Let G be a context-free grammar with start variable S and the following productions:

$$S \to aSbS \mid bSaS \mid \Lambda$$

a. Show that $L(G) = AEqB = \{x \in \{a,b\}^* \mid n_a(x) = n_b(x)\}$
b. Is G ambiguous? Motivate your answer.

Solutions

- exercises
- homework
- exams

unwanted in CFG:
– variables not used in successful derivations $S \Rightarrow^* x \in \Sigma^*$

CFG $G = (V, \Sigma, S, P)$

Definition

variable A is *live* if $A \Rightarrow^* x$ for some $x \in \Sigma^*$.

variable A is *reachable* if $S \Rightarrow^* \alpha A \beta$ for some $\alpha, \beta \in (\Sigma \cup V)^*$.

variable A is *useful* if there is a derivation of the form $S \Rightarrow^* \alpha A \beta \Rightarrow^* x$ for some string $x \in \Sigma^*$.

useful implies live and reachable.

conversely, ...

[M] Exercise 4.51, 4.52, 4.53

CFG $G = (V, \Sigma, S, P)$

Definition

variable A is *live* if $A \Rightarrow^* x$ for some $x \in \Sigma^*$.

variable A is *reachable* if $S \Rightarrow^* \alpha A \beta$ for some $\alpha, \beta \in (\Sigma \cup V)^*$.

variable A is *useful* if there is a derivation of the form $S \Rightarrow^* \alpha A \beta \Rightarrow^* x$ for some string $x \in \Sigma^*$.

useful implies live and reachable.
For $S \to AB \mid b$ and $A \to a$, variable A is live and reachable, not useful.
[M] Exercise 4.51, 4.52, 4.53

Live variables

Construction

- $N_0 = \varnothing$
- $N_{i+1} = N_i \cup \{ A \in V \mid A \to \alpha \text{ in } P, \text{ with } \alpha \in (N_i \cup \Sigma)^* \}$

$N_1 = \{ A \in V \mid A \to x \text{ in } P, \text{ with } x \in \Sigma^* \}$
$N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq V$
there exists a k such that $N_k = N_{k+1}$
A is live iff $A \in \bigcup_{i \geq 0} N_i = N_k$
(minimal) depth of derivation tree $A \Rightarrow^* x$

Live variables

Construction

– $N_0 = \varnothing$
– $N_{i+1} = N_i \cup \{ A \in V \mid A \to \alpha \text{ in } P, \text{ with } \alpha \in (N_i \cup \Sigma)^* \}$

Exercise 4.53(c_i).

$S \to ABC \mid BaB$ $\qquad A \to aA \mid BaC \mid aaa$
$B \to bBb \mid a$ $\qquad C \to CA \mid AC$

Reachable variables

Construction

- $N_0 = \{S\}$
- $N_{i+1} = N_i \cup \{\ A \in V \mid B \to \alpha_1 A \alpha_2 \text{ in } P,\ \text{with } B \in N_i\ \}$

$N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq V$
there exists a k such that $N_k = N_{k+1}$
A is reachable iff $A \in \bigcup_{i \geq 0} N_i = N_k$
(minimal) length of derivation $S \Rightarrow^* \alpha A \beta$

Reachable variables

Construction

- $N_0 = \{S\}$
- $N_{i+1} = N_i \cup \{\ A \in V \mid B \to \alpha_1 A \alpha_2 \text{ in } P, \text{ with } B \in N_i\ \}$

$N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq V$
there exists a k such that $N_k = N_{k+1}$
A is reachable iff $A \in \bigcup_{i \geq 0} N_i = N_k$
(minimal) length of derivation $S \Rightarrow^* \alpha A \beta$

- remove all non-live variables (and productions that contain them)
- remove all unreachable variables (and their productions)

then all variables are useful

Reachable variables

Construction

- $N_0 = \{S\}$
- $N_{i+1} = N_i \cup \{\ A \in V \mid B \to \alpha_1 A \alpha_2 \text{ in } P,\ \text{with } B \in N_i\ \}$

Exercise 4.53(c_i)., ctd

$S \to BaB$ $\qquad$ $A \to aA \mid aaa$ $\qquad$ $B \to bBb \mid a$

– remove all non-live variables (and productions that contain them)
– remove all unreachable variables (and productions)

then all variables are useful

does not work the other way around . . .

