Process Semantics of P/T-Nets with Inhibitor Arcs* **

H.C.M. Kleijn[1] and M. Koutny[2]

[1] LIACS, Leiden University, P.O.Box 9512, NL-2300 RA Leiden, The Netherlands
[2] Dept. of Comp. Sci., University of Newcastle, Newcastle upon Tyne NE1 7RU, U.K.

Abstract. In this paper, we define a process semantics of P/T-nets with inhibitor arcs (PTI-nets). For PTI-nets with bounded inhibiting places, we combine the existing approaches for ordinary P/T-nets and for elementary net systems with inhibitor arcs. To deal with unbounded inhibiting places, a new feature has to be added to the underlying occurrence nets. In either case we show how to construct a process from a step sequence and give a complete characterization of all processes which can be obtained in this way. Using these processes it is possible to express the causal relationships between events in a PTI-net behaviour.
Keywords: Causality/partial order theory of concurrency; analysis and synthesis, structure and behaviour of nets.

1 Introduction

Petri nets with inhibitor arcs have been around for quite some time now and as stated in [12], 'Petri nets with inhibitor arcs are intuitively the most direct approach to increasing the modelling power of Petri nets'. Unlike a 'normal' Petri net, a Petri net with inhibitor arcs has the possibility of testing whether a place is empty in the current marking (*zero testing*). Thus inhibitor arcs are very well suited to model situations involving testing for a specific condition, rather than producing and consuming resources. Place/Transition nets with inhibitor arcs (PTI-nets) are strictly more expressive than ordinary Place/Transition-nets (P/T-nets). They can simulate the computations of Turing machines and several important problems like reachability and liveness which are decidable for P/T-nets are undecidable for PTI-nets.

This paper is concerned with the description of the causal relationships in (concurrent) runs of PTI-nets. The research presented here is a natural continuation of the work of [8] regarding elementary net systems with inhibitor arcs. There, so-called stratified order structures are employed to provide a causality semantics which is consistent with the operational semantics in terms of step sequences. Whereas for an elementary net system, an abstract causality semantics can be given in terms of partial orders alone, the presence of inhibitor arcs requires more information on the relationships between event occurrences. As

* This research was supported by a grant from the Netherlands Organization for Scientific Research (NWO) and the British Council.
** LIACS Technical Report 99-13. To appear in LNCS, proceedings of ICATPN 2000.

an example (borrowed from [8]), consider the net with the two events, e and f, shown in figure 1.

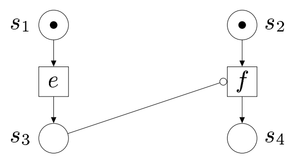

Fig. 1. An elementary net system with inhibitor arc.

In addition to the normal arcs, there is an inhibitor arc from condition s_3 to f. This implies that f can only occur if s_3 is empty (and the standard enabling conditions in an elementary net system are fulfilled). This net has three non-empty firing sequences: $\omega_1 = e$, $\omega_2 = f$ and $\omega_3 = fe$. Note that the occurrence of e is completely independent of the occurrence of f. However, f is disabled after the occurrence of e. This implies that independence of events is no longer symmetric. In the a priori semantics of [8], e and f may also be executed simultaneously, since the inhibiting condition s_3 of f does not hold *prior* to the occurrence of f. Thus also the step $\{e, f\}$ may be executed. This implies that independence and absence of ordering are no longer the same.
Stratified order structures take care of these more involved relations between event occurrences by providing next to a partial order a weak partial order. The partial order describes the strict causal relationships between event occurrences whereas the weak partial order describes weak causal relationships as the above: f *may precede* e but not vice versa and hence the step $\{e, f\}$ may be sequentialised to fe, but not to ef.
For elementary net systems (without inhibitor arcs), an abstract partial order semantics follows immediately from their process semantics (see, e.g., [14]). A process is constructed by unfolding the system according to a given run represented by a firing sequence. The result is an occurrence net: a (labelled) acyclic net with non-branching conditions, since conflicts are resolved during the run. By abstracting from the conditions of the occurrence net, one obtains a (labelled) partial order which describes precisely the causal relationships between the events in the given run: all linearisations of the partial order are firing sequences of the elementary net system and they include the firing sequence on the basis of which the process was constructed.
Also in [8], first a process semantics is given. Since in the a priori semantics not all concurrent runs of the system can be sequentialised to a firing sequence, this process semantics is based on step sequences. (Consider again the elementary net system in figure 1, with an additional inhibitor arc from s_4 to e. Now, $\omega_3 = fe$ is no longer a firing sequence, although $\sigma = \{e, f\}$ is still a legal step sequence.) Given a step sequence, the system is unfolded into a (labelled) occurrence net with additional arcs to represent the zero testing. Testing if a condition does not hold (inhibitor arc) is in the unfolding represented by testing if its complement condition does hold (activator arc). In the resulting activator occurrence net the

