LIACS
Leiden University

Examination Logic

13 June 2016, 14:00 - 17:00

1. **[1 point]** Find a formula ϕ of propositional logic which contains all and only the atoms p and q and r, and which is true only when p, q and r are all true or when $\neg p \wedge q$ is true. Give its truth table.

The truth table for the formula ϕ must be as follows:

p	q	r	$\neg p \wedge q$	ϕ
F	F	F	F	F
F	F	T	F	F
F	T	F	T	T
F	T	T	T	T
T	F	F	F	F
T	F	T	F	F
T	T	F	F	F
T	T	T	F	T

The resulting ϕ in CNF is obtained from those line where ϕ evaluates to true, that is ϕ is
$(p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee \neg r)$.

2. **[2 points]** Give a proof in *natural deduction* for each of the following sequents:

a) $(p \wedge q) \vee (\neg r \wedge p) \vdash r \to p$

1	$(p \wedge q) \vee (\neg r \wedge p)$	premise		
2	r	assumption		
3	$p \wedge q$	assumption	$\neg r \wedge p$	assumption
4	p	$\wedge e_L$ 3	p	$\wedge i_R$ 3
5	p	$\vee e$ 1,3-4,3-4		
6	$r \to p$	$\to i$ 2-5		

b) $p \to q \vdash q \vee \neg p$

	$p \to q$	premise		
2	$p \vee \neg p$	LEM		
3	p	assumption	$\neg p$	assumption
4	q	$\to e$ 3,1	$q \vee \neg p$	$\vee i_R$ 3
5	$q \vee \neg p$	$\vee i_L$ 4		
6	$q \vee \neg p$	$\vee e$ 2,3-5,3-5		

c) $p \wedge q, \neg(p \wedge r) \vdash \neg r$

1	$p \wedge q$	premise
2	$\neg(p \wedge r)$	premise
3	r	assumption
4	p	$\wedge e_L$ 1
5	$p \wedge r$	$\wedge i$ 3,4
6	$\perp$	$\neg e$ 2,5
7	$\neg r$	$\neg i$ 3-6

c) $p \to (q \to r) \vdash (p \to q) \to (p \to r)$

1	$p \to (q \to r)$	premise
2	$p \to q$	assumption
3	p	assumption

4	q	$\rightarrow e$ 3,2
5	$q \rightarrow r$	$\rightarrow e$ 3,1
6	r	$\rightarrow e$ 4,5
7	$p \rightarrow r$	$\rightarrow i$ 3-6
8	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 2-7

3. **[1,5 points]** Compute the *conjunctive normal form* of the following formulas and check if they are valid. Explain your answers and state which laws (de Morgan law, distributive law, …) you have applied.

a) $(p \wedge \neg q) \vee (p \wedge q)$.

We have $(p \wedge \neg q) \vee (p \wedge q) \equiv (p \vee (p \wedge q)) \wedge (\neg q \vee (p \wedge q))$ (distributive laws)

$\equiv (p \vee p) \wedge (p \vee q) \wedge (\neg q \vee p) \wedge (\neg q \vee q)$ (distributive laws)

Since, for example, in the first conjunct there is no atom appearing both positively and negatively, the entire formula is not valid.

b) $\neg(p \wedge \neg q) \wedge (q \vee \neg p)$.

We have $\neg(p \wedge \neg q) \wedge (q \vee \neg p) \equiv (\neg p \vee \neg\neg q) \wedge (q \vee \neg p)$ (De Morgan's laws)

$\equiv (\neg p \vee q) \wedge (q \vee \neg p)$ (double negation)

$\equiv (\neg p \vee (q \vee \neg p)) \wedge (q \vee (q \vee \neg p))$ (distributive laws)

$\equiv (\neg p \vee q \vee \neg p) \wedge (q \vee q \vee \neg p)$ (associativity)

Since, for example, in the first conjunct there is no atom appearing both positively and negatively, the entire formula is not valid.

c) $((p \rightarrow q) \vee p) \wedge (p \vee \neg(r \wedge \neg r \wedge q))$.

