Fundamentele Informatica 3

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http://www.liacs.leidenuniv.nl/~vlietrvan1/fi3/

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10. Computable Functions
10.1. Primitive Recursive Functions

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Exercise 10.1.

Let F be the set of partial functions from $\mathbb{N}$ to $\mathbb{N}$. Then $F = C \cup U$, where the functions in C are computable and the ones in U are not.

Show that C is countable and U is not.

Example.

Let L be language that is not recursive, e.g. $L = \mathsf{SA}$

Then χ_L is not computable.

Exercise 7.37.

Show that if there is a TM T computing the function $f : \mathbb{N} \to \mathbb{N}$, then there is another one, T', whose tape alphabet is $\{1\}$.

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Suggestion: Suppose T has tape alphabet $\Gamma = \{a_1, a_2, \ldots, a_n\}$. Encode Δ and each of the a_i's by a string of 1's and Δ's of length $n+1$ (for example, encode Δ by $n+1$ blanks, and a_i by $1^i\Delta^{n+1-i}$). Have T' simulate T, but using blocks of $n+1$ tape squares instead of single squares.

Exercise.

How many Turing machines are there having n nonhalting states $q_0, q_1, \ldots, q_{n-1}$ and tape alphabet $\{0, 1\}$?

Exercise 10.2.

The *busy-beaver function* $b : \mathbb{N} \to \mathbb{N}$ is defined as follows.
The value $b(0)$ is 0.
For $n > 0$, there are only a finite number of Turing machines having n nonhalting states $q_0, q_1, \ldots, q_{n-1}$ and tape alphabet $\{0, 1\}$. Let $T_0, T_1, \ldots, T_m$ be the TMs of this type that eventually halt on input 1^n, and for each i, let n_{T_i} be the number of 1's that T_i leaves on its tape when it halts after processing the input string 1^n. The number $b(n)$ is defined to be the maximum of the numbers $n_{T_0}, n_{T_1}, \ldots, n_{T_m}$.

Show that the total function $b : \mathbb{N} \to \mathbb{N}$ is not computable.

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Show that the total function $b : \mathbb{N} \to \mathbb{N}$ is not computable.
Suggestion: Suppose for the sake of contradiction that T_b is a TM that computes b. Then we can assume without loss of generality that T_b has tape-alfabet $\{0, 1\}$.

Definition 10.1. Initial Functions

The initial functions are the following:

1. *Constant* functions: For each $k \geq 0$ and each $a \geq 0$, the constant function $C_a^k : \mathbb{N}^k \to \mathbb{N}$ is defined by the formula

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3. *Projection* functions: For each $k \geq 1$ and each i with $1 \leq i \leq k$, the projection function $p_i^k : \mathbb{N}^k \to \mathbb{N}$ is defined by the formula

$$p_i^k(x_1, x_2, \ldots, x_k) = x_i$$

Composition:

$$h(x) = f(g(x))$$

Definition 10.2. The Operations of Composition and Primitive Recursion

1. Suppose f is a partial function from $\mathbb{N}^k$ to $\mathbb{N}$, and for each i with $1 \leq i \leq k$, g_i is a partial function from $\mathbb{N}^m$ to $\mathbb{N}$.
   The partial function obtained from f and $g_1, g_2, \ldots, g_k$ by composition is the partial function h from $\mathbb{N}^m$ to $\mathbb{N}$ defined by the formula

$$h(X) = f(g_1(X), g_2(X), \ldots, g_k(X)) \text{ for every } X \in \mathbb{N}^m$$

Recursion: if $f(k) = k!$, then

$f(0) = 0! = 1 \qquad f(k+1) = (k+1)! = (k+1)k! = (k+1)f(k)$

Definition 10.2. The Operations of Composition and Primitive Recursion (continued)

2. Suppose $n \geq 0$ and g and h are functions of n and $n + 2$ variables, respectively. (By "a function of 0 variables," we mean simply a constant.)
   The function obtained from g and h by the operation of *primitive recursion* is the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by the formulas

$$\begin{aligned} f(X, 0) &= g(X) \\ f(X, k+1) &= h(X, k, f(X, k)) \end{aligned}$$

   for every $X \in \mathbb{N}^n$ and every $k \geq 0$.

