

Seminar (Combinatorial) Algorithms



Universiteit
Leiden

Hendrik Jan Hoogeboom & Walter Kosters

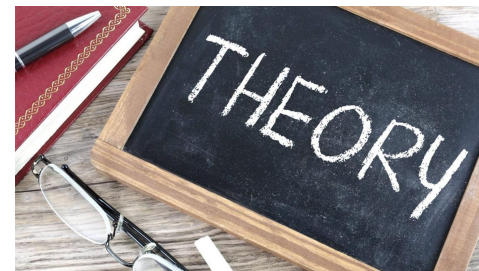
Spring 2024, Feb 6–May 21

Snellius 402

Tuesday February 6, 2023, 11:00–12:45

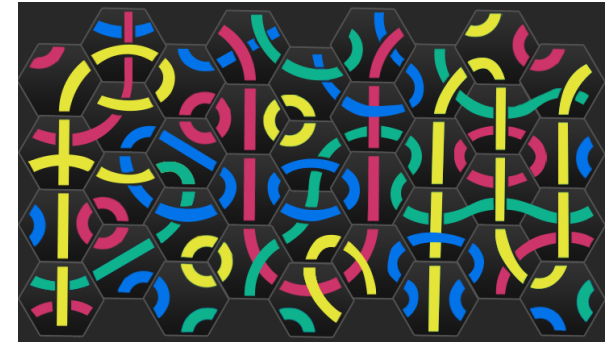
www.liacs.leidenuniv.nl/~kosterswa/semalg/

- this year's topic slides 3–4
- example presentation from 2020 slides 5–24
- want to participate? slide 25
 - try to solve problem (*) from slide 7
 - look at this year's topic
 - decide y/n
 - send e-mail



We discuss papers from [FUN2010/12/14/16/18/21/22](#), the 5–11th International Conference on Fun with Algorithms. See the [list of selected papers](#) from

- [link 2010](#) (*) [link 2016](#)
- [link 2012](#) (*) [link 2018](#)
- [link 2014](#) (*) [link 2021](#)
- [link 2022](#)



(*) only accessible within university network

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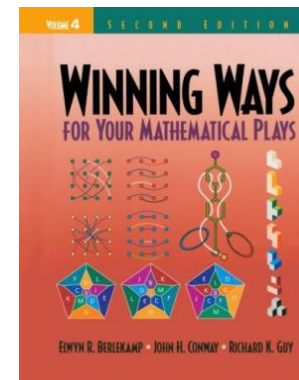
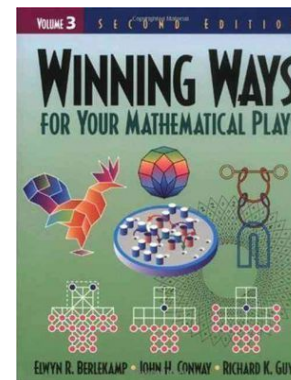
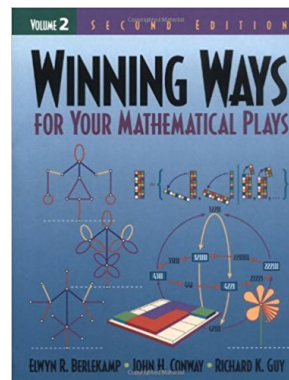
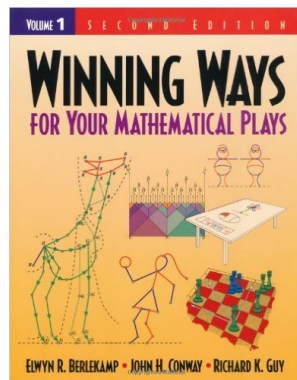
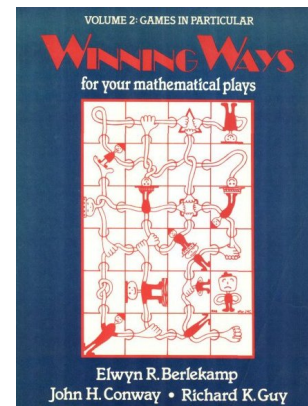
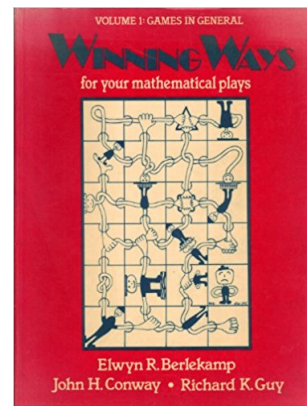
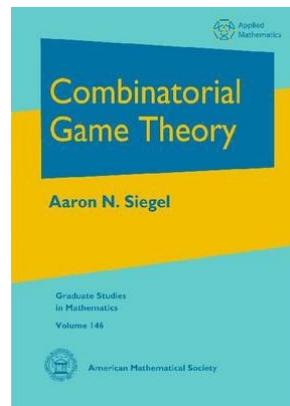
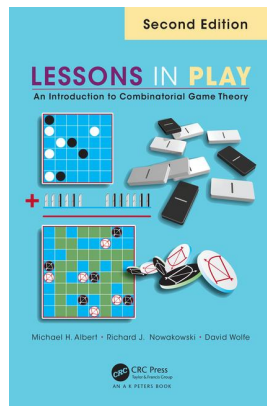
Example (FUN2016): Cohen et al., [Spy-game on graphs](#)

