Fundamentele Informatica 3

voorjaar 2016

http://www.liacs.leidenuniv.nl/~vlietrvan1/fi3/

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college 15, 17 mei 2016

10. Computable Functions
10.3. Gödel Numbering
10.4. All Computable Functions are μ-Recursive
10.5. Other Approaches to Computability

A slide from lecture 14:

Definition 10.17.
The Gödel Number of a Sequence of Natural Numbers

For every $n \geq 1$ and every finite sequence $x_0, x_1, \ldots, x_{n-1}$ of n natural numbers, the *Gödel number* of the sequence is the number

$$\mathit{gn}(x_0, x_1, \ldots, x_{n-1}) = 2^{x_0} 3^{x_1} 5^{x_2} \ldots (\mathit{PrNo}(n-1))^{x_{n-1}}$$

where $\mathit{PrNo}(i)$ is the ith prime (Example 10.13).

Configuration of Turing machine determined by

- state
- position on tape
- tape contents

A slide from lecture 4:

Assumptions:

1. Names of the states are irrelevant.

2. Tape alphabet Γ of every Turing machine T is subset of infinite set $\mathcal{S} = \{a_1, a_2, a_3, \ldots\}$, where $a_1 = \Delta$.

A slide from lecture 4:

Definition 7.33. An Encoding Function

Assign numbers to each state:
$n(h_a) = 1$, $n(h_r) = 2$, $n(q_0) = 3$, $n(q) \geq 4$ for other $q \in Q$.

Assign numbers to each tape symbol:
$n(a_i) = i$.

Assign numbers to each tape head direction:
$n(R) = 1$, $n(L) = 2$, $n(S) = 3$.

Now different numbering

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be Turing machine

States:

h_a	h_r	q_0	$\ldots$	$.$
0	1	2	$\ldots$	s_T

with $s_T = \ldots$

Tape symbols:

Δ	$\ldots$	$.$
0	$\ldots$	ts_T

with $ts_T = \ldots$

Now different numbering

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be Turing machine

States:

h_a	h_r	q_0	$\ldots$	$.$
0	1	2	$\ldots$	s_T

with $s_T = |Q| + 1$

Tape symbols:

Δ	$\ldots$	$.$
0	$\ldots$	ts_T

with $ts_T = |\Gamma|$

$$\begin{aligned} \textit{tapenumber}(\Delta 1 a \Delta b 1 \Delta) &= 2^0 3^1 5^2 7^0 11^3 13^1 17^0 \ldots \\ \textit{confignumber} &= 2^q 3^P 5^{\textit{tapenumber}} \end{aligned}$$

10.4. All Computable Functions are μ-Recursive

A slide from lecture 13

Definition 10.15. μ-Recursive Functions

The set $\mathcal{M}$ of μ-recursive, or simply *recursive*, partial functions is defined as follows.

1. Every initial function is an element of $\mathcal{M}$.

2. Every function obtained from elements of $\mathcal{M}$ by composition or primitive recursion is an element of $\mathcal{M}$.

3. For every $n \geq 0$ and every total function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ in $\mathcal{M}$, the function $M_f : \mathbb{N}^n \to \mathbb{N}$ defined by
$$M_f(X) = \mu\, y[f(X, y) = 0]$$
is an element of $\mathcal{M}$.

$$X = (x_1, x_2, \ldots, x_n)$$

$$q_0 \quad \boxed{\underline{\Delta} 1^{x_1} \Delta 1^{x_2} \Delta \cdots \Delta 1^{x_n} \Delta \cdots}$$

$$\vdash$$

$$q_{\ldots} \quad \boxed{\Delta \underline{1^{x_1}} \Delta 1^{x_2} \Delta \cdots \Delta 1^{x_n} \Delta \cdots}$$

$$\vdash$$

$$\cdots$$

$$\vdash$$

$$h_a \quad \boxed{\underline{\Delta} 1^{f(X)} \Delta \cdots}$$

$$f(X)$$

$$X = (x_1, x_2, \ldots, x_n)$$

$$q_0 \; \boxed{\underline{\underline{\Delta}}1^{x_1}\Delta 1^{x_2}\Delta \cdots \Delta 1^{x_n}\Delta \cdots} \xrightarrow{\textit{confignumber}} m_0$$

$$\vdash$$

$$q_{\ldots} \; \boxed{\Delta \underline{1^{x_1}}\Delta 1^{x_2}\Delta \cdots \Delta 1^{x_n}\Delta \cdots} \xrightarrow{\textit{confignumber}} m_1$$

