Feedback homework 3: end of this week?

Deadline homework 4: Tuesday, 19 December, 23:59

Exam: Thursday, 21 December, 09:00-12:00. Registration required

Q&A session for exam: Monday, 18 December, 13:00 - 15:00

From lecture 12:

Theorem (Pumping Lemma for context-free languages)

$\forall$ *for every context-free language* L

$\exists$ *there exists a constant* $n \geq 2$

such that

$\forall$ *for every* $u \in L$

with $|u| \geq n$

$\exists$ *there exists a decomposition* $u = vwxyz$

such that

(1) $|wy| \geq 1$

(2) $|wxy| \leq n$,

$\forall$ *(3) for all* $m \geq 0$, $vw^m xy^m z \in L$

[M] Thm. 6.1

From lecture 12:

Example

AnBnCn is not context-free.

[M] E 6.3

$u = a^n b^n c^n$

$\{\ x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x)\ \}$

Example

XX is not context-free.

[M] E 6.4

$u = a^n b^n a^n b^n$

$\{\ a^i b^j a^i b^j \mid i, j \geq 0\ \}$

Example

$\{\ x \in \{a, b, c\}^* \mid n_a(x) < n_b(x) \text{ and } n_a(x) < n_c(x)\ \}$ is not context-free.

[M] E 6.5

From lecture 3:

Prove the following generalization of the pumping lemma, which can sometimes make it unnecessary to break the proof into cases.

If L can be accepted by an FA,
then there is an integer n
such that for any $x \in L$ with $|x| \geq n$
and for any way of writing x as $x_1 x_2 x_3$ with $|x_2| = n$,
there are strings u, v and w such that

a. $x_2 = uvw$
b. $|v| \geq 1$
c. For every $m \geq 0$, $x_1 u v^m w x_3 \in L$

Ogden's Lemma

Generalization of pumping lemma for CFL:
pump at distinguished positions in u

Ogden's lemma does not have to be known for the exam.

From lecture 2:

FA $M_i = (Q_i, \Sigma, q_i, A_i, \delta_i) \quad i = 1, 2$

Product construction

construct FA $M = (Q, \Sigma, q_0, A, \delta)$ such that

- $Q = Q_1 \times Q_2$
- $q_0 = (q_1, q_2)$
- $\delta(\ (p,q), \sigma) = (\ \delta_1(p,\sigma), \delta_2(q,\sigma)\ )$
- A as needed

Theorem (2.15 Parallel simulation)

- $A = \{(p,q) \mid p \in A_1 \text{ or } q \in A_2\}$, *then* $L(M) = L(M_1) \cup L(M_2)$
- $A = \{(p,q) \mid p \in A_1 \text{ and } q \in A_2\}$, *then* $L(M) = L(M_1) \cap L(M_2)$
- $A = \{(p,q) \mid p \in A_1 \text{ and } q \notin A_2\}$, *then* $L(M) = L(M_1) - L(M_2)$

Proof. . .

[M] Sect 2.2

From lecture 6:

Regular languages are closed under

– Boolean operations (complement, union, intersection)
– Regular operations (union, concatenation, star)
– Reverse (mirror)
– [inverse] Homomorphism

From lecture 7:
Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

$G_i = (V_i, \Sigma, S_i, P_i)$, having no variables in common.

Construction

$G = (V_1 \cup V_2 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup P_2 \cup \{S \to S_1, S \to S_2\}$ $L(G) = L(G_1) \cup L(G_2)$
– $P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$ $L(G) = L(G_1) L(G_2)$

$G = (V_1 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup \{S \to S S_1, S \to \Lambda\}$ $L(G) = L(G_1)^*$

[M] Thm 4.9

How about

- $L_1 \cap L_2$
- $L_1 - L_2$
- L_1'

for CFLs L_1 and L_2 ?

From lecture 8:

Example

AnBnCn is intersection of two context-free languages.

$L_1 = \{a^i b^i c^k \mid i, k \geq 0\}$
$L_2 = \{a^i b^k c^k \mid i, k \geq 0\}$
[M] E 6.10
Hence, CFL is not closed under intersection

Example

AnBnCn is intersection of two context-free languages.

[M] E 6.10

Hence, CFL is not closed under intersection

$L_1 \cap L_2 = (L_1' \cup L_2')'$

Hence, CFL is not closed under complement

$L_1' = \Sigma^* - L_1$

Hence, CFL is not closed under setminus

Example

Complement of XX
$= \{ x \in \{a,b\}^* \mid |x| \text{ is odd } \} \cup \{ x\, y \mid x, y \in \{a,b\}^*, |x| = |y|, x \neq y \}$
is context-free

[M] E 6.11

Indeed, CFL is not closed under complement

Example

Complement of *AnBnCn* is context-free.

[M] E 6.12

Example

Complement of *AnBnCn* is context-free.

$AnBnCn = L_1 \cap L_2 \cap L_3$, with
$L_1 = \{a^i b^j c^k \mid i \leq j\}$
$L_2 = \{a^i b^j c^k \mid j \leq k\}$
$L_3 = \{a^i b^j c^k \mid k \leq i\}$
[M] E 6.12

Example

Complement of $\{x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x)\}$ is context-free.