- Definitions, ideas, . . .
- Start with placing guards or spy?
Prove some sort of equivalence.
- Show that deciding whether at most k guards “suffice” for certain graphs is NP-hard, by reduction from the Set Cover problem.

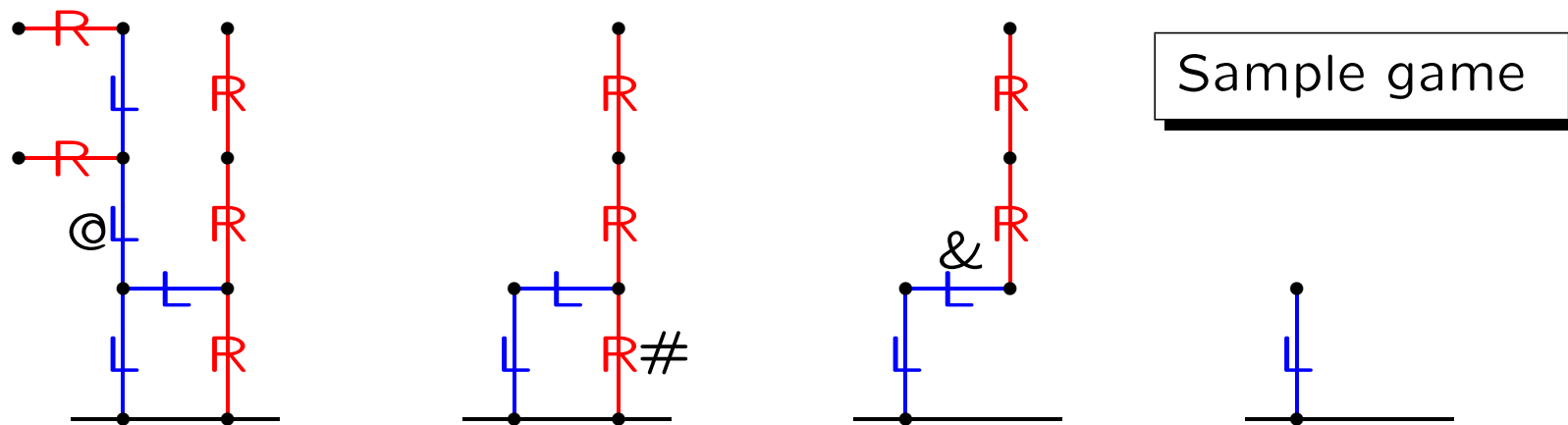


[link to the paper](#)