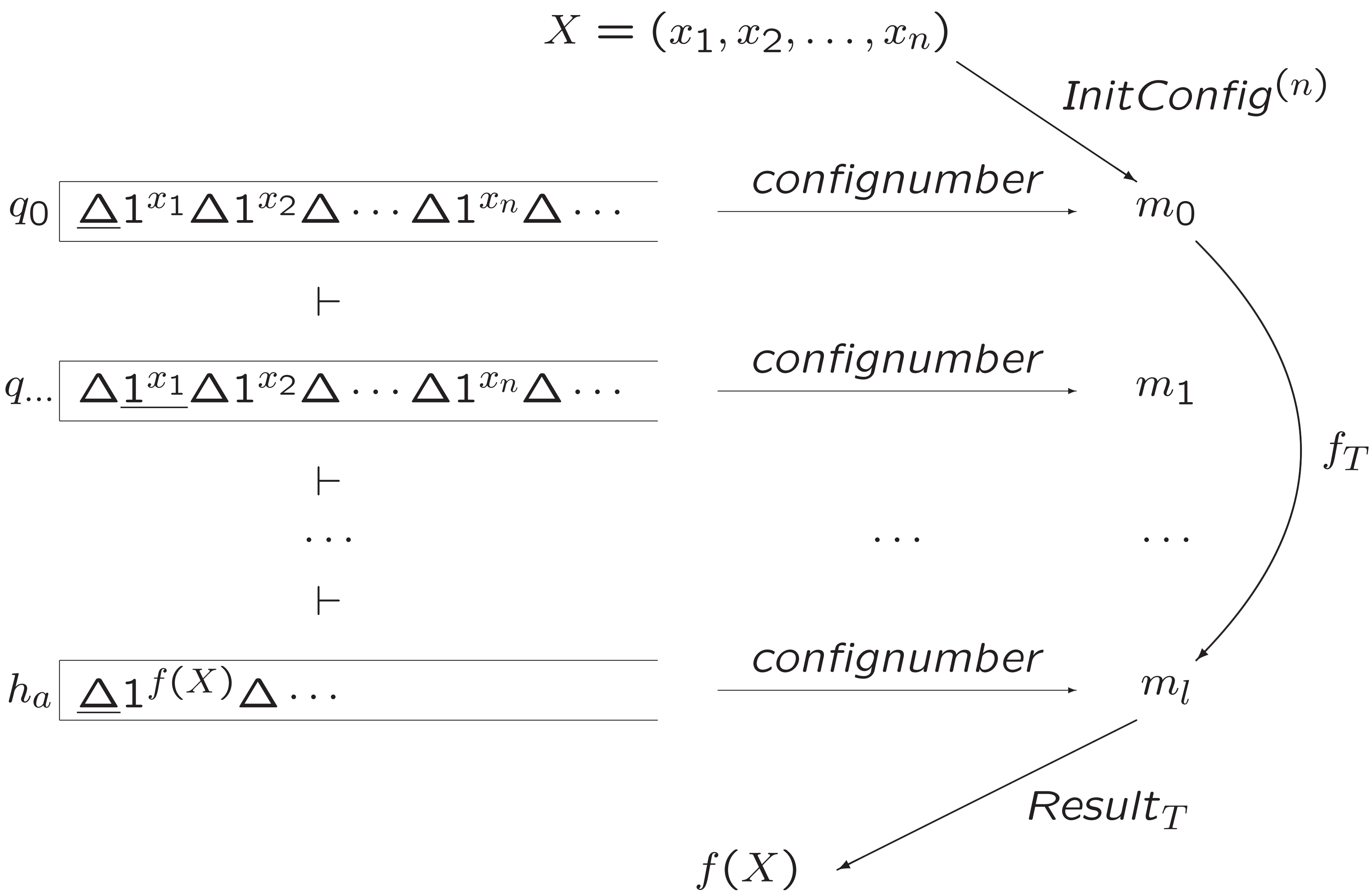
$$\vdash$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\vdash$$

