Distinguishing strings

From lecture 3:

Definition

Let L be language over Σ, and let $x, y \in \Sigma^*$.
Then x, y are *distinguishable* wrt L (*L-distinguishable*),
if there exists $z \in \Sigma^*$ with

$$xz \in L \text{ and } yz \notin L \quad \text{or} \quad xz \notin L \text{ and } yz \in L$$

Such z *distinguishes* x and y wrt L.

Equivalent definition:
let $L/x = \{ z \in \Sigma^* \mid xz \in L \}$

x and y are L-*distinguishable* if $L/x \neq L/y$.
Otherwise, they are L-*indistinguishable*.

The strings in a set $S \subseteq \Sigma^*$ are *pairwise L-distinguishable*, if for every pair x, y of distinct strings in S, x and y are L-distinguishable.

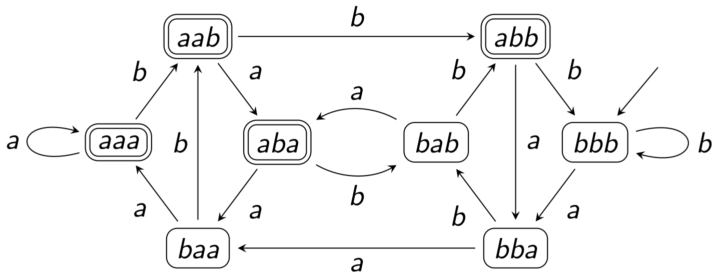
Definition independent of FAs

[M] D 2.20

Strings with a in the nth symbol from the end

[M] E. 2.24

Strings with *a* in the *n*th symbol from the end



[M] E. 2.24

From lecture 3:

Theorem

Suppose $M = (Q, \Sigma, q_0, A, \delta)$ is an FA accepting $L \subseteq \Sigma^*$.

If $x, y \in \Sigma^*$ are L-distinguishable, then $\delta^*(q_0, x) \neq \delta^*(q_0, y)$.

For every $n \geqslant 2$, if there is a set of n pairwise L-distinguishable strings in Σ^*, then Q must contain at least n states.

Hence, indeed: if $\delta^*(q_0, x) = \delta^*(q_0, y)$, then x and y are not L-distinguishable.

Proof. . .

[M] Thm 2.21

Theorem

For every language $L \subseteq \Sigma^*$,
if there is an infinite set S of pairwise L-distinguishable strings,
then L cannot be accepted by a finite automaton.

Proof...

`[M]` Thm 2.26

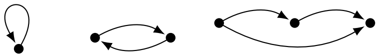
Pal

$Pal = \{x \in \{a, b\}^* \mid x = x^r\}$

[M] E. 2.27

Equivalence relation

R equivalence relation on A
– reflexive xRx for all ...
– symmetric xRy then yRx
– transitive xRy and yRz then xRz



equivalence class $[x]_R = \{ y \in A \mid yRx \}$
short: $[x]$
partition A

[M] Sect. 1.3

Definition

For a language $L \subseteq \Sigma^*$, we define the relation $\equiv_L$ (an equivalence relation) on Σ^* as follows: for $x, y \in \Sigma^*$

$$x \equiv_L y \qquad \text{if and only if } x \text{ and } y \text{ are } L\text{-indistinguishable}$$

Equivalence relation. . .

right invariant $\quad x \equiv_L y$ implies $xz_1 \equiv_L yz_1$

Book uses I_L for $\equiv_L$

Example

$L_1 = \{\ x \in \{a, b\}^* \mid x \text{ ends with } aa\ \}$

L/x for $x = \Lambda, a, b, aa \ldots$
$L/\Lambda = \mathrm{L}$
$L/a = \{a\} \cup L$
$L/b = L$
$L/aa = \{\Lambda, a\} \cup L$

Equivalence classes / partitioning of
$\{a, b\}^* = \{\Lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, \ldots\} \ldots$

Example

Equivalence classes of $\equiv_L$, where $L = AnBn = \{a^i b^i \mid i \geqslant 0\}$