conditions are again non-branching (with respect to the normal arcs). Moreover, it is acyclic in a sense which includes the activator arcs ($\diamondsuit$-acyclic) and thus allows to extract a (labelled) stratified order structure which describes precisely the causality and weak causality relationships between the events in the given run: all step sequences which obey the constraints imposed by the stratified order structure are step sequences of the system and they include the step sequence on basis of which the process was constructed.

In this paper we propose a process semantics for PTI-nets with the aim to provide a basis for their abstract causality semantics. Since the nets are no longer necessarily safe (markings may assign more than one token to a place), we combine the ideas of [8] with the definition of processes for (finite) P/T-nets as discussed in, e.g., [6] and [1]. In these processes each token in a place of the original P/T-net is represented by a distinct condition in the process net. Consequently, unfolding the net according to a step sequence in general yields more than one occurrence net. However, the same occurrence nets as employed for elementary net systems are used in the process definition of P/T-nets.
First we consider the case of PTI-nets in which the number of tokens in an inhibiting place cannot grow arbitrarily large (the inhibiting places are bounded). We refer to these nets as PTBI-nets. For them, using complementary places for the inhibiting places and activator arcs in the processes, the ideas of [8] can be combined with the approach of [1] which relates process axioms and inductively defined unfoldings. We define the processes of PTBI-nets and give an unfolding construction based on step sequences. We show that these definitions are consistent with each other, and that they can be used to extract the causal relationships between the events in a run of a PTBI-net.
Next we turn to the unbounded case. In this case, the classical place complementation can no longer be applied. Instead we introduce a new feature in the form of additional conditions (z-conditions) to the occurrence nets. A z-condition represents an empty inhibiting place and is connected by an activator arc to the events representing transitions which test that place for zero tokens. Z-conditions are introduced 'on-demand' during the construction of a process for a given step sequence, and with their introduction an up-date of the occurrence net has to take place. This differs from the standard unfolding procedures discussed above which do not refer to the past and are purely local (based on the neighbourhood of the transitions in the original net). Moreover, z-conditions may be branching (with respect to the normal arcs). Still, the resulting z-activator occurrence nets can be fully (axiomatically) characterised, and they provide us with an abstract causal semantics for the unbounded case.

Both the process semantics and the causal semantics for PTI-nets are consistent with those for PTBI-nets, which in their turn generalise the semantics of P/T-nets as defined in [1] and the semantics of elementary net systems with inhibitor arcs from [8].

This paper is largely self-contained, although it may be an advantage for the reader to be acquainted with the 'classical' process theory as presented in [1, 6] and [14]. Due to the page limit, some proofs are either only sketched or omitted.

2 Preliminaries

$\mathbf{N}$ denotes the set $\{0, 1, 2, \ldots\}$ of natural numbers. All functions considered in this paper are total. For a finite set X, we denote by $|X|$ its cardinality.