$$h_a \; \boxed{\underline{\underline{\Delta}}1^{f(X)}\Delta \cdots} \xrightarrow{\textit{confignumber}} m_l$$

$$f(X)$$



$X = (x_1, x_2, \ldots, x_n)$
$InitConfig^{(n)}$
q_0 $\underline{\Delta}1^{x_1}\Delta 1^{x_2}\Delta \cdots \Delta 1^{x_n}\Delta \cdots$
confignumber
m_0
$\vdash$
$q_{\ldots}$ $\Delta\underline{1^{x_1}}\Delta 1^{x_2}\Delta \cdots \Delta 1^{x_n}\Delta \cdots$
confignumber
m_1
f_T
$\vdash$
$\cdots$
$\cdots$
$\cdots$
$\vdash$
h_a $\underline{\Delta}1^{f(X)}\Delta \cdots$
confignumber
m_l
$Result_T$
$f(X)$

We must show that $f : \mathbb{N}^n \to \mathbb{N}$ defined by

$$f(X) = \mathit{Result}_T(f_T(\mathit{InitConfig}^{(n)}(X)))$$

is μ-recursive.

Step 1

The function *InitConfig*$^{(n)} : \mathbb{N}^n \to \mathbb{N}$

Exercise 10.34.

Show using mathematical induction that if *tn*$^{(n)}(x_1, \ldots, x_n)$ is the tape number containing the string

$$\Delta 1^{x_1} \Delta 1^{x_2} \Delta \ldots \Delta 1^{x_n}$$

then *tn*$^{(n)} : \mathbb{N}^n \to \mathbb{N}$ is primitive recursive.

Use *nr*$(\Delta) = 0$ and *nr*$(1) = 1$.

A slide from lecture 12

Definition 10.2. The Operations of Composition and Primitive Recursion (continued)

2. Suppose $n \geq 0$ and g and h are functions of n and $n+2$ variables, respectively. (By "a function of 0 variables," we mean simply a constant.)
   The function obtained from g and h by the operation of *primitive recursion* is the function $f : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ defined by the formulas

$$\begin{aligned} f(X, 0) &= g(X) \\ f(X, k+1) &= h(X, k, f(X, k)) \end{aligned}$$

   for every $X \in \mathbb{N}^n$ and every $k \geq 0$.

Exercise 10.34.