[M] E 2.37

Example

Equivalence classes of $\equiv_L$, where $L = AnBn = \{a^i b^i \mid i \geqslant 0\}$

$\{\Lambda\}, \{a\}, \{a^2\}, \{a^3\}, \ldots$
$\{a^i b^i \mid i \geqslant 1\}$
$\{a^{i+1} b^i \mid i \geqslant 1\}, \{a^{i+2} b^i \mid i \geqslant 1\}, \{a^{i+3} b^i \mid i \geqslant 1\}, \ldots$
$\{x \in \{a, b\}^* \mid x \text{ is not a prefix of any element of } L\}$
$= \{b, ba, bb, aba, abb, baa, \ldots\}$

Infinitely many equivalence classes

quotients

- $L/a^i = \{\, a^k b^{i+k} \mid k \geqslant 0 \,\}$
- $L/a^{i+k} b^i = \{\, b^k \,\} \quad i > 0, k \geqslant 0$
- $L/a^i b^j = L/xbay = \varnothing \quad j > i$

[M] E 2.37

Example

$L_1 = \{\, x \in \{a,b\}^* \mid x \text{ ends with } aa \,\}$

L/x for $x = \Lambda, a, b, aa \ldots$

Equivalence classes / partitioning of
$\{a,b\}^* = \{\Lambda, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, \ldots\}$:

$\{\Lambda, b, ab, bb, aab, abb, \ldots\}$
$\{a, ba, aba, \ldots\}$
$\{aa, aaa, baa, \ldots\}$

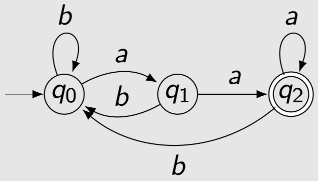
Finitely many equivalence classes

x ends with aa

From lecture 1:

Example

$L_1 = \{ x \in \{a, b\}^* \mid x \text{ ends with } aa \}$



[M] E. 2.1

State q in FA $\approx$ $L_q = \{x \in \Sigma^* \mid \delta^*(q_0, x) = q\}$

From lecture 3:

Theorem

Suppose $M = (Q, \Sigma, q_0, A, \delta)$ is an FA accepting $L \subseteq \Sigma^*$.

If $x, y \in \Sigma^*$ are L-distinguishable, then $\delta^*(q_0, x) \neq \delta^*(q_0, y)$.

For every $n \geqslant 2$, if there is a set of n pairwise L-distinguishable strings in Σ^*, then Q must contain at least n states.

Proof...

[M] Thm 2.21

In other words: if $\delta^*(q_0, x) = \delta^*(q_0, y)$, then x, y are L-indistinguishable.
Each L_q is subset of equivalence class

Myhill-Nerode

Theorem

If $L \subseteq \Sigma^*$ can be accepted by a finite automaton, then the set Q_L of equivalence classes of the relation $\equiv_L$ is finite.

Conversely, if the set Q_L is finite,
the finite automaton $M_L = (Q_L, \Sigma, q_0, A, \delta)$ accepts L, where
$q_0 = \dots$
$A = \dots$
$\delta([x], \sigma) = \dots$

[M] Thm 2.36

Myhill-Nerode

Theorem

If $L \subseteq \Sigma^*$ can be accepted by a finite automaton, then the set Q_L of equivalence classes of the relation $\equiv_L$ is finite.

Conversely, if the set Q_L is finite,
the finite automaton $M_L = (Q_L, \Sigma, q_0, A, \delta)$ accepts L, where
$q_0 = [\Lambda]$
$A = \{q \in Q_L \mid q \subseteq L\}$
$\delta([x], \sigma) = [x\sigma]$

Finally, M_L has the fewest states of any FA accepting L.