Let X be a set. A *multiset* (over X) is a function $m : X \to \mathbf{N}$. The sum of two multisets m_1 and m_2 over X is denoted by $m_1 + m_2$ and is defined by $(m_1 + m_2)(x) = m_1(x) + m_2(x)$, for all $x \in X$. The *empty multiset*, denoted by $\underline{0}$, is defined by $\underline{0}(x) = 0$, for all $x \in X$. Note that a multiset m over X may be seen as the subset $\{x \in X \mid m(x) \geq 1\}$ of X, the elements of which are equipped with multiplicities. Conversely, every subset of X may be viewed through its characteristic function as a multiset over X. We denote $x \in m$ if $m(x) \geq 1$.

A *step sequence* (over X) is a finite sequence $m_1 \ldots m_n$ of non-empty multisets m_i (over X). The empty sequence is denoted by λ. If each of the multisets m_i in a step sequence $\sigma = m_1 \ldots m_n$ is a singleton set $\{x_i\}$ (i.e., $m_i(x_i) = 1$ and $m_i(y) = 0$, for all $y \neq x_i$), then σ may be written as $x_1 \ldots x_n$. Thus X^*, the set of all finite sequences of occurrences of elements from X, is a subset of the set of all step sequences over X.
Now assume that X is finite. A *labelling* of X is a function $l : X \to A$, where A is some set of labels (the labelling alphabet). It is extended to step sequences over X in the following way: For $m : X \to \mathbf{N}$, we define $l(m) : A \to \mathbf{N}$ by $l(m)(a) = \sum_{\{x \mid l(x)=a\}} m(x)$, for all $a \in A$. For $\sigma = m_1 \ldots m_n$, we set $l(\sigma) = l(m_1) \ldots l(m_n)$. In particular, $l(\lambda) = \lambda$. Hence step sequences over X are mapped to step sequences over A. Observe that $l(\sigma)$ is in A^*, whenever $\sigma = x_1 \ldots x_n$ is in X^*. In general, however, a set is mapped to a multiset.

For two relations $P, Q \subseteq X \times X$, their composition $P \circ Q$ is also a binary relation over X, defined by $P \circ Q = \{(x, z) \mid \exists y \in X : (x, y) \in P \wedge (y, z) \in Q\}$. Let $id_X = \{(x, x) \mid x \in X\}$ be the identity relation in X. A binary relation P over X is reflexive if $id_X \subseteq P$; it is irreflexive if $id_X \cap P = \emptyset$; and it is transitive if $P \circ P \subseteq P$. The transitive closure of P is denoted by P^+, and its transitive and reflexive closure by $P^\star$.

2.1 Partially ordered sets

A *partial order* on X is an irreflexive and transitive binary relation over X. If $\prec \subseteq X \times X$ is a partial order, then the pair $(X, \prec)$ is referred to as a *partially ordered set*, or *poset* for short. In this paper we will only consider *finite* posets (X is finite).

A *labelled poset* is a triple $(X, \prec, l)$ such that $(X, \prec)$ is a poset and $l : X \to A$ is a labelling of X. As we will be mainly dealing with labelled posets, all terminology is introduced directly for labelled posets. If need be, it can be carried over to posets by identifying the poset $(X, \prec)$ with the labelled poset $(X, \prec, id_X)$.

Let $(X, \prec, l)$ be a labelled poset. As usual, for $x, y \in X$, we write $x \prec y$ rather than $(x, y) \in \prec$ and we use $x \preceq y$ to denote that $x = y$ or $x \prec y$. The notation $x \not\leftrightarrow y$ indicates that x and y are distinct incomparable elements ($x \neq y \wedge x \not\prec y \wedge y \not\prec x$).
The labelled poset $(X, \prec, l)$ is *linear* (or *total*), if every two distinct elements are