$\{x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x)\} = A_1 \cap A_2 \cap A_3$, with
$A_1 = \{x \in \{a, b, c\}^* \mid n_a(x) \leq n_b(x)\}$
$A_2 = \{x \in \{a, b, c\}^* \mid n_b(x) \leq n_c(x)\}$
$A_3 = \{x \in \{a, b, c\}^* \mid n_c(x) \leq n_a(x)\}$
[M] E 6.12

Example

$L_1 = \{\ a^{2n} b^n \mid n \geq 1\ \}^*$

$a^{16} b^8 a^8 b^4 a^4 b^2 a^2 b^1$

$L_2 = a^* \{\ b^n a^n \mid n \geq 1\ \}^* \{\ b\ \}$

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Product construction

construct FA $M = (Q, \Sigma, q_0, A, \delta)$ such that
- $Q = Q_1 \times Q_2$
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- $A = \{(p, q) \mid p \in A_1$ *and* $q \notin A_2\}$, *then* $L(M) = L(M_1) - L(M_2)$

Proof...

[M] Sect 2.2

Theorem

If L_1 is a CFL, and L_2 in REG, then $L_1 \cap L_2$ is CFL.

[M] Thm 6.13

product construction

PDA $M_1 = (Q_1, \Sigma, \Gamma, q_1, Z_1, A_1, \delta_1)$
FA $M_2 = (Q_2, \Sigma, q_2, A_2, \delta_2)$

$Q = Q_1 \times Q_2 \qquad q_0 = \langle q_1, q_2 \rangle \qquad A = A_1 \times A_2$

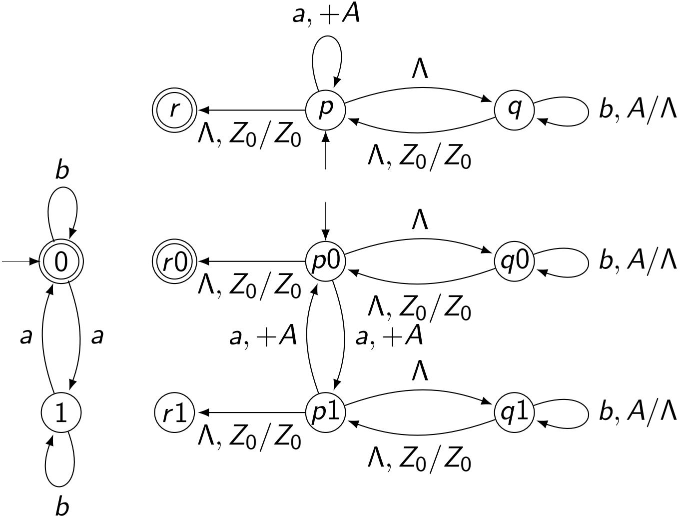
$\delta(\langle p_1, q_1 \rangle, \sigma, X) \ni (\langle p_2, q_2 \rangle, \alpha)$
whenever $\delta_1(p_1, \sigma, X) \ni (p_2, \alpha)$ and $\delta_2(q_1, \sigma) = q_2$

$\delta(\langle p_1, q \rangle, \Lambda, X) \ni (\langle p_2, q \rangle, \alpha)$
whenever $\delta_1(p_1, \Lambda, X) \ni (p_2, \alpha)$ and $q \in Q_2$

The inductive proof that this construction works does not have to be known for the exam.

Also CFG proof

$\{ a^n b^n \mid n \geq 1 \}^* \cap \{ w \in \{a,b\}^* \mid n_a(x) \text{ even} \}$



Non-determinism of PDA

- enables $L(M_1) \cup L(M_2)$
- 'prevents' $L(M_1)'$ (also Λ-transitions)

If L is accepted by DPDA without Λ-transitions, then so is L'

Even: if L is accepted by DPDA, then so is L'

Hence, if L is CFL and L' is not, then there is no DPDA for L
Not reversed (see Pal)

"given a CFL L, does it have property ... ?" yes/no
input CFG G

Given CFG G [G_1 and G_2]
– and given a string x, is $x \in L(G)$? membership problem
convert G to ChNF, and try all derivations of length $2|x| - 1$
(special case if $x = \Lambda$)
Cocke, Younger, and Kasami (1967): n^3 (with DP)
Earley (1970): n^3 (and n^2 if G is unambiguous)

– is $L(G) \neq \varnothing$? non-emptiness
is S useful?
pumping lemma
– is $L(G)$ infinite?
pumping lemma

– is $L(G_1) \cap L(G_2)$ nonempty? [M] Thm 9.20
– is $L(G) = \Sigma^*$? [M] Thm 9.23
– is $L(G_1) \subseteq L(G_2)$?

$$L(G) = \Sigma^*, \text{ if and only if } \Sigma^* \subseteq L(G)$$

All undecidable

Given context-free L and regular R

– is $R \subseteq L$?

– is $L \subseteq R$?

ABOVE

$R \subseteq L$?

Special case $R = \Sigma^*$

$\Sigma^* \subseteq L$ iff $L = \Sigma^*$ undecidable

$L \subseteq R$?

iff $L \cap R' = \varnothing$

regular languages are closed under complement

CFL closed under intersection with regular languages

emptiness context-free language decidable

Section 7

Course Computability

Contents

END.

Thanks to HJH for the slides