LIACS | Examination Logic
Leiden University | 10 July 2015, 10:00 - 13:00

1. **[1 point]** Prove by induction that $\sum_{k=1}^{n}(2k-1) = n^2$ for all positive integers n ≥ 1.

2. **[2 points]** Give a proof in natural deduction for each of the following sequents:
   a) $\neg p \wedge (q \vee r) \vdash q \to (r \to \neg p)$
   b) $p \to q, \neg(q \vee r) \vdash \neg p$
   c) $p \wedge q, \neg p \vdash \neg q \to p$
   d) $p \to (p \to (p \to q)) \vdash \neg q \to \neg p$

3. **[1,5 points]** Give a semantic tableau to show that the following sequents are not valid:
   a) $p \vee q \vdash \neg p \wedge \neg q$
   b) $p \vee q \vdash q \to p$
   c) $p \to q \vdash \neg p \to \neg q$

4. **[1 point]** Consider the following truth table for the formulas ϕ and ψ:

p	q	ϕ	ψ
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	F

   Find propositional logic formulas in *conjunctive normal form* equivalent to ϕ and ψ, respectively.

5. **[1,5 points]**
   a) Give a predicate logic formula ϕ expressing the fact that there are at least two elements.
   b) Give a predicate logic formula ϕ expressing the fact that there are exactly two elements.
   c) Give a predicate logic formula ϕ such that $\phi[y/x]$ is not the same as $(\phi[z/x])[y/z]$.

6. **[1 point]** Write a formula ϕ in predicate logic such that, for each of the following pair of models M and N, ϕ holds in the model M but not in the model N.
   a) $M = (\mathbb{Q}, P^M)$ and $N = (\mathbb{Z}, ,P^N)$. Here $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integers, and P^M is the strict (thus not equal) order relation < between rational numbers, and, P^N is the strict order relation < between integer number.
   b) $M = (\mathbb{Z}, P^M)$ and $N = (\mathbb{Z}, P^N)$. Here $\mathbb{Z}$ is the set of integers, and P^M is the strict (thus not equal) order relation < between integers, and, P^N is the less or equal order relation ≤ between integers.

7. **[2 points]** Show the validity of each of the following sequent by means of a proof in natural deduction, where P, Q, are predicates of arity 1, and R is a predicate of arity 2:
   a) $\forall y \neg P(y), (Q(y) \vee R(y,y)) \to P(x) \vdash \exists x \neg(Q(x) \vee R(x,x))$
   b) $\exists x \forall y R(x,y) \vdash \forall y \exists x R(x,y)$
   c) $P(x) \to \forall y Q(y) \vdash \forall y(P(x) \to Q(y))$

The final score is given by the sum of the points obtained.