	$\neg p \to p$	premise		
3	$p \vee \neg p$	LEM		
4	p	assumption	$\neg p$	assumption
5	$\neg p$	$\to$e 4,1	p	$\to$e 4,2
6	$\bot$	$\neg$e 4,5	$\bot$	$\neg$e 4,5
7	$\bot$	$\vee$e 3, 4-6		

3. **[1,5 points]** Use mathematical induction to prove that $1 + 2^2 + ... + 2^{n-1} = 2^n - 3$ for all integers $n \geq 3$.

Proof:
Let $n = 3$. Then $2^3 - 3 = 8 - 3 = 5$. And on the left hand side we get $1 + 2^2 = 1 + 4 = 5$Thus the statement we need to prove works for $n = 3$.

Assume now the statement holds for $n = k \geq 3$; that is,
$1 + 2^2 + 2^3 + 2^4 + ... + 2^{k-1} = 2^k - 3$

Let us consider the case when $n = k + 1$:

$1 + 2^2 + 2^3 + 2^4 + \ldots + 2^{k-1} + 2^k$
$= (2 + 2^2 + 2^3 + 2^4 + \ldots + 2^{k-1}) + 2^k$
$= (2^k - 3) + 2^k$ (here we use the induction hypothesis!)
$= 2 \times 2^k - 3$
$= 2^{k+1} - 3$

4. **[1,5 points]** Find which of the following formula is valid by computing the conjunctive normal form. Explain your answer.
   a) $(p \land \neg q) \lor (p \land q)$.
   We have $(p \land \neg q) \lor (p \land q) \equiv (p \lor (p \land q)) \land (\neg q \lor (p \land q))$ (distributive laws)
   $\equiv (p \lor p) \land (p \lor q) \land (\neg q \lor p) \land (\neg q \lor q)$ (distributive laws)
   Since the first three conjuncts are not valid, the entire formula is not valid.
   b) $\neg(p \land \neg q) \land (q \lor \neg p)$.
   We have $\neg(p \land \neg q) \land (q \lor \neg p) \equiv (\neg p \lor \neg\neg q) \land (q \lor \neg p)$ (De Morgan's laws)
   $\equiv (\neg p \lor q) \land (q \lor \neg p)$ (double negation)
   Since the first (or the second) conjunct is not valid, the entire formula is not valid.
   c) $((p \to q) \lor p) \land (p \lor \neg(r \land \neg r \land q))$.
   We have $((p \to q) \lor p) \land (p \lor \neg(r \land \neg r \land q)) \equiv ((\neg p \lor q) \lor p) \land (p \lor \neg(r \land \neg r \land q))$ (implication)
   $\equiv (\neg p \lor q \lor p) \land (p \lor (\neg r \lor \neg\neg r \lor \neg q))$ (De Morgan)
   $\equiv (\neg p \lor q \lor p) \land (p \lor \neg r \lor r \lor \neg q)$ (double negation)
   Since the both conjuncts are valid, the entire formula is valid.

5. **[1,5 points]** Apply the marking algorithm to check if the following Horn formulas are satisfiable:
   a) $(\top \to q) \land ((p \land q) \to r) \land (q \to p)$.
   Let us mark the propositions by using subscripts indicating the marking round. We have
   $$(\top_1 \to q_2) \land ((p_3 \land q_2) \to r_4) \land (q_2 \to p_3)$$
   Thus the formula is satisfiable under any valuations mapping p,q and r to T.
   b) $(\top \to p) \land ((p \land q) \to r) \land (p \to q) \land ((r \land p) \to q)$.
   Let us mark the propositions by using subscripts indicating the marking round. We have
   $$(\top_1 \to p_2) \land ((p_2 \land q_3) \to r_4) \land (p_2 \to q_3) \land ((r_4 \land p_2) \to q_3) .$$
   Thus the formula is satisfiable under any valuations mapping p,q and r to T.
   c) $(\top \to p) \land (p \to q) \land ((p \land q) \to r) \land (q \to \bot) \land (\top \to r)$.
   Let us mark the propositions by using subscripts indicating the marking round. We have
   $$(\top_1 \to p_2) \land (p_2 \to q_3) \land ((p_2 \land q) \to r_2) \land (q_3 \to \bot_4) \land (\top_1 \to r_2)$$
   Thus the formula is not satisfiable.

6. **[2 points]** Show the validity by means of natural deduction of the following sequents:
   a) $\forall x\, P(x) \vdash P(a) \to P(b)$.

1	$\forall x P(x)$	premise
2	$P(a)$	assumption
3	$P(b)$	$\forall$e 1
4	$P(a) \to P(b)$	$\to$i 2-3

   b) $a = b \land \neg P(a,b) \vdash \neg\forall x\, P(x,x)$.

1	$a = b \land \neg P(a,b)$	assumption
2	$a = b$	$\land$eL 1

3	$\neg P(a,b)$	$\wedge eR$ 1
4	$\forall x\, P(x,x)$	assumption
5	$P(a,a)$	$\forall e$ 4
6	$\neg P(a,a)$	$=e$ 2,3
7	$\bot$	$\neg e$ 5,6
8	$\neg\forall x\, P(x,x)$	$\neg i$ 4-7
9	$\exists x\, P(x)$	$\exists e$ 2,3-8

c) $\vdash \forall x \forall y(x = x \vee x = y)$.

1	x_0	$x_0 = x_0$	$=i$
2	y_0	$x_0 = x_0 \vee x_0 = y_0$	$\vee i_L$ 1
3		$\forall y(x_0 = x_0 \vee x_0 = y)$	$\forall i$ 2-2
4		$\forall x \forall y\, (x = x \vee x = y))$	$\forall i$ 1-3

d) $\vdash \neg\exists x \neg(x = x)$.

1		$\exists x \neg(x = x)$	assumption
2	x_0	$\neg(x_0 = x_0)$	assumption
3		$x_0 = x_0$	$=i$
4		$\bot$	$\neg e$ 2,3
5		$\bot$	$\exists e$ 1,2-4
6		$\neg\exists x \neg(x = x)$	$\neg i$ 1,5

7. **[1,5 points]** Consider the predicate formula $\forall x \exists y\, (P(x,y) \rightarrow f(x,c) = y)$, where c is a constant, P is a binary predicate and f is a binary function. Find a model which makes the formula true.
Take the model M with the set $\mathbb{N}$ of natural number as universe, P^M the usual order relation $<$, f^M the usual addition $+$, and $c^M = 1$ (the number 1). Then for every number n in $\mathbb{N}$ we can find a number m by taking $m = n+1$ so that $n < m$ and $n+1 = m$. Thus our model M makes the formula true.

The final score is given by the sum of the points obtained.