Example 10.5. Addition, Multiplication and Subtraction

$$\mathit{Add}(x, y) = x + y$$

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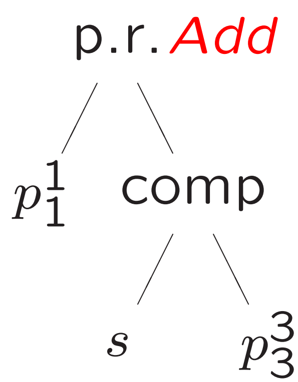
Structure tree:



Definition 10.3. Primitive Recursive Functions

The set *PR* of *primitive recursive* functions is defined as follows.

1. All initial functions are elements of *PR*.

2. For every $k \geq 0$ and $m \geq 0$, if $f : \mathbb{N}^k \to \mathbb{N}$ and $g_1, g_2, \ldots, g_k : \mathbb{N}^m \to \mathbb{N}$ are elements of *PR*, then the function $f(g_1, g_2, \ldots, g_k)$ obtained from f and $g_1, g_2, \ldots, g_k$ by composition is an element of *PR*.

3. For every $n \geq 0$, every function $g : \mathbb{N}^n \to \mathbb{N}$ in *PR*, and every function $h : \mathbb{N}^{n+2} \to \mathbb{N}$ in *PR*, the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ obtained from g and h by primitive recursion is in *PR*.

In other words, the set *PR* is the smallest set of functions that contains all the initial functions and is closed under the operations of composition and primitive recursion.

Example 10.5. Addition, Multiplication and Subtraction

$$\mathsf{Mult}(x, y) = x * y$$

Example 10.5. Addition, Multiplication and Subtraction

$$Sub(x,y) = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

$x \dot{-} y$

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$$\begin{aligned} Sub(x,0) &= x && (\text{so } g = p_1^1) \\ Sub(x,k+1) &= Pred(Sub(x,k)) \\ & && (= h(x,k,Sub(x,k)),\ \text{so } h = Pred(p_3^3)) \end{aligned}$$

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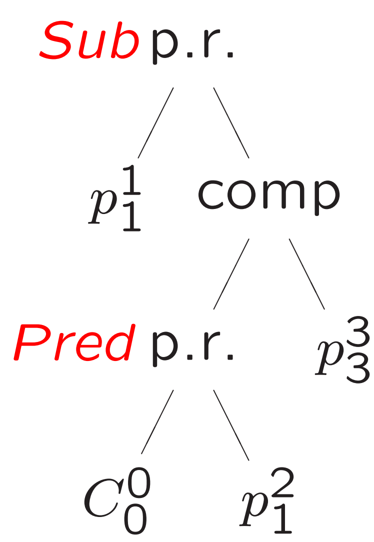

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$$\begin{aligned} f(X, 0) &= g(X) \\ f(X, k+1) &= h(X, k, f(X, k)) \end{aligned}$$

   for every $X \in \mathbb{N}^n$ and every $k \geq 0$.

Theorem 10.4.

Every primitive recursive function is total and computable.

Proof, part 1

```
for (i=1;i<=k;i++)
{ yi = gi(x1,x2,...,xm)
}
return f(y1,y2,...,yk);
```

Theorem 10.4.

Every primitive recursive function is total and computable.

Proof, part 2

```
i = 0;
v = g(x);
while (i<k)
{ v = h(x,i,v);
  i ++;
}
return v;
```

Theorem 10.4.

Every primitive recursive function is total and computable.

PR:
total and computable

Turing-computable functions:
not necessarily total

Example 10.5. Addition, Multiplication and Subtraction

$$Sub(x, y) = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

$x \dot{-} y$

n-*place predicate* P is function from $\mathbb{N}^n$ to $\{\text{true}, \text{false}\}$

characteristic function χ_P defined by

$$\chi_P(X) = \begin{cases} 1 & \text{if } P(X) \text{ is true} \\ 0 & \text{if } P(X) \text{ is false} \end{cases}$$

We say P is primitive recursive. . .

Theorem 10.6.

The two-place predicates *LT*, *EQ*, *GT*, *LE*, *GE*, and *NE* are primitive recursive.
(*LT* stands for “less than,” and the other five have similarly intuitive abbreviations.)
If P and Q are any primitive recursive n-place predicates, then $P \wedge Q$, $P \vee Q$ and $\neg P$ are primitive recursive.

Proof. . .