Note:
If $x \in L$, then $[x] \subseteq L$ (L is union of equivalence classes)
Right invariant $x \equiv_L y$ implies $x\sigma \equiv_L y\sigma$

[M] Thm 2.36

From lecture 3:

Theorem

For every language $L \subseteq \Sigma^*$,
if there is an infinite set S of pairwise L-distinguishable strings,
then L cannot be accepted by a finite automaton.

Proof. . .

[M] Thm 2.26

Exercise 2.36.
For a certain language $L \subseteq \{a, b\}^*$, $\equiv_L$ has exactly four equivalence classes. They are $[\Lambda]$, $[a]$, $[ab]$ and $[b]$.

It is also true that the three strings a, aa, and abb are all equivalent, and that the two strings b and aba are equivalent.

Finally, $ab \in L$, but Λ and a are not in L, and b is not even a prefix of any element of L.

Draw an FA accepting L.

Minimizing states

1. Remove unreachable states
2. Merge ‘equivalent’ states

Equivalent states

Equivalence relation $\equiv_L$ induces equivalence relation on states
Each L_q is subset of equivalence class
L_p and L_q may be subset of same equivalent class
$p \equiv q \iff L_p$ and L_q are subset of same equivalent class
$p \not\equiv q \iff$ for some $z \in \Sigma^*$ exactly one of $\delta^*(p, z)$ and $\delta^*(q, z)$ is in A

Definition

S_M: set of pairs (p, q) such that $p \not\equiv q$

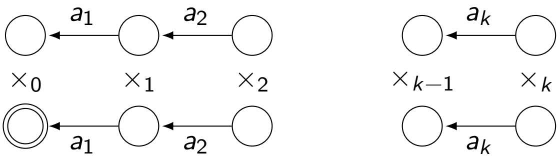
1. If exactly one of p and q is in A, then $(p, q) \in S_M$
2. If for some $\sigma \in \Sigma$, $(\delta(p, \sigma), \delta(q, \sigma) \in S_M$, then $(p, q) \in S_M$

Minimizing states

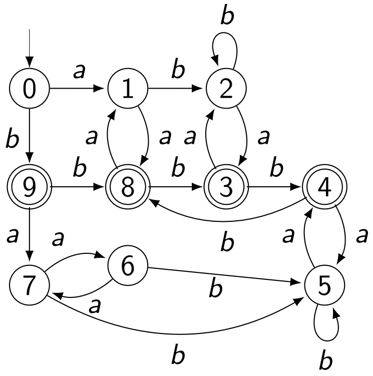
ALGORITHM mark pairs of non-equivalent states

```
start by marking pairs $(p, q)$ where exactly one $p, q$ in $A$

repeat
    for each unmarked pair $(p, q)$
        check whether there is a $\sigma$ such that $(\, \delta(p, \sigma), \delta(p, \sigma)\, )$ is marked
        then mark $(p, q)$
until this pass does not mark new pairs
```

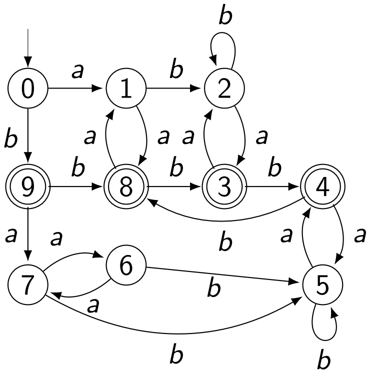


[M] Algo 2.40



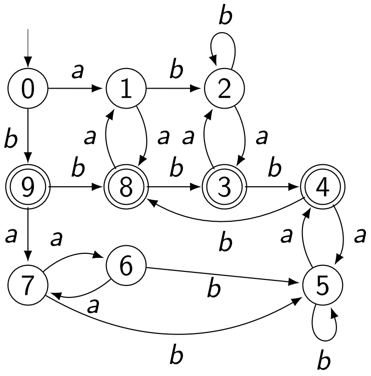
	0	1	2	3	4	5	6	7	8
1	·								
2	·	·							
3	·	·	·						
4	·	·	·	·					
5	·	·	·	·	·				
6	·	·	·	·	·	·			
7	·	·	·	·	·	·	·		
8	·	·	·	·	·	·	·	·	
9	·	·	·	·	·	·	·	·	·