Exercise 4.53(c_i)., revisited

$S \to ABC \mid BaB \qquad A \to aA \mid BaC \mid aaa$

$B \to bBb \mid a \qquad C \to CA \mid AC$

unwanted in CFG:
– variables not used in successful derivations $S \Rightarrow^* x \in \Sigma^*$

And also:
– $A \to \Lambda$ A variable Λ-productions

$$S \to AB \mid aB \qquad A \to BS \mid bS \qquad B \to bb \mid \Lambda$$

$$S \Rightarrow AB \Rightarrow BSB \Rightarrow SB \Rightarrow S$$

unwanted in CFG:
– variables not used in successful derivations $S \Rightarrow^* x \in \Sigma^*$
– $A \to \Lambda$ A variable Λ-productions

And also:
– $A \to B$ A, B variables unit productions [chain rules]

$$S \to A \mid aB \qquad A \to B \mid bS \qquad B \to S \mid \Lambda$$

$$S \Rightarrow A \Rightarrow B \Rightarrow S$$

Let l be length of a string in a derivation
Let t be number of terminals in a string in a derivation

If G has no Λ-productions, and no unit productions,
then $l + t$ strictly increases in every step of a derivation
Proof ...

Hence, a string $x \in \Sigma^*$ can only be generated in derivations of at most $2|x| - 1$ steps

unwanted in CFG:
– variables not used in successful derivations $S \Rightarrow^* x \in \Sigma^*$
– $A \to \Lambda$ A variable Λ-productions
– $A \to B$ A, B variables unit productions [chain rules]

restricted CFG, with 'nice' form
Chomsky normal form $A \to BC$, $A \to \sigma$
Greibach normal form (⊠) $A \to \sigma B_1 \ldots B_k$

Removing Λ-productions

Idea:

Example

$A \to BCDCB$
$B \to b \mid \Lambda$
$C \to c \mid \Lambda$
$D \to d$

Definition

variable A is *nullable* iff $A \Rightarrow^* \Lambda$

Theorem

– if $A \to \Lambda$ then A is nullable
– if $A \to B_1 B_2 \dots B_k$ and all B_i are nullable, then A is nullable

[M] Def 4.26 / Exercise 4.48

Construction

– $N_0 = \varnothing$
– $N_{i+1} = N_i \cup \{ A \in V \mid A \to \alpha \text{ in } P, \text{ with } \alpha \in N_i^* \}$

$N_1 = \{ A \in V \mid A \to \Lambda \text{ in } P \}$
$N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq V$
there exists a k such that $N_k = N_{k+1}$
A is nullable iff $A \in \bigcup_{i \geq 0} N_i = N_k$

Construction

– identify nullable variables
– for every production $A \to \alpha$ add $A \to \beta$,
  where β is obtained from α by removing one or more nullable variables
– remove all Λ-productions

Grammar for $\{ a^i b^j c^k \mid i = j \text{ or } i = k \}$

$S \to TU \mid V$
$T \to aTb \mid \Lambda$
$U \to cU \mid \Lambda$
$V \to aVc \mid W$
$W \to bW \mid \Lambda$

Grammar for $\{ a^i b^j c^k \mid i = j \text{ or } i = k \}$

$S \to TU \mid V$
$T \to aTb \mid \Lambda$
$U \to cU \mid \Lambda$
$V \to aVc \mid W$
$W \to bW \mid \Lambda$

$N_1 = \{T, U, W\}$, variables with Λ at right-hand side productions

$N_2 = \{T, U, W\} \cup \{S, V\}$, variables with $\{T, U, W\}^*$ at rhs productions

$N_3 = N_2 = \{T, U, W, S, V\}$, all variables found, no new add all productions, where (any number of) nullable variables are removed...