comparable (the relation $\not\leftrightarrow$ is empty). It is *stratified* [4] if $x \not\leftrightarrow y$ and $y \not\leftrightarrow z$ imply that $x \not\leftrightarrow z$ whenever $x \neq z$. Thus a linear labelled poset is always stratified. Note that $(X, \prec, l)$ is stratified if and only if $\not\leftrightarrow \cup\, id_X$ is an equivalence relation. If $(X, \prec, l)$ is stratified it defines a unique (ordered) sequence of subsets $X_1 \ldots X_k$ of X, the equivalence classes of $\not\leftrightarrow \cup\, id_X$, with the property: $\prec = \bigcup_{i<j} X_i \times X_j$, and $\not\leftrightarrow = (\bigcup_{i=1}^{k} X_i \times X_i) \setminus id_X$. Hence each labelled stratified poset $po = (X, \prec, l)$ as above defines a unique step sequence $\mho_{po} = l(X_1) \ldots l(X_k)$. Conversely, if $po = (X, \prec, l)$ is such that X can be partitioned into non-empty sets $X_1, \ldots, X_k$ satisfying the above conditions, then it is stratified and $\mho_{po} = l(X_1) \ldots l(X_k)$.

Two labelled posets $po_1 = (X_1, \prec_1, l_1)$ and $po_2 = (X_2, \prec_2, l_2)$ are *isomorphic* if there is a bijection $f : X_1 \to X_2$ such that for all $x, y \in X_1$, $l_1(x) = l_2(f(x))$, and $x \prec_1 y$ if and only if $f(x) \prec_2 f(y)$.
Note that every step sequence σ defines an isomorphism class of labelled stratified posets po with the property that $\mho_{po} = \sigma$. In the sequel, however, we are not really interested in the underlying set which is only used to carry labels and we will simply use po_σ to denote any labelled stratified poset po such that $\mho_{po} = \sigma$.

2.2 Stratified order structures

A *relational structure* is a triple $\mathcal{S} = (X, \prec, \sqsubset)$, where $\prec$ and $\sqsubset$ are two binary relations over a finite set X. $\mathcal{S}$ is called a *stratified order structure* [5, 7], or an *so-structure* for short, if for all $x, y, z \in X$ the following hold (again using the infix notation):

$$\begin{array}{lcl} & & x \not\sqsubset x \quad \text{C1} \\ x \prec y & \Longrightarrow & x \sqsubset y \quad \text{C2} \\ x \sqsubset y \sqsubset z \;\wedge\; x \neq z & \Longrightarrow & x \sqsubset z \quad \text{C3} \\ x \sqsubset y \prec z \;\vee\; x \prec y \sqsubset z & \Longrightarrow & x \prec z \quad \text{C4.} \end{array}$$

It is easily seen that $(X, \prec)$ is a poset and, furthermore, that $x \prec y$ implies $y \not\sqsubset x$. Furthermore, if $(X, \prec)$ is a poset, then $(X, \prec, \prec)$ is an so-structure.
In diagrams, $\prec$ is represented by solid arcs, and $\sqsubset$ by dashed arcs. We can omit arcs that can be deduced using C1-C4.

The elements of a relational structure $(X, \prec, \sqsubset)$ will usually be labelled. Thus we consider structures $\mathcal{S} = (X, \prec, \sqsubset, l)$, such that $(X, \prec, \sqsubset)$ is a relational structure and $l : X \to A$ is a labelling of X. All remaining terminology is now introduced directly for labelled relational structures. (It can be carried over to the non-labelled case by identifying $(X, \prec, \sqsubset)$ with $(X, \prec, \sqsubset, id_X)$.) In diagrams, we do not name the nodes but only give their labels.

Concurrency theory employs partial orders $\prec$ to model both specifications and observations of behaviours. On the level of observations, they are used to define operational semantics; $\prec$ is then interpreted as the *earlier than* relation, and $\not\leftrightarrow$ as (potential) *simultaneity*. On the level of behaviour specifications, $\prec$ is usually interpreted as *causality*, and $\not\leftrightarrow$ as *independence*. The first relation $\prec_\mathcal{S}$ in an so-structure $\mathcal{S}$, should be interpreted as the standard *causality*, and the second relation, $\sqsubset_\mathcal{S}$, as a *weak causality*. While causality is an abstraction of the 'earlier than' relation, weak causality is a similar abstraction of the 'not later than' relation (this should be clearer if one looks at the formula (1) where $\prec_{po}$ represents the former, and $\prec_{po} \cup \not\leftrightarrow$ the latter relation). For a detailed discussion of so-structures the reader is referred to [7].