[M] Fig 2.42



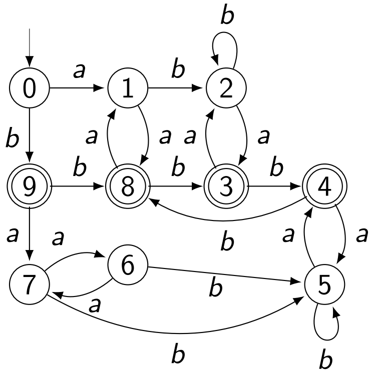
1	·								
2	·	·							
3	1	1	1						
4	1	1	1	·					
5	·	·	·	1	1				
6	·	·	·	1	1	·			
7	·	·	·	1	1	·	·		
8	1	1	1	·	·	1	1	1	
9	1	1	1	·	·	1	1	1	·
	0	1	2	3	4	5	6	7	8

[M] Fig 2.42



1	2								
2	2	·							
3	1	1	1						
4	1	1	1	·					
5	2	·	·	1	1				
6	2	2	2	1	1	2			
7	2	2	2	1	1	2	·		
8	1	1	1	·	·	1	1	1	
9	1	1	1	2	·	1	1	1	2
	0	1	2	3	4	5	6	7	8

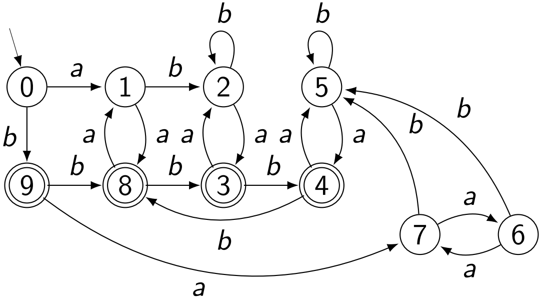
[M] Fig 2.42

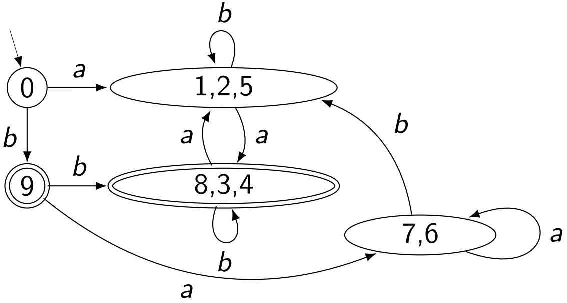


1	2								
2	2	·							
3	1	1	1						
4	1	1	1	·					
5	2	·	·	1	1				
6	2	2	2	1	1	2			
7	2	2	2	1	1	2	·		
8	1	1	1	·	·	1	1	1	
9	1	1	1	2	3	1	1	1	2
	0	1	2	3	4	5	6	7	8

Resulting (minimal) FA. . .

[M] Fig 2.42





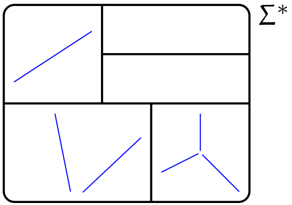
[M] Fig 2.42

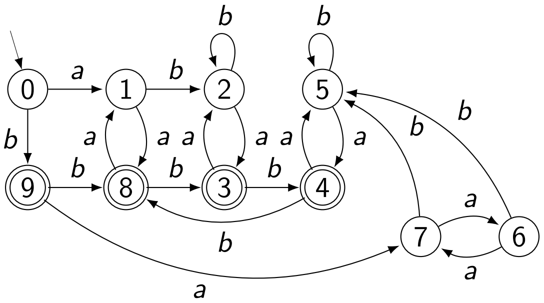
$L = L(M)$

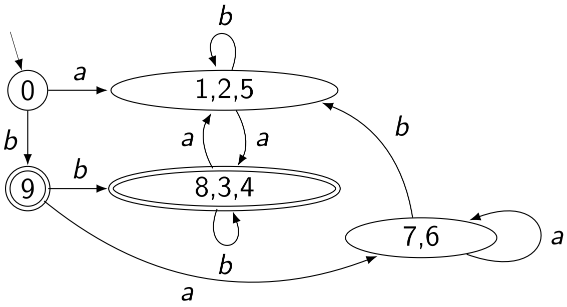
$\equiv_M$ state $\delta^*(q_0, x)$

$\equiv_L$ "future" L/x

$x \equiv_M y$, then $x \equiv_L y$.