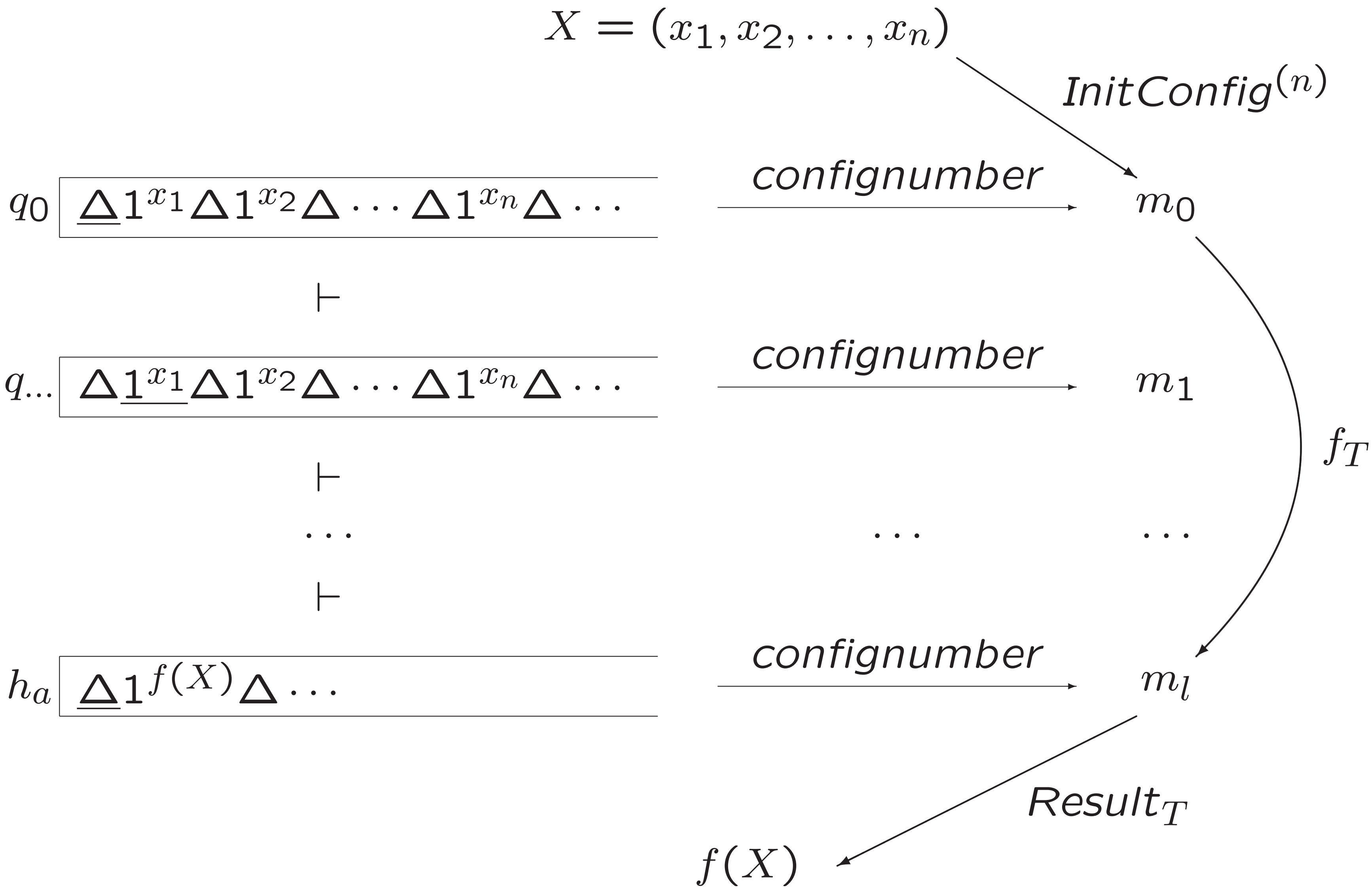
Show using mathematical induction that if *tn*$^{(n)}(x_1, \ldots, x_n)$ is the tape number containing the string

$$\Delta 1^{x_1} \Delta 1^{x_2} \Delta \ldots \Delta 1^{x_n}$$

then *tn*$^{(n)} : \mathbb{N}^n \to \mathbb{N}$ is primitive recursive.
Suggestion: In the induction step, show that

$$tn^{(m+1)}(X, x_{m+1}) = tn^{(m)}(X) * \prod_{j=1}^{x_{m+1}} PrNo(m + \textstyle\sum_{i=1}^{m} x_i + j)$$

Use $nr(\Delta) = 0$ and $nr(1) = 1$.



X = (x1, x2, ..., xn)
InitConfig(n)
confignumber
m0
q0
Δ1x1Δ1x2Δ ... Δ1xnΔ ...
⊢
q...
confignumber
m1
fT
...
ha
Δ1f(X)Δ ...
confignumber
ml
ResultT
f(X)

A slide from lecture 13

Definition 10.9. Bounded Quantifications

Let P be an $(n+1)$-place predicate. The *bounded existential quantification* of P is the $(n+1)$-place predicate E_P defined by

$E_P(X, k) =$ (there exists y with $0 \leq y \leq k$ such that $P(X, y)$ is true)

The *bounded universal quantification* of P is the $(n+1)$-place predicate A_P defined by

$$A_P(X, k) = \text{(for every } y \text{ satifying } 0 \leq y \leq k,\ P(X, y) \text{ is true)}$$

A slide from lecture 13

Theorem 10.10.

If P is a primitive recursive $(n+1)$-place predicate,
both the predicates E_P and A_P are also primitive recursive.

Proof...