Overview: Hackenbush — Surreal numbers — Nim — +



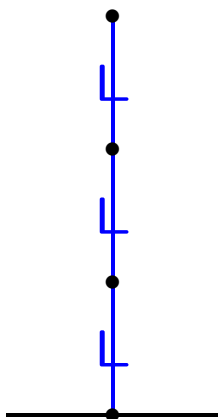
In the game (Blue-Red-)Hackenbush the two players **Left** = she and **Right** = he alternately remove a **blue** or a **Red** edge. All edges that are no longer connected to the ground, are also removed. *If you cannot move, you lose!*



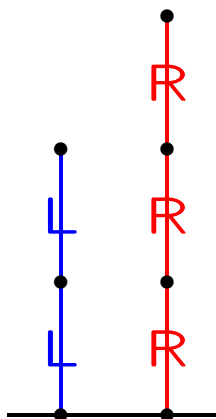
Left chooses @, **Right** # (stupid), **Left** &. Now **Left** wins because **Right** cannot move.

By the way, **Right** can win here, whoever starts!

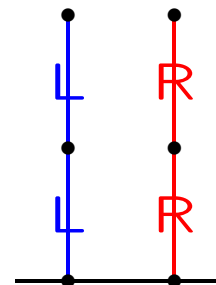
When playing Hackenbush, what is the **value of a position**?



value 3



value $2 - 3 = -1$



value $2 - 2 = 0$

value > 0 : **Left** wins (whoever starts)

\mathcal{L}

value $= 0$: first player loses

\mathcal{P}

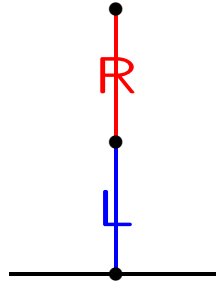
value < 0 : **Right** wins (whoever starts)

\mathcal{R}

(*) Hackenbush has no “first player wins”!

\mathcal{N}

But what is the value of this position?



If **Left** begins, she wins immediately. If **Right** begins, **Left** can still move, and also wins. So **Left** always wins. Therefore, the value is > 0 .

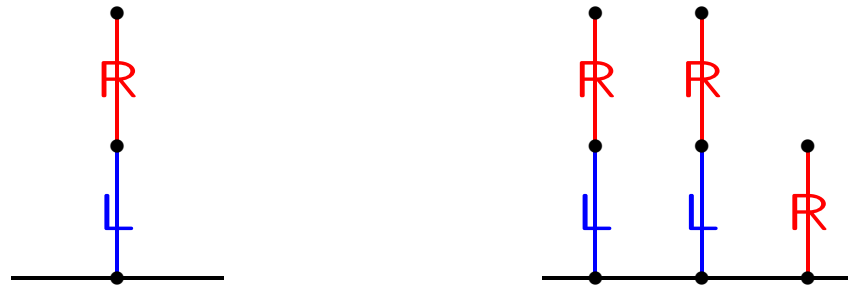
Is the value equal to 1?

If the value in the left hand side position x would be 1, the value of the right hand side position would be $1 + (-1) = 0$, and the first player should lose. Is this true?



No! If **Left** begins, **Left** loses, and if **Right** begins **Right** can also win. So **Right** always wins (i.e., can always win), and therefore the right hand side position is < 0 , and $x + (-1) < 0$, and the left one is between 0 and 1.

The left hand side position is denoted by $\{ 0 \mid 1 \}$.

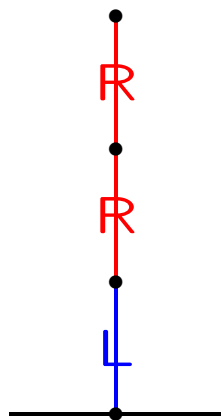


Note that the right hand side position does have value 0: the first player loses. And so we have:

$$\{ 0 \mid 1 \} + \{ 0 \mid 1 \} + (-1) = 0,$$

and “apparently” $\{ 0 \mid 1 \} = 1/2$.

We denote the value of a position where **Left** can play to (values of) positions from the set L and **Right** can play to (values of) positions from the set R by $\{ L \mid R \}$.



