Logica (I&E)

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`http://liacs.leidenuniv.nl/~vlietrvan1/logica/`

Rudy van Vliet
kamer 140 Snellius, tel. 071-527 2876
rvvliet(at)liacs(dot)nl

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1.4 Semantics of propositional logic
1.2 Natural deduction

Voordat ik een fout maak, maak ik die fout niet.

A slide from lecture 3:

Definition 1.10.
Logical formulas ϕ with valid sequent $\vdash \phi$ are *theorems*.

Example 1.11.

$$\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$$

Proof. . .

Boxproof

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	¬¬i 3
5	$\neg\neg q$	MT 2,4
6	q	¬¬e 5
7	r	→ e 1,6
8	$p \to r$	→ i 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	→ i 2–8
10	$(q \to r) \to ((\neg q \to \neg p) \to (p \to r))$	→ i 1–9

Remark 1.12.
This way, we may transform any proof of

$$\phi_1, \phi_2, \phi_3, \ldots, \phi_n \vdash \psi$$

into a proof of

$$\vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots(\phi_n \to \psi)\ldots)))$$

A slide from lecture 3:

Or-elimination

$$\dfrac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$$

Example 1.18.

Disjunctions distribute over conjunctions.

$$p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$$

$$(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$$

Proof. . .

The rule ‘copy’

$$\vdash p \to (q \to p)$$

Proof. . .

The rules for negation

Definition 1.19.
Contradictions are expressions of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$, where ϕ is any formula.

$$p \wedge \neg p \vdash q$$

p: The moon is made of green cheese.
q: I like pepperoni on my pizza.

Bottom-elimination:

$$\frac{\bot}{\phi}\ \bot\mathrm{e}$$

Not-elimination:

$$\frac{\phi \qquad \neg\phi}{\bot}\ \neg\mathrm{e}$$

Example 1.20.

$$\neg p \vee q \vdash p \to q$$

Proof. . .

Not-introduction:

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathsf{i}$$

Example 1.21.

$$p \to q, p \to \neg q \vdash \ldots$$

Example 1.21.

$$p \to q, p \to \neg q \vdash \neg p$$

Proof. . .

A slide from lecture 3:

Example 1.7.

$$p \to (q \to r), p, \neg r \vdash \neg q$$

Proof. . .

Example 1.22.

$$p \to (q \to r), p, \neg r \vdash \neg q$$

Proof without Modus Tollens. . .

A slide from lecture 2:

Propositional logic

Example 1.1. If the train arrives late and there are no taxis at the station, then John is late for his meeting. John is not late for his meeting. The train did arrive late.
Therefore, there were taxis at the station.

Example 1.2. If it is raining and Jane does not have her umbrella with her, then she will get wet. Jane is not wet. It is raining.
Therefore, Jane has her umbrella with her.

General structure:
$p \wedge \neg q \to r, \neg r, p \vdash q$

Example 1.23.

$$p \wedge \neg q \to r, \neg r, p \vdash q$$

Proof. . .

1.2.2. Derived rules

Modus tollens

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi} \text{ MT}$$

Proof. . .

1.2.2. Derived rules

Double negation-introduction

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\mathsf{i}$$

Proof. . .

1.2.2. Derived rules

Proof by contradiction

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{PBC}$$

Proof. . .

1.2.2. Derived rules

Law of the excluded middle

$$\frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

Proof. . .

Example 1.24.

$$p \to q \vdash \neg p \vee q$$

Proof (using LEM)...

Basic rules of natural induction

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \qquad \psi}{\phi\wedge\psi}\ \wedge\mathsf{i}$	$\dfrac{\phi\wedge\psi}{\phi}\ \wedge\mathsf{e}_R \qquad \dfrac{\phi\wedge\psi}{\psi}\ \wedge\mathsf{e}_L$
$\vee$	$\dfrac{\phi}{\phi\vee\psi}\ \vee\mathsf{i}_R \qquad \dfrac{\psi}{\phi\vee\psi}\ \vee\mathsf{i}_L$	$\dfrac{\phi\vee\psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathsf{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi\rightarrow\psi}\ \rightarrow\mathsf{i}$	$\dfrac{\phi \qquad \phi\rightarrow\psi}{\psi}\ \rightarrow\mathsf{e}$

Basic rules of natural induction

	introduction	*elimination*
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i}$	$\dfrac{\phi \qquad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$		$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$		$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

Some useful derived rules

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}\ \text{MT} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \text{PBC} \qquad\qquad \frac{}{\phi \vee \neg\phi}\ \text{LEM}$$

1.2.4 Provable equivalence

Definition 1.25.
Let ϕ and ψ be formulas of propositional logic.
We say that ϕ and ψ are *provably equivalent*,
if and only if the sequents $\phi \vdash \psi$ and $\psi \vdash \phi$ are valid;
Notation: $\phi \dashv\vdash \psi$

1.2.4 Provable equivalence

Examples:

$$\neg(p \wedge q) \dashv\vdash \neg q \vee \neg p \qquad \neg(p \vee q) \dashv\vdash \neg q \wedge \neg p$$

$$p \rightarrow q \dashv\vdash \neg q \rightarrow \neg p \qquad p \rightarrow q \dashv\vdash \neg p \vee q$$

$$p \wedge q \rightarrow p \dashv\vdash r \vee \neg r \qquad p \wedge q \rightarrow r \dashv\vdash p \rightarrow (q \rightarrow r)$$

1.2.5. An aside: proof by contradiction

Intuitionistic logicians do not accept

$$\dfrac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \mathsf{PBC} \qquad \dfrac{}{\phi \vee \neg\phi}\ \mathsf{LEM} \qquad \dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathsf{e}$$

Theorem 1.26.
There exist irrational numbers a and b such that a^b is rational.

Proof. . .