$S \to TU \mid V$
$T \to aTb \mid \Lambda$
$U \to cU \mid \Lambda$
$V \to aVc \mid W$
$W \to bW \mid \Lambda$

[M] Ex. 4.31

add all productions, where (any number of) nullable variables are removed

$S \to TU \mid V$	$S \to T \mid U \mid \Lambda$
$T \to aTb \mid \Lambda$	$T \to ab$
$U \to cU \mid \Lambda$	$U \to c$
$V \to aVc \mid W$	$V \to ac \mid \Lambda$
$W \to bW \mid \Lambda$	$W \to b$

remove all Λ-productions. . .

[M] Ex. 4.31

add all productions, where (any number of) nullable variables are removed

$S \to TU \mid V$	$S \to T \mid U \mid \Lambda$
$T \to aTb \mid \Lambda$	$T \to ab$
$U \to cU \mid \Lambda$	$U \to c$
$V \to aVc \mid W$	$V \to ac \mid \Lambda$
$W \to bW \mid \Lambda$	$W \to b$

remove all Λ-productions

$$
\begin{aligned}
S &\to TU \mid V \mid T \mid U \\
T &\to aTb \mid ab \\
U &\to cU \mid c \\
V &\to aVc \mid W \mid ac \\
W &\to bW \mid b
\end{aligned}
$$

[M] Ex. 4.31

Theorem

For every CFG G *there is* CFG G_1 *without* Λ*-productions such that* $L(G_1) = L(G) - \{\Lambda\}$.

Proof $L(G) - \{\Lambda\} \subseteq L(G_1) \ldots$

[M] Thm 4.27

Theorem

For every CFG G there is CFG G_1 without Λ-productions such that $L(G_1) = L(G) - \{\Lambda\}$.

Proof $L(G) - \{\Lambda\} \subseteq L(G_1)$

$G = (V, \Sigma, S, P)$
Consider arbitrary $x \in L(G) - \{\Lambda\}$
$S \Rightarrow_G^* x$, i.e., $S \Rightarrow_G^n x$ for some $n \geq 1$
Needed: $S \Rightarrow_{G_1}^* x$
We prove more general statement:
For all $A \in V$, $n \geq 1$ and $x \in \Sigma^* - \{\Lambda\}$, if $A \Rightarrow_G^n x$, then $A \Rightarrow_{G_1}^* x$,
using induction on n
Basis, $n = 1$: If $A \Rightarrow_G x$, then also $A \Rightarrow_{G_1} x$

Theorem

For every CFG *G there is* CFG *G_1 without Λ-productions such that $L(G_1) = L(G) - \{\Lambda\}$.*

Proof $L(G) - \{\Lambda\} \subseteq L(G_1)$ (continued)

Induction hypothesis: Let $k \geq 1$, and suppose that for all $A \in V$, $n \leq k$ and $x \in \Sigma^* - \{\Lambda\}$, if $A \Rightarrow_G^n x$, then $A \Rightarrow_{G_1}^* x$

Induction step: Consider $A \Rightarrow_G^{k+1} x$

then $A \Rightarrow_G X_1 X_2 \ldots X_m \Rightarrow_G^k x = x_1 x_2 \ldots x_m$, for some $m \geq 1$ and $X_1, X_2, \ldots, X_m \in V \cup \Sigma$

Three cases:

1. X_i is terminal
2. X_i is variable and $x_i \neq \Lambda$
3. X_i is variable and $x_i = \Lambda$

[M] Thm 4.27

Removing unit productions

Idea:

Example

$A \to B \mid aCb$
$B \to C \mid Bb \mid Bc$
$C \to c \mid ABC$

Assume Λ-productions have been removed

Variable B is *A-derivable*, if
– $B \neq A$, and
– $A \Rightarrow^* B$ (using only unit productions)

Construction

– $N_1 = \{\ B \in V \mid B \neq A \text{ and } A \rightarrow B \text{ in } P\ \}$
– $N_{i+1} = N_i \cup \{\ C \in V \mid C \neq A \text{ and } B \rightarrow C \text{ in } P, \text{ with } B \in N_i\ \}$

$N_1 \subseteq N_2 \subseteq \cdots \subseteq V$
there exists a k such that $N_k = N_{k+1}$
B is A-derivable iff $B \in \bigcup_{i \geq 1} N_i = N_k$