A slide from lecture 14

Definition 10.11. Bounded Minimalization

For an $(n+1)$-place predicate P, the *bounded minimalization* of P is the function $m_P : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$m_P(X,k) = \begin{cases} \text{min}\{y \mid \ 0 \leq y \leq k \text{ and } P(X,y)\} & \text{if this set is not empty} \\ k+1 & \text{otherwise} \end{cases}$$

The symbol μ is often used for the minimalization operator, and we sometimes write

$$m_P(X,k) = \overset{k}{\mu}\, y[P(X,y)]$$

An important special case is that in which $P(X,y)$ is $(f(X,y)=0)$, for some $f : \mathbb{N}^{n+1} \to \mathbb{N}$. In this case m_P is written m_f and referred to as the bounded minimalization of f.

A slide from lecture 14

Theorem 10.12.

If P is a primitive recursive $(n+1)$-place predicate,
its bounded minimalization m_P is a primitive recursive function.

Proof. . .

Step 2

The predicate *IsConfig*$_T$ defined by

$$\mathit{IsConfig}_T(m) = (m \text{ is configuration number for } T)$$

Now different numbering

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be Turing machine

States:

h_a	h_r	q_0	$\ldots$	.
0	1	2	$\ldots$	s_T

with $s_T = |Q| + 1$

Tape symbols:

Δ	$\ldots$	.
0	$\ldots$	ts_T

with $ts_T = |\Gamma|$

$$\begin{aligned} \mathit{tapenumber}(\Delta 1 a \Delta b 1 \Delta) &= 2^0 3^1 5^2 7^0 11^3 13^1 17^0 \ldots \\ \mathit{confignumber} &= 2^q 3^P 5^{\mathit{tapenumber}} \end{aligned}$$

Step 2 (continued)

The function $\mathit{IsAccepting}_T$ defined by

$$\mathit{IsAccepting}_T(m) = \begin{cases} 0 & \text{if } m \text{ represents accepting config. of } T \\ 1 & \text{otherwise} \end{cases}$$

Step 2 (continued)

The function $\mathit{IsAccepting}_T$ defined by

$$\mathit{IsAccepting}_T(m) = \begin{cases} 0 & \text{if } \mathit{IsConfig}_T(m) \land \mathit{Exponent}(0, m) = 0 \\ 1 & \text{otherwise} \end{cases}$$