When used as a tool for representing concurrent behaviours, so-structures are derived from locally defined information involving events which directly interact with one another. This local information then needs to be combined into a global relationship involving all the event occurrences. For this a closure operation is applied which builds an so-structure from representative local relations. The $\diamondsuit$–closure of a relational structure was introduced in [8] to serve such a purpose.

Let $\mathcal{S} = (X, \prec, \sqsubset, l)$ be a labelled relational structure. The *$\diamondsuit$–closure* of $\mathcal{S}$ is the labelled relational structure $\mathcal{S}^{\diamondsuit} = (X, \prec_{\mathcal{S}^\diamond}, \sqsubset_{\mathcal{S}^\diamond}, l)$, where

$$\prec_{\mathcal{S}^\diamond} = (\prec \cup \sqsubset)^\star \circ \prec \circ (\prec \cup \sqsubset)^\star \quad \text{and} \quad \sqsubset_{\mathcal{S}^\diamond} = (\prec \cup \sqsubset)^\star \setminus id_X .$$

We also say that a labelled relational structure $\mathcal{S}$ is *$\diamondsuit$–acyclic* if $\prec_{\mathcal{S}^\diamond}$ is irreflexive. The property of $\prec_{\mathcal{S}^\diamond}$ being irreflexive, which holds when the structure $\mathcal{S}^{\diamondsuit}$ obtained from $\mathcal{S}$ is an so-structure, has a straightforward interpretation in operational terms. Basically, it means that in any single system history as described by $\mathcal{S}$, there are no event occurrences $e_1, e_2, \ldots, e_k$ such that each e_i has occurred *before or simultaneously with* e_{i+1}, while e_k has occurred *before* e_1.

Proposition 1. *[8] Let $\mathcal{S} = (X, \prec, \sqsubset, l)$ be a labelled relational structure.*

1. *$\mathcal{S}^{\diamondsuit}$ is a labelled so-structure if and only if $\prec_{\mathcal{S}^\diamond}$ is irreflexive.*
2. *If $\mathcal{S}$ is an so-structure, then $\mathcal{S}^{\diamondsuit} = \mathcal{S}$.* □

We now turn to the relationship between so-structures and stratified posets which resembles that between partial orders and their linear extensions.

A labelled stratified poset $po = (X_{po}, \prec_{po}, l_{po})$ is an *extension* of a labelled so-structure $\mathcal{S} = (X_{\mathcal{S}}, \prec_{\mathcal{S}}, \sqsubset_{\mathcal{S}}, l_{\mathcal{S}})$ if they have the same domain $X_{po} = X_{\mathcal{S}}$ and the same labelling $l_{po} = l_{\mathcal{S}}$, and moreover, $\prec_{\mathcal{S}} \subseteq \prec_{po}$ and $\sqsubset_{\mathcal{S}} \subseteq \prec_{po} \cup \not\leftrightarrow_{po}$. We denote this by $po \in strat(\mathcal{S})$. If $\mathcal{S} = (X, \prec, \sqsubset, l)$ is a labelled so-structure then we have [8]:

$$\mathcal{S} = \Big(X \,, \bigcap_{po \in strat(\mathcal{S})} \prec_{po} \,, \bigcap_{po \in strat(\mathcal{S})} (\prec_{po} \cup \not\leftrightarrow_{po}) \,, l \Big). \tag{1}$$

Thus $\mathcal{S}$ can be derived from its poset extensions. Recall that Szpilrajn's theorem [13] states that each poset is unambiguously identified by its linear extensions. A similar result does not hold for so-structures since these do not necessarily have total order extensions, e.g., $\mathcal{S} = (\{a, b\}, \emptyset, \{(a, b), (b, a)\})$. For them one needs to consider stratified poset extensions [9].
Again, we are not interested in the actual carriers of the labels in a poset and so in the sequel we will use the notation $strat(\mathcal{S})$ to denote the set of all isomorphic copies of the labelled stratified poset extensions of $\mathcal{S}$.