We have $((p \rightarrow q) \vee p) \wedge (p \vee \neg(r \wedge \neg r \wedge q)) \equiv ((\neg p \vee q) \vee p) \wedge (p \vee \neg(r \wedge \neg r \wedge q))$ (implication)

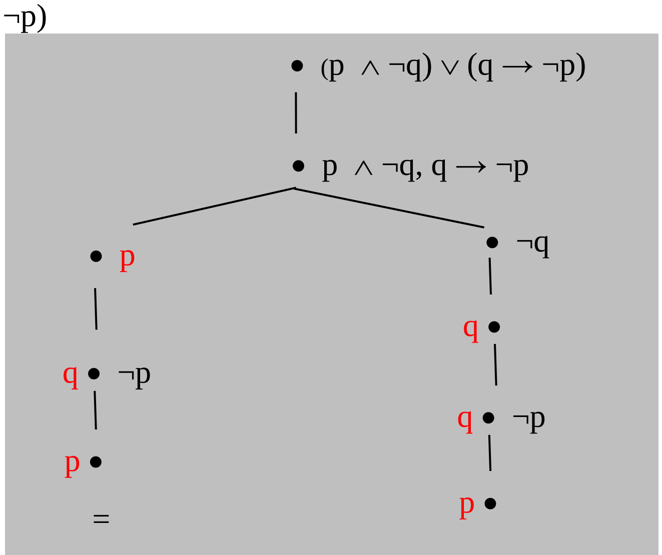
$\equiv (\neg p \vee q \vee p) \wedge (p \vee (\neg r \vee \neg\neg r \vee \neg q))$ (De Morgan)

$\equiv (\neg p \vee q \vee p) \wedge (p \vee \neg r \vee r \vee \neg q)$ (double negation)

Since the both conjuncts have one atomic proposition appearing positively and negatively, the entire formula is valid.

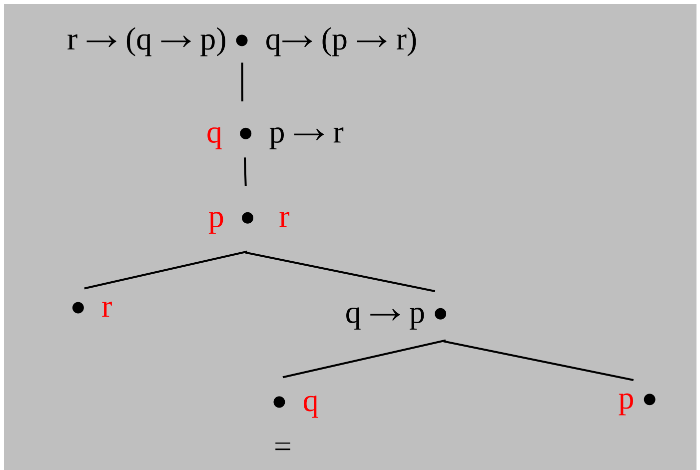
4. **[2 points]** Use the *tableau method* to find a counterexample for the validity of each of the following sequent

a) $\vdash (p \wedge \neg q) \vee (q \rightarrow \neg p)$



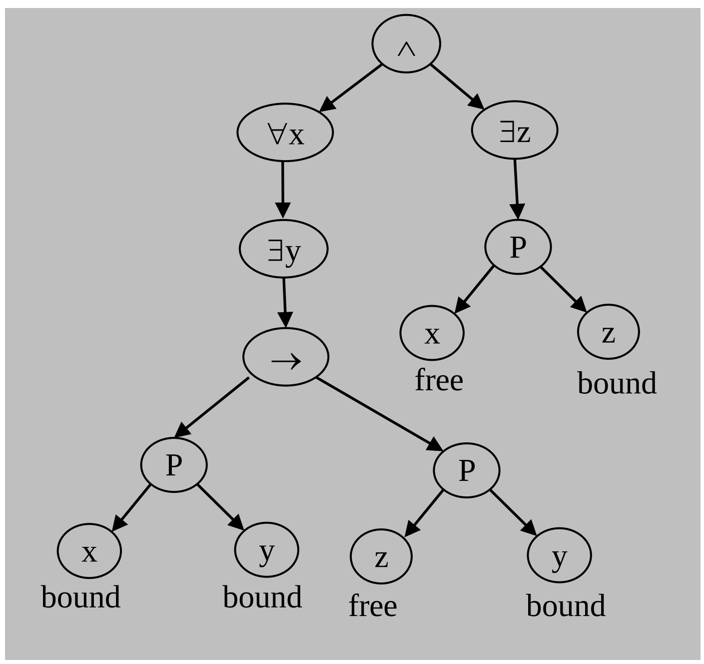
The open branch gives us the counterexample: p is true and q is true.