Step 3

The function $\mathit{Result}_T \ldots$

Step 3

The function Result_T

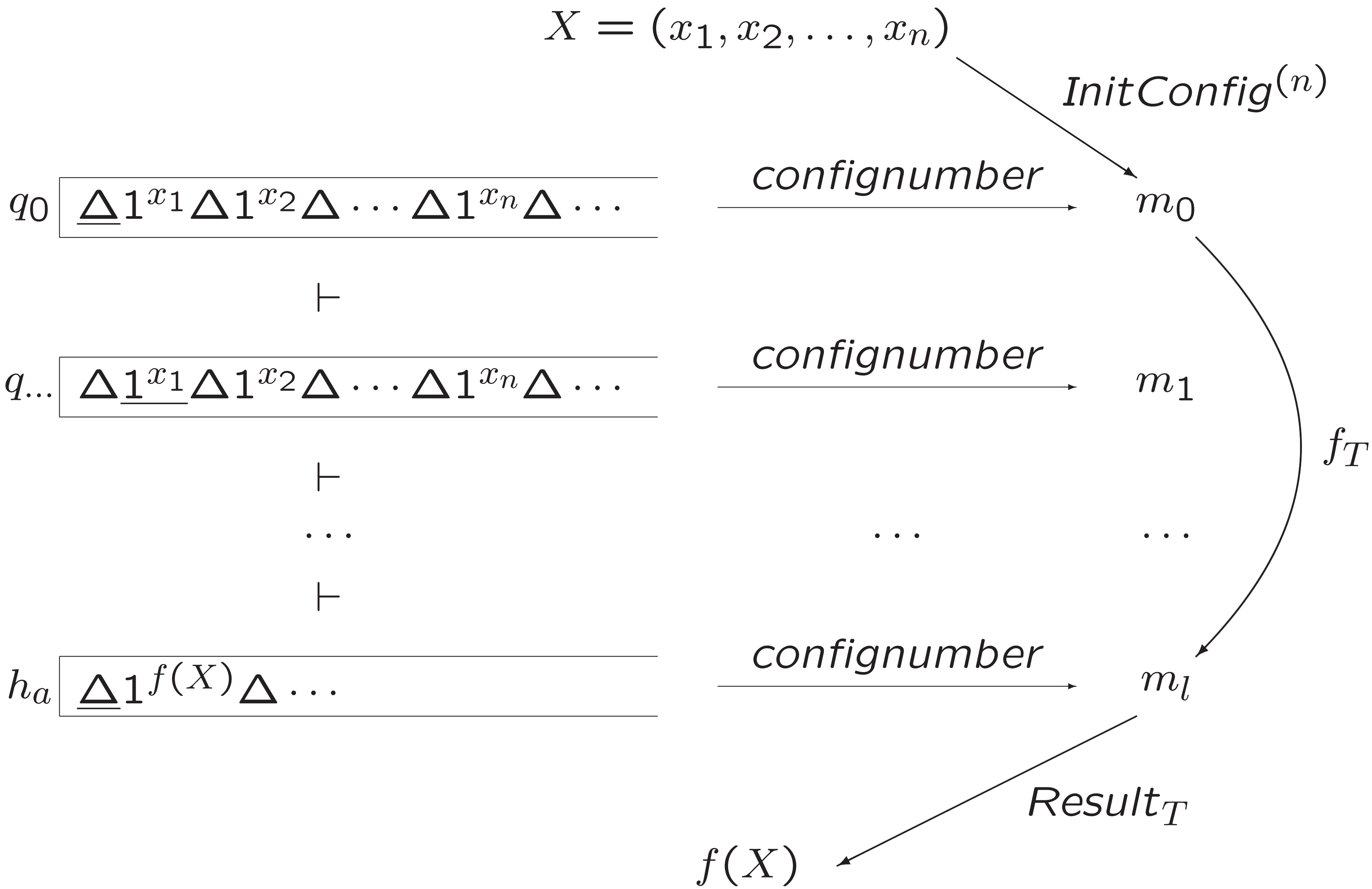
$$\mathit{Result}_T(m) = \begin{cases} \mathit{HighestPrime}(\mathit{Exponent}(2, m)) & \text{if } \mathit{IsConfig}_T(m) \\ 0 & \text{otherwise} \end{cases}$$

An exercise from exercise class 14

Exercise 10.22.

Show that the function *HighestPrime* introduced in Section 10.4 is primitive recursive.

$$\mathit{HighestPrime}(k) = \begin{cases} 0 & \text{if } k \leq 1 \\ \max\{i \mid \mathit{Exponent}(i,k) > 0\} & \text{if } k \geq 2 \end{cases}$$


$X = (x_1, x_2, \ldots, x_n)$
$InitConfig^{(n)}$
q_0
$\underline{\underline{\Delta}}1^{x_1}\Delta 1^{x_2}\Delta \cdots \Delta 1^{x_n}\Delta \cdots$
confignumber
m_0
$\vdash$
$q_{\ldots}$
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confignumber
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f_T
$\vdash$
$\cdots$
$\cdots$
$\cdots$
$\vdash$
h_a
$\underline{\underline{\Delta}}1^{f(X)}\Delta \cdots$
confignumber
m_l
$Result_T$
$f(X)$

Step 4

$$\begin{aligned}
\mathit{State}(m) &= \mathit{Exponent}(0, m) \\
\mathit{Posn}(m) &= \mathit{Exponent}(1, m) \\
\mathit{TapeNumber}(m) &= \mathit{Exponent}(2, m) \\
\mathit{Symbol}(m) &= \mathit{Exponent}(\mathit{Posn}(m), \mathit{TapeNumber}(m))
\end{aligned}$$

Step 4

$$\begin{aligned} \mathit{NewState}(m) &= \ldots \\ \mathit{NewSymbol}(m) &= \ldots \\ \mathit{NewPosn}(m) &= \ldots \\ \mathit{NewTapeNumber}(m) &= \ldots \end{aligned}$$

Exercise 10.35.

Show that the function *NewTapeNumber* discussed in Section 10.4 is primitive recursive.

Suggestion: Determine the prime factor of *TapeNumber*(m) that may change by a move of the Turing machine, when the tape head is at position *Posn*(m).