We say that two labelled relational structures, $\mathcal{S}_1 = (X_1, \prec_1, \sqsubset_1, l_1)$ and $\mathcal{S}_2 = (X_2, \prec_2, \sqsubset_2, l_2)$, are *isomorphic* if there is a bijection $f : X_1 \to X_2$ such that for all $x, y \in X_1$, $l_1(x) = l_2(f(x))$, and $x \prec_1 y$ if and only if $f(x) \prec_2 f(y)$, and $x \sqsubset_1 y$ if and only if $f(x) \sqsubset_2 f(y)$.

3 Place/Transition nets with inhibitor arcs

This section introduces the notation and terminology for P/T-nets with inhibitor arcs (PTI-nets, for short) and discusses their operationally defined a priori step sequence semantics. PTI-nets have an underlying structure consisting of a net augmented with inhibitor arcs.

A *net* is a triple $N = (S, T, F)$ such that S and T are disjoint finite sets, and $F \subseteq (T \times S) \cup (S \times T)$. The elements of S and T are respectively called *places* and *transitions*, and F is called the *flow relation*. We assume that, for every $t \in T$, $\{s \mid (s,t) \in F\} \neq \emptyset$ and $\{s \mid (t,s) \in F\} \neq \emptyset$ (nets are *T-restricted*).

An *inhibitor net* is a net together with a (possibly empty) set of *inhibitor arcs* leading from places to transitions. (In diagrams, inhibitor arcs have small circles as arrowheads.) Thus an inhibitor net N is specified as a tuple (S, T, F, I) such that (S, T, F) is a net (the underlying net of N) and $I \subseteq S \times T$ is its set of inhibitor arcs. A net (S, T, F) (without inhibitor arcs) is considered as a special instance of an inhibitor net and identified with the inhibitor net $(S, T, F, \emptyset)$.

Given an inhibitor net $N = (S, T, F, I)$ and $x \in S \cup T$, the *post-set* of x, denoted by $x^\bullet$, is defined by $x^\bullet = \{y \mid (x,y) \in F\}$ and the *pre-set* of x, denoted by ${}^\bullet x$, is defined by ${}^\bullet x = \{y \mid (y,x) \in F\}$. In addition, for all $t \in T$, ${}^\circ t = \{s \in S \mid (s,t) \in I\}$ denotes the set of *inhibiting places* of t. These notations are extended to multisets over $S \cup T$ in the following way: For a multiset $U : S \cup T \to \mathbf{N}$, $U^\bullet = \{y \mid \exists x \in U : (x,y) \in F\}$ and ${}^\bullet U = \{y \mid \exists x \in U : (y,x) \in F\}$; and for a multiset $U : T \to \mathbf{N}$, ${}^\circ U = \{s \in S \mid \exists t \in U : (s,t) \in I\}$.

A PTI-net is an inhibitor net equipped with an initial state. The states of an inhibitor net are given in the form of markings.

Let $N = (S, T, F, I)$ be an inhibitor net. A *marking* of N is a multiset of places. Following standard terminology, given a marking M of N and a place $s \in S$, we say that s is marked (under M) if $M(s) \geq 1$ and that $M(s)$ is the number of tokens in s under M.

Transitions represent actions which may occur at a given marking and then lead to a new marking. Here we define this dynamics in the more general terms of multisets of (concurrently occurring) transitions. A *step* is a multiset of transitions, $U : T \to \mathbf{N}$. It is *enabled* at a marking M if, for all $s \in S$:

$$M(s) \geq \sum_{t \in s^\bullet} U(t) \quad \text{and} \quad [s \in {}^\circ U \Longrightarrow M(s) = 0].$$

Thus, by the first condition, in order for U to be enabled at M, for each place s, the number of tokens in s under M should be at least equal to the total number of occurrences of transitions in U that have s as an input place. By the second condition, if a place s is an inhibiting place of some transition occurring in U, then s should be empty in M. Note that the enabledness of a step is based on an *a priori* condition: the inhibiting places of transitions occurring in that step