The value is $\{ 0 \mid \frac{1}{2}, 1 \} = \frac{1}{4}$.

Simplicity rule: The value is always the “simplest” number between left and right set: the smallest integer — or the dyadic number with the lowest denominator (power of 2).

Give a position with value $3/8$.

Show that $\{ 0 \mid 100 \} = 1$.

The following problem is NP-complete:
Decide on the value of a “Redwood Bed”.

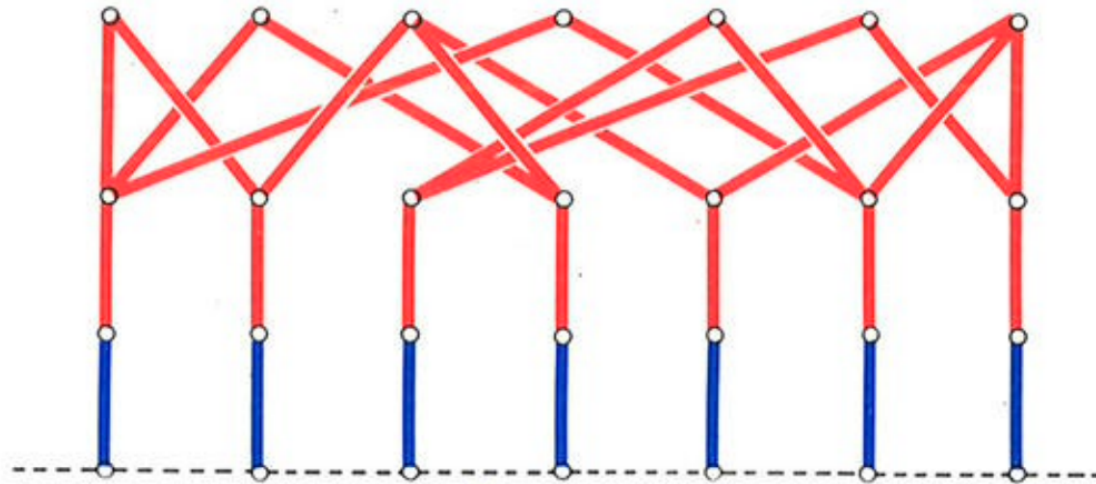


Figure 31. A Moderately Hard Bed.

From WinningWays, p. 210.



Donald E. (Ervin) Knuth

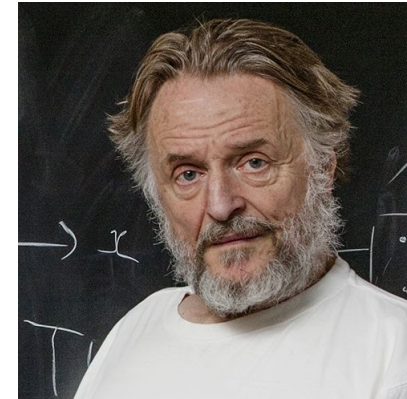
1938, US

NP; KMP

TEX

change-ringing; 3:16

The Art of Computer
Programming



John H. (Horton) Conway

1937–2020, UK → US

C_01 , C_02 , C_03

Doomsday algorithme

game of Life; Angel problem

Winning Ways for your
Mathematical Plays

Surreal numbers

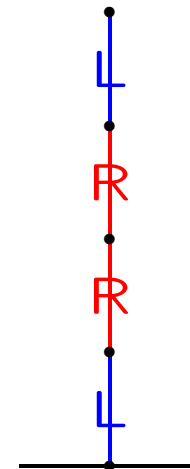
In this way we define **surreal numbers**: “decent” pairs of sets of previously defined surreal numbers: all elements from the left set are smaller than those from the right set.

Start with $0 = \{ \emptyset \mid \emptyset \} = \{ \text{nothing} \mid \text{nothing} \} = \{ \mid \}$: the game where both **Left** and **Right** have no moves at all, and therefore the first player loses: born on day 0.

And then $1 = \{ 0 \mid \}$ en $-1 = \{ \mid 0 \}$, born on day 1.

And $42 = \{ 41 \mid \}$, born on day 42.