Step 5

The function $\mathit{Move}_T : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathit{Move}_T(m) = \begin{cases} \mathit{gn}(\mathit{NewState}(m), \mathit{NewPosn}(m), \mathit{NewTapeNumber}(m)) \\ \quad \text{if } \mathit{IsConfig}_T(m) \\ 0 \quad \text{otherwise} \end{cases}$$

Step 6

The function $\mathit{Moves}_T : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$\mathit{Moves}_T(m, 0) = \begin{cases} m & \text{if } \mathit{IsConfig}_T(m) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathit{Moves}_T(m, k+1) = \begin{cases} \mathit{Move}_T(\mathit{Moves}_T(m, k)) & \text{if } \mathit{IsConfig}_T(m) \\ 0 & \text{otherwise} \end{cases}$$

A slide from lecture 12

Definition 10.2. The Operations of Composition and Primitive Recursion (continued)

2. Suppose $n \geq 0$ and g and h are functions of n and $n+2$ variables, respectively. (By "a function of 0 variables," we mean simply a constant.)
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   for every $X \in \mathbb{N}^n$ and every $k \geq 0$.

Step 7

The function $\mathit{NumberOfMovesToAccept}_T : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathit{NumberOfMovesToAccept}_T(m) = \mu\, y[\mathit{IsAccepting}_T(\mathit{Moves}_T(m, y)) = 0]$$