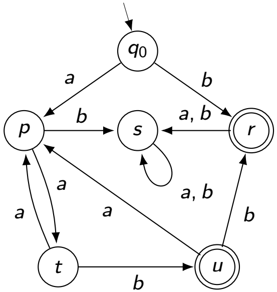


[M] Fig 2.42

Distinguishing states

From lecture 3:

$L = \{aa, aab\}^*\{b\}$

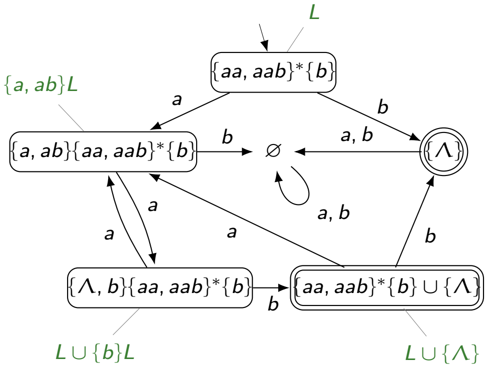


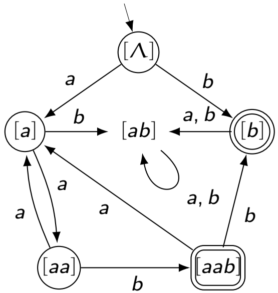
[M] E 2.22

Distinguishing states

$L = \{aa, aab\}^*\{b\}$

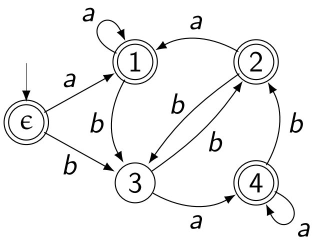
$L/\sigma = \{\, z \in \Sigma^* \mid \sigma z \in L \,\}$ $\quad L \xrightarrow{\sigma} L/\sigma$ $\qquad [x] \xrightarrow{\sigma} [x\sigma]$

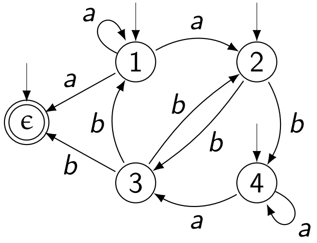


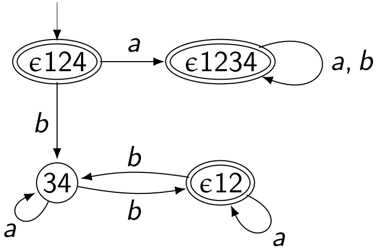


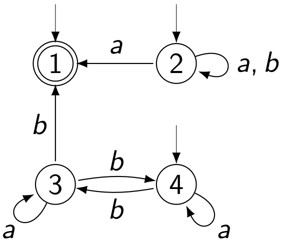
[M] E 2.22 see ↪E 3.6

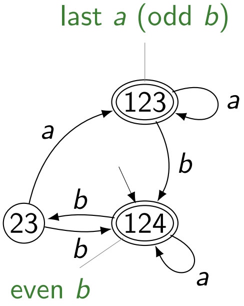
Example: Brzozowski minimization











ABOVE

Brzozowski observes that one can minimize an FA by performing the following operations twice: invert (mirror), then determinize.
It is rather magical that this indeed works.
The method is in theory rather unfavourable, because of the exponentiation when determinizing, but in practice seems not too slow.