And $\frac{3}{8} = \{ \frac{1}{4} \mid \frac{1}{2} \}$, born on day 4.



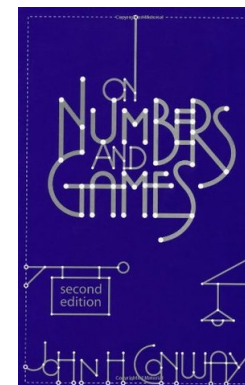
Sets can be infinite: $\pi = \{ 3, 3\frac{1}{8}, 3\frac{9}{64}, \dots \mid 4, 3\frac{1}{2}, 3\frac{1}{4}, 3\frac{3}{16}, \dots \}$.

We define: $\varepsilon = \{ 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \}$, an “incredibly small number”, and $\omega = \{ 0, 1, 2, 3, \dots \mid \} = \{ \mathbf{N} \mid \emptyset \}$, a “terribly large number, some sort of ∞ ”.

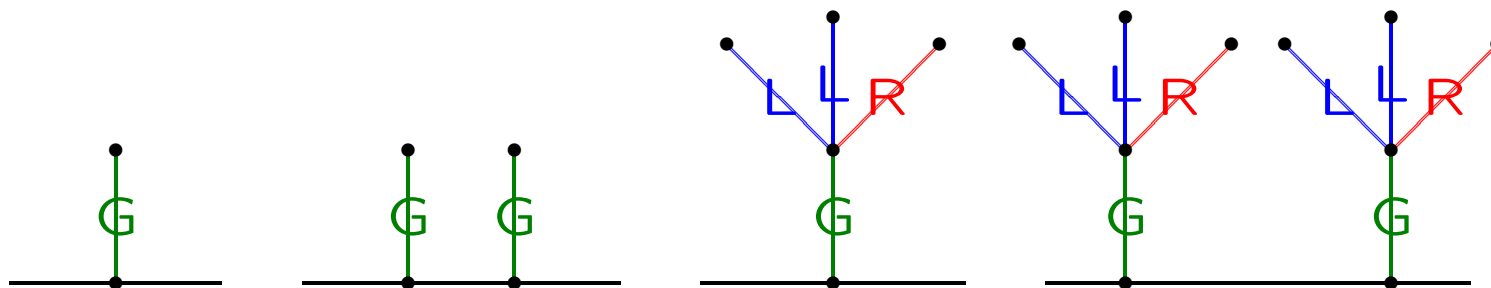
Then we have $\varepsilon \cdot \omega = 1$ — if you know how to multiply.

And then $\omega + 1$, $\sqrt{\omega}$, ω^ω , $\varepsilon/2$,
and so on!

But we will stick to
“games” like $\{ 1 \mid -1 \}$.



In **Red-Green-Blue-Hackenbush** we also have **Green** edges, that can be removed by both players.



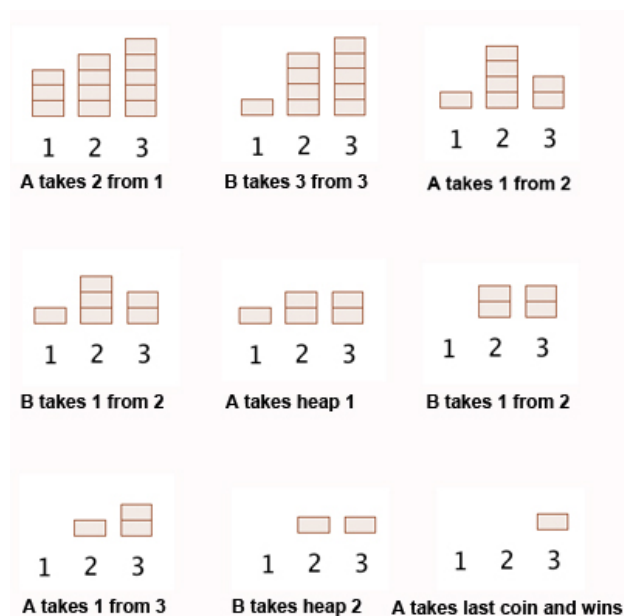
The first position has value $*$ $= \{ 0 \mid 0 \}$ (not a surreal number), because the player to move can win.