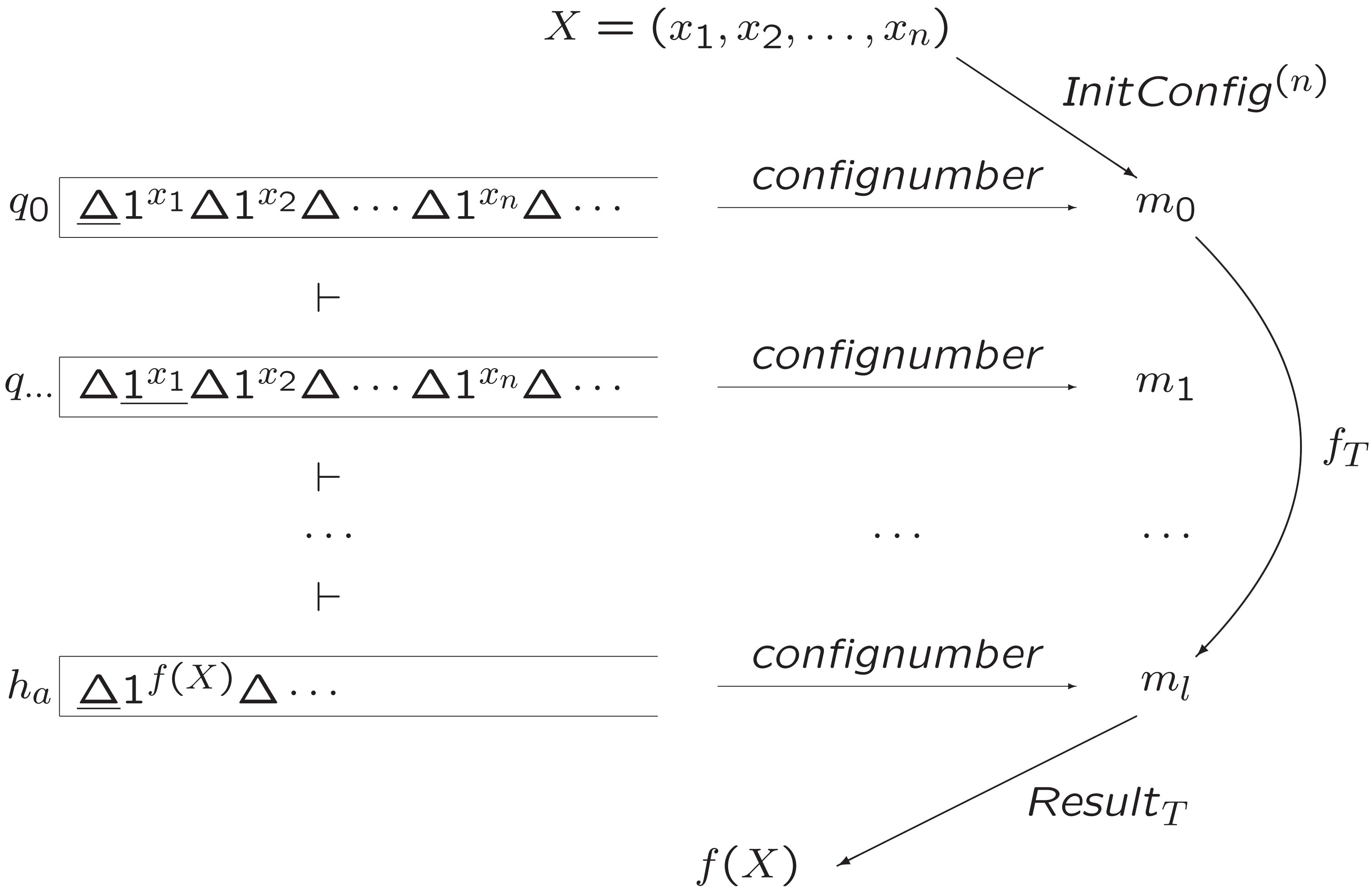
Step 7

The function $\mathit{NumberOfMovesToAccept}_T : \mathbb{N} \to \mathbb{N}$ defined by

$$\mathit{NumberOfMovesToAccept}_T(m) = \mu\, y[\mathit{IsAccepting}_T(\mathit{Moves}_T(m, y)) = 0]$$

The function $f_T : \mathbb{N} \to \mathbb{N}$ defined by

$$f_T(m) = \mathit{Moves}_T(m, \mathit{NumberOfMovesToAccept}_T(m))$$


$X = (x_1, x_2, \ldots, x_n)$
$InitConfig^{(n)}$
q_0 $\underline{\underline{\Delta}} 1^{x_1} \Delta 1^{x_2} \Delta \cdots \Delta 1^{x_n} \Delta \cdots$
confignumber
m_0
$\vdash$
$q_{\ldots}$ $\Delta \underline{1^{x_1}} \Delta 1^{x_2} \Delta \cdots \Delta 1^{x_n} \Delta \cdots$
confignumber
m_1
f_T
$\vdash$
$\cdots$
$\cdots$
$\cdots$
$\vdash$
h_a $\underline{\underline{\Delta}} 1^{f(X)} \Delta \cdots$
confignumber
m_l
$Result_T$
$f(X)$

We must show that $f : \mathbb{N}^n \to \mathbb{N}$ defined by

$$f(X) = \mathit{Result}_T(f_T(\mathit{InitConfig}^{(n)}(X)))$$

is μ-recursive.

Theorem 10.20.

Every Turing computable partial function from $\mathbb{N}^n$ to $\mathbb{N}$ is μ-recursive.

The Rest of the Proof. . .

A slide from lecture 13

Definition 10.15. μ-Recursive Functions

The set $\mathcal{M}$ of μ-recursive, or simply *recursive*, partial functions is defined as follows.

1. Every initial function is an element of $\mathcal{M}$.

2. Every function obtained from elements of $\mathcal{M}$ by composition or primitive recursion is an element of $\mathcal{M}$.

3. For every $n \geq 0$ and every total function $f : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ in $\mathcal{M}$, the function $M_f : \mathbb{N}^n \rightarrow \mathbb{N}$ defined by

$$M_f(X) = \mu\, y[f(X, y) = 0]$$

is an element of $\mathcal{M}$.

10.5. Other Approaches to Computability

Computer programs vs. Turing machines

Computer programs vs. μ-recursive functions

Let

- $G = (V, \Sigma, S, P)$ be unrestricted grammar
- f be partial function from Σ^* to Σ^*

Then G is said to compute f, if there are $A, B, C, D \in V$, such that for every x and y in Σ^*

$$f(x) = y \text{ if and only if } AxB \Rightarrow^* CyD$$

This definition (and simple examples of it) must be known for the exam

Exercise.

Describe an unrestricted grammar that computes the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2^n$.

Both the input n and the answer 2^n are unary numbers.

En verder. . .

Tentamen: vrijdag 3 juni 2016, 14:00–17:00

Vragenuur. . . ?