Construction

- for each $A \in V$, identify A-derivable variables
- for every pair (A, B) where B is A-derivable, and every production $B \to \alpha$ add $A \to \alpha$
- remove all unit productions

Grammar for $\{ a^i b^j c^k \mid i = j \text{ or } i = k \}$

$S \to TU \mid V \mid T \mid U$
$T \to aTb \mid ab$
$U \to cU \mid c$
$V \to aVc \mid W \mid ac$
$W \to bW \mid b$

$$S \to TU \mid V \mid T \mid U$$
$$T \to aTb \mid ab$$
$$U \to cU \mid c$$
$$V \to aVc \mid W \mid ac$$
$$W \to bW \mid b$$

S-derivable: $\{V, T, U\}, \{V, T, U, W\}$ V-derivable: $\{W\}$

New productions:

$$S \to aTb \mid ab \quad S \to cU \mid c \quad S \to aVc \mid W \mid ac \quad S \to bW \mid b$$
$$V \to bW \mid b$$

Remove unit productions:

$$S \to TU \mid aTb \mid ab \mid cU \mid c \mid aVc \mid ac \mid bW \mid b$$
$$T \to aTb \mid ab$$
$$U \to cU \mid c$$
$$V \to aVc \mid ac \mid bW \mid b$$
$$W \to bW \mid b$$

Definition

CFG in *Chomsky normal form*
productions are of the form
– $A \to BC$ variables A, B, C
– $A \to \sigma$ variable A, terminal σ

Theorem

For every CFG G there is CFG G_1 in CNF such that $L(G_1) = L(G) - \{\Lambda\}$.

[M] Def 4.29, Thm 4.30

Construction

1. remove Λ-productions
2. remove unit productions
3. introduce variables for terminals $X_\sigma \to \sigma$
4. split long productions

$$A \to aBabA$$

is replaced by

$$X_a \to a \qquad X_b \to b \qquad A \to X_a B X_a X_b A$$

$$A \to ACBA$$

is replaced by

$$A \to AY_1 \qquad Y_1 \to CY_2 \qquad Y_2 \to BA$$

Mind the order

Grammar for $\{ a^i b^j c^k \mid i = j \text{ or } i = k \}$

$S \to TU \mid V$

$T \to aTb \mid \Lambda \qquad U \to cU \mid \Lambda$

$V \to aVc \mid W \qquad W \to bW \mid \Lambda$

After removing Λ-productions and unit productions, we obtain (see before)

$S \to TU \mid aTb \mid ab \mid cU \mid c \mid aVc \mid ac \mid bW \mid b$

$T \to aTb \mid ab \qquad U \to cU \mid c$

$V \to aVc \mid ac \mid bW \mid b \qquad W \to bW \mid b$

Now introduce productions for the terminals. . .

Grammar for $\{ a^i b^j c^k \mid i = j \text{ or } i = k \}$

$S \to TU \mid V$

$T \to aTb \mid \Lambda \qquad U \to cU \mid \Lambda$

$V \to aVc \mid W \qquad W \to bW \mid \Lambda$

After removing Λ-productions and unit productions, we obtain (see before)

$S \to TU \mid aTb \mid ab \mid cU \mid c \mid aVc \mid ac \mid bW \mid b$

$T \to aTb \mid ab \qquad U \to cU \mid c$

$V \to aVc \mid ac \mid bW \mid b \qquad W \to bW \mid b$

Now introduce productions for the terminals:

$X_a \to a \qquad X_b \to b \qquad X_c \to c$

$S \to TU \mid X_a T X_b \mid X_a X_b \mid X_c U \mid c \mid X_a V X_c \mid X_a X_c \mid X_b W \mid b$

$T \to X_a T X_b \mid X_a X_b$

$U \to X_c U \mid c$

$V \to X_a V X_c \mid X_a X_c \mid X_b W \mid b$

$W \to X_b W \mid b$

Only a few productions that are too long:

$S \to X_a T X_b \mid X_a V X_c$

$T \to X_a T X_b$

$V \to X_a V X_c$

Split these long productions...