The second position is $* + * = 0$ (player to move loses).

The third position is a first player win.

The fourth position is a win for **Left** (whoever begins), and is therefore > 0 .

In the **Nim** game we have several stacks of tokens = coins = matches. A move consists of taking a nonzero number of tokens from one of the stacks. If you cannot move, you lose (“normal play”).



The game is **impartial**: both players have the same moves. (And for the **misère** version: if you cannot move, you win.)

For Nim we have **Bouton's analysis** from 1901.

We define the **nim-sum** $x \oplus y$ of two positive integers x and y as the bitwise XOR of their binary representations: addition without carry. With two stacks of equal size the first player loses ($x \oplus x = 0$): use the “mirror strategy”.

A nim game with stacks of sizes a_1, a_2, \dots, a_k is a first player loss exactly if $a_1 \oplus a_2 \oplus \dots \oplus a_k = 0$. And this sum is the **Sprague-Grundy value**.

We denote a nim game with value m by $*m$ (the same as a stalk of m green Hackenbush edges; not a surreal number). And $*1 = *$. So if $m \neq 0$ the first player loses.

The **Sprague-Grundy Theorem** roughly says that every impartial game is a Nim game.

With stacks of sizes 29, 21 and 11, we get $29 \oplus 21 \oplus 11 = 3$:

| | |
|-------|----|
| 11101 | 29 |
| 10101 | 21 |
| 1011 | 11 |
| ----- | -- |
| 00011 | 3 |

So a first player win, with unique winning move $11 \rightarrow 8$.

Why this move, and why is it unique?

How to add these “games” (we already did)? Like this:

$$a + b = \{ A_L + b, a + B_L \mid A_R + b, a + B_R \}$$

if $a = \{ A_L \mid A_R \}$ and $b = \{ B_L \mid B_R \}$.

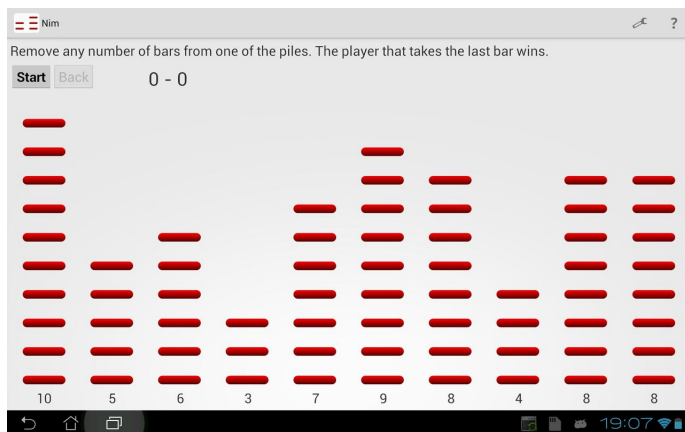
We put $u + \emptyset = \emptyset$ and (more general) $u + V = \{u + v : v \in V\}$.

This corresponds with the following: you play two games in parallel, and in every move you must play in exactly one of these two games: the **disjunctive sum**.

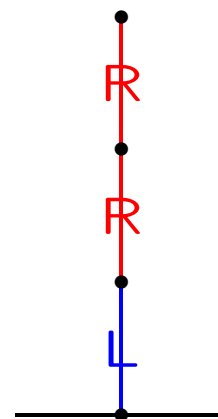
$$\text{Verify that } 1 + \frac{1}{2} = \{ 1 \mid 2 \} = \frac{3}{2}.$$

See [Claus Tøndering's paper](#).

Now consider this addition of two game positions, with on the left a Nim position and on the right a Hackenbush position:



+



Then this sum is > 0 , it is a win for **Left**!

And we even have: $*m + 1/2^{10000} > 0$, and therefore $-1/2^{10000} < * < 1/2^{10000}$, but $*$ is not comparable to 0.