b) $r \rightarrow (q \rightarrow p) \vdash q \rightarrow (p \rightarrow r)$



The two open branches give us the same counterexample: p is true, q is true, and r is false.

5. **[1 point]** Let P be a predicate symbol of arity 2, and f, g be two function symbols of arity 1 and 2, respectively.
   a) Draw the *parse tree* of the formula ϕ given by $\forall x \exists y(P(x,y) \rightarrow P(z,y)) \wedge \exists z P(x,z)$, where x,y,z are three variables.



   b) Compute the *substitutions* $\phi[t/x]$ and $\phi[t/y]$ where $t = g(y,f(x))$. Is the term t *free for* z in ϕ?
   $\phi[t/x] = \forall x \exists y(P(x,y) \rightarrow P(z,y)) \wedge \exists z P(g(y,f(x)),z)$, and $\phi[t/y] = \phi$ as y always occurs as a bound variable ϕ. The term t is not free in z, because there exists a leave z free in ϕ but with the variable x (but also y) of t falling in the scope of a quantifier.

6. **[1 point]** Let P be a unary predicate symbol, R a binary predicate symbol and c be a constant. Consider the model M with A = {c,d,e,f}, $P^M = \{c\}$, $R^M = \{(c,d),(d,e),(e.f)\}$, and $c^M = c$.
   a) Does $M \vDash P(c)$ hold? Explain your answer.
   Yes, because $c \in P^M$.
   b) Does $M \vDash \forall x \exists y R(x,y)$ hold? Explain your answer.
   No, because there is no pair with f as first component (i.e. (f,…)) in R^M.
   c) Does $M \vDash \forall x(P(x) \rightarrow \exists y R(x,y))$ hold? Explain your answer.

Yes, because c is the only element making P(x), and the pair (c,d) is in R^M.

d) Does there exist a look-up table ℓ such that $M \vDash_\ell R(c,x)$ hold? Explain your answer.

Yes, take any look-up table with $\ell(x) = d$, so that the pair (c,d) is in R^M.

7. **[1,5 points]** Show the validity of each of the following sequent by means of a proof in *natural deduction*, where P is a predicate of arity 1, R is a predicate of arity 2, and a, b are two constants:

a) $\forall x P(x) \vdash \forall x (P(x) \wedge P(x))$.

1		$\forall x P(x)$	premise
2	x_0	$P(x_0)$	$\forall$e 1
3		$P(x_0) \wedge P(x_0)$	$\wedge$i 2, 2
4		$\forall x\,(P(x) \wedge P(x))$	$\forall$i 2-3

b) $a = b, \neg R(a,b) \vdash \neg \forall x R(x,x)$.

1	$a = b$	assumption
2	$\neg R(a,b)$	assumption
3	$\forall x R(x,x)$	assumption
4	$R(b,b)$	$\forall$e 3
5	$\neg R(b,b)$	=e 1,2 {using $\neg R(x,b)$}
6	$\bot$	$\neg$e 4,5
7	$\neg \forall x R(x,x)$	$\neg$i 3-6

c) $\vdash \forall x \exists y (P(x) \rightarrow P(y))$.

1	x_0		
2		$P(x_0)$	assumption
3		$P(x_0)$	copy 2
4		$P(x_0) \rightarrow P(x_0)$	$\rightarrow$i 2-3
5		$\exists y (P(x_0) \rightarrow P(y))$	$\exists$i 4
6		$\forall x \exists y (P(x) \rightarrow P(y))$	$\forall$i 1-5

The final score is given by the sum of the points obtained.