Only a few productions that are too long:

$S \rightarrow X_a T X_b \mid X_a V X_c$
$T \rightarrow X_a T X_b$
$V \rightarrow X_a V X_c$

Split these long productions:

$S \rightarrow X_a Y_1 \mid X_a Y_2$
$Y_1 \rightarrow T X_b$ $\qquad$ $Y_2 \rightarrow V X_c$
$T \rightarrow X_a Y_3$
$V \rightarrow X_a Y_4$
$Y_3 \rightarrow T X_b$ $\qquad$ $Y_4 \rightarrow V X_c$

Note that we can reuse Y_1 ($\approx Y_3$) and Y_2 ($\approx Y_4$) for two productions

From lecture 8:

Definition

regular grammar (or *right-linear grammar*)
productions are of the form
– $A \to \sigma B$ variables A, B, terminal σ
– $A \to \Lambda$ variable A

Theorem

A language L is regular,
if and only if there is a regular grammar generating L.

Proof. . .

[M] Def 4.13, Thm 4.14

Definition

CFG in *Chomsky normal form*

productions are of the form

– $A \to BC$ variables A, B, C

– $A \to \sigma$ variable A, terminal σ

[M] Def 4.29

Outlook

Chomsky NF for pumping lemma (later)

$\text{even}(L) = \{ \ w \in L \mid |w| \text{ even } \}$

idea: new variables for even/odd length strings
Chomsky normal form to reduce number of possibilities.

grammar $G = (V, \Sigma, P, S)$ for L, in ChNF
new grammar $G = (V', \Sigma, P', S')$ for $\text{even}(L)$
variables: $V' = \{X_e, X_o \mid X \in V\}$
axiom: $S' = S_e$
productions: – for every $A \to BC$ in P we have in P':

$$A_e \to B_e C_e \mid B_o C_o \qquad A_o \to B_e C_o \mid B_o C_e$$

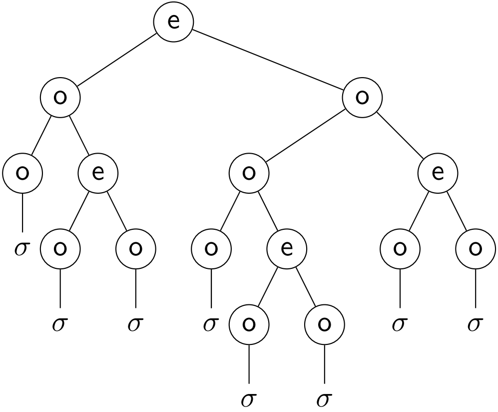
– for every $A \to \sigma$ in P we have in P': $A_o \to \sigma$

ABOVE

We consider closure properties: given an operation X show that whenever L is regular/context-free, then also $X(L)$ is regular/context-free.

This is done as follows: if L is regular/context-free, then we know there is a regular/context-free grammar G for L, and we show how to construct a new grammar G' (of the same type) for $X(L)$, in terms of the original grammar G.

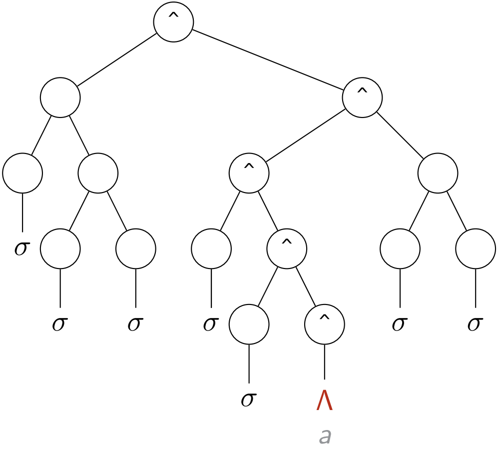
Even/odd markings



$L \subseteq \{a,b\}^*$, $\quad$ $\text{chop}(L) = \{\, xy \mid xay \in L\}$ $\quad$ remove some a in each string