We finally play **Clobber**, on an m times n board, with white (Right) and black (Left) stones. A stone can capture = “clobber” a directly horizontally/vertically adjacent stone from the other color. If you cannot move, you lose.

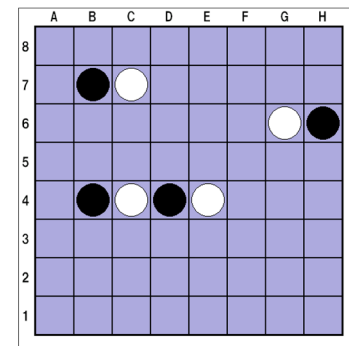
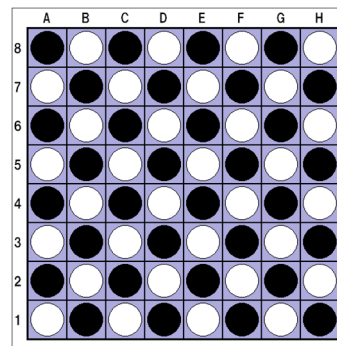
Some examples:

$$\boxed{\bullet} \boxed{\circ} = \{ 0 \mid 0 \} = *$$

$$\boxed{\bullet} \boxed{\bullet} \boxed{\circ} = \{ 0 \mid * \} = \uparrow > 0$$

$$\boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} = 0$$

$$\boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} = \pm(\uparrow, \uparrow^{[2]} *, \{0 \mid \uparrow, \pm(*, \uparrow)\}, \{\uparrow * \mid \downarrow, \pm(*, \uparrow)\})$$



Three main references:

LessonsInPlay:

M.H. Albert, R.J. Nowakowski and D. Wolfe, Lessons in Play, **second edition**, CRC Press, 2019.

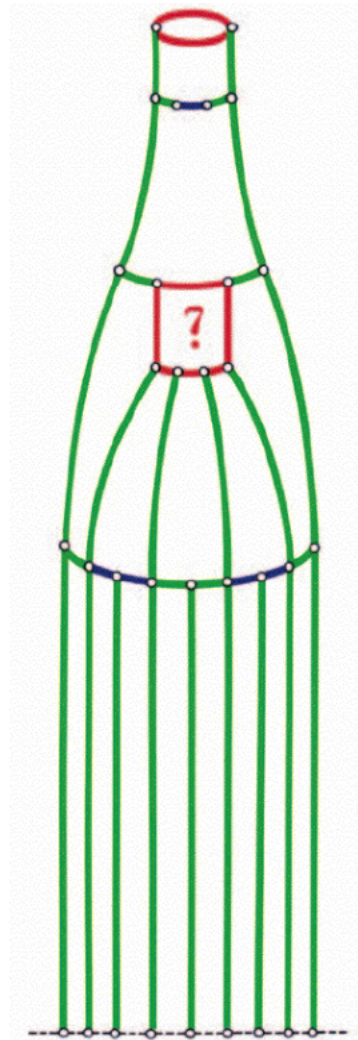
Siegel:

A.N. Siegel, Combinatorial Game Theory, AMS, 2013.

WinningWays:

E.R. Berlekamp, J.H. Conway and R.K. Guy, Winning Ways for your Mathematical Plays, 1982/2001.

(Note that there are two editions: the first has two volumes, the second has four volumes.)



From WinningWays, p. 213.

How is the seminar organized? Do the following twice:

Present a (chosen) “paper” during a 45/30 minutes **lecture**. Make slides, and use the “blackboard”.

Produce a 7–10 page **paper/report in L^AT_EX/PDF**. Use your own words, no copy-paste; English.

The grading is based on the four **P**s: **p**reparation (2×), **p**resentation (2×), **p**aper (2×) and **p**articipation (including presence: discussions, questions).

Apply for participation: send e-mail[†] with proof of (*) from slide 7 before Friday afternoon February 9, 2024. At most ≈ 12 participants.

[†] w.a.kosters@liacs.leidenuniv.nl

