Logica (I&E)

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1.4 Semantics of propositional logic
1.2 Natural deduction

Als je een speler ziet sprinten, is hij te laat vertrokken.

A slide from lecture 2:

Truth tables

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	ψ	$\phi \to \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	$\neg\phi$
T	F
F	T

1.4.3. Soundness of propositional logic

Definition 1.34.
If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to T, ψ evaluates to T as well we say that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds and $\vDash$ the *semantic entailment* relation.

Examples semantic entailment

1. $p \wedge q \vDash p$?

2. $p \vee q \vDash p$?

3. $\neg q, p \vee q \vDash p$?

4. $p \vDash q \vee \neg q$?

A slide from lecture 2:

1.2. Natural deduction

Proof rules

Premises $\phi_1, \phi_2, \ldots, \phi_n$

Conclusion ψ

Sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

A slide from lecture 2:

The rules for conjunction

And-introduction:

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \ \wedge\mathsf{i}$$

And-elimination:

$$\frac{\phi \wedge \psi}{\phi} \ \wedge\mathsf{e}_1 \qquad\qquad \frac{\phi \wedge \psi}{\psi} \ \wedge\mathsf{e}_2$$

Alternative notation

And-elimination:

$$\frac{\phi \land \psi}{\phi} \land\mathsf{e}_R \qquad\qquad \frac{\phi \land \psi}{\psi} \land\mathsf{e}_L$$

The rules of double negation

It is not true that it does not rain.

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\mathsf{e} \qquad\qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\mathsf{i}$$

Example 1.5.

$$p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$$

Proof. . .

The rule for eliminating implication

= Modus ponens

$$\frac{\phi \qquad \phi \to \psi}{\psi} \to \mathsf{e}$$

Je ziet een speler sprinten.
Als je een speler ziet sprinten, is hij te laat vertrokken.

Example.

$$p \to (q \to r), p \to q, p \vdash r$$

Proof. . .

Another rule for eliminating implication

= Modus tollens

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}\ \mathsf{MT}$$

Als je een speler ziet sprinten, is hij te laat vertrokken.
De speler is niet te laat vertrokken.

Example 1.7.

$$p \to (q \to r), p, \neg r \vdash \neg q$$

Proof. . .

Examples 1.8.

$$\neg p \to q, \neg q \vdash p$$

Proof. . .

$$p \to \neg q, q \vdash \neg p$$

Proof. . .

The rule implies introduction

Example.

$$p \to q \vdash \neg q \to \neg p$$

Proof. . .

The rule implies introduction

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \to \psi} \to \mathsf{i}$$

We can only use a formula ϕ in a proof at a given point, if . . .

Example 1.9.

$$\neg q \to \neg p \vdash p \to \neg\neg q$$

Proof. . .

One-line argument

1	p	premise

Definition 1.10.
Logical formulas ϕ with valid sequent $\vdash \phi$ are *theorems*.

Example 1.11.

$$\vdash (q \to r) \to ((\neg q \to \neg p) \to (p \to r))$$

Proof. . .

Boxproof

1	$q \to r$	assumption
2	$\neg q \to \neg p$	assumption
3	p	assumption
4	$\neg\neg p$	¬¬i 3
5	$\neg\neg q$	MT 2,4
6	q	¬¬e 5
7	r	→ e 1,6
8	$p \to r$	→ i 3–7
9	$(\neg q \to \neg p) \to (p \to r)$	→ i 2–8
10	$(q \to r) \to ((\neg q \to \neg p) \to (p \to r))$	→ i 1–9

Structure of possible proof

Structure of formula (tree)

Example 1.13.

$$p \wedge q \rightarrow r \vdash p \rightarrow (q \rightarrow r)$$

Proof. . .

Example 1.14.

$$p \to (q \to r) \vdash p \land q \to r$$

Proof. . .

Hence, equivalent formulas:

$$p \land q \to r \dashv\vdash p \to (q \to r)$$

The rules for disjunction

Or-introduction:

$$\frac{\phi}{\phi \vee \psi} \ \vee\mathsf{i}_1 \qquad \frac{\psi}{\phi \vee \psi} \ \vee\mathsf{i}_2$$

Regardless of ψ / ϕ . . .

Alternative notation

Or-introduction:

$$\frac{\phi}{\phi \lor \psi} \lor\mathsf{i}_R \qquad \frac{\psi}{\phi \lor \psi} \lor\mathsf{i}_L$$

Or-elimination

How to infer χ from $\phi \vee \psi$?

Or-elimination

$$\dfrac{\phi \lor \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \lor\mathrm{e}$$

Example.

$$p \lor q \vdash q \lor p$$

Proof. . .

Example 1.16.

$$q \to r \vdash p \vee q \to p \vee r$$

Proof. . .