idea: new variables for the task of removing letter a

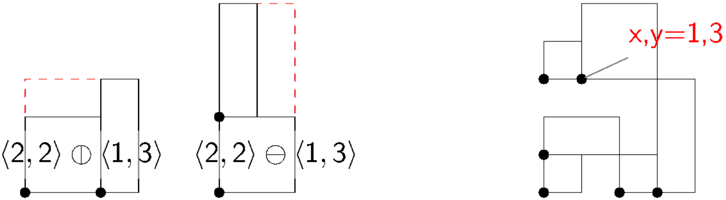
grammar $G = (V, \{a,b\}, P, S)$ for L, in ChNF
new grammar $G = (V', \{a,b\}, P', S')$ for $\text{chop}(L)$
variables: $V' = V \cup \{\hat{X} \mid X \in V\}$
axiom: $S' = \hat{S}$
productions: keep all productions from P, and
– for every $A \to BC$ add $\hat{A} \to \hat{B}C \mid B\hat{C}$
– for every $A \to a$ add $\hat{A} \to \Lambda$


^
^
^
^
^
σ
σ
σ
σ
σ
σ
σ
Λ
a

$E \to E + T \mid T$
$T \to T * F \mid F$
$F \to (\, E \,) \mid int$

$E \to E_1 + T_1$	$E.\text{val} = \text{E}_1.\text{val} + \text{T}_1.\text{val}$
$E \to T_1$	$E.\text{val} = \text{T}_1.\text{val}$
$T \to T_1 * F_1$	$T.\text{val} = \text{T}_1.\text{val} \cdot \text{F}_1.\text{val}$
$T \to F_1$	$T.\text{val} = \text{F}_1.\text{val}$
$F \to (\, E_1 \,)$	$F.\text{val} = \text{E}_1.\text{val}$
$F \to int$	$F.\text{val} = \text{IntVal}(\, \text{int} \,)$

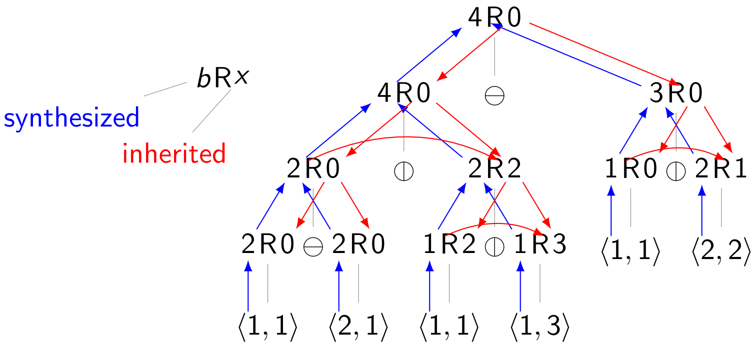
D.E. Knuth. Semantics of Context-Free Languages.
Math. Systems Theory (1968) 127–145 doi:10.1007/BF01692511



$$((\langle 1,1\rangle \ominus \langle 2,1\rangle) \obar (\langle 1,1\rangle \obar \langle 1,3\rangle)) \ominus (\langle 1,1\rangle \obar \langle 2,2\rangle)$$

production	semantic rule
$R \to \langle E_1, E_2\rangle$	$R.b = E_1.\mathrm{val} \quad \mathrm{R.h} = \mathrm{E_2.val}$
$R \to (R_1 \obar R_2)$	$R.b = R_1.b + R_2.b$
	$R.h = \max\{R_1.h, R_2.h\}$
	$R_1.x = R.x \quad R_2.x = R.x + R_1.b$
	$R_1.y = R.y \quad R_2.y = R.y$
$R \to (R_1 \ominus R_2)$	$R.b = \max\{R_1.b, R_2.b\}$
	$R.h = R_1.h + R_2.h$
	$R_1.x = R.x \quad R_2.x = R.x$
	$R_1.y = R.y \quad R_2.y = R.y + R_1.h$

$$R \to (R_1 \oplus R_2) \quad R.b = R_1.b + R_2.b$$
$$R_1.x = R.x \quad R_2.x = R.x + R_1.b$$
$$R \to (R_1 \ominus R_2) \quad R.b = \max\{R_1.b, R_2.b\}$$
$$R_1.x = R.x \quad R_2.x = R.x$$



Quiz

Homework 3! (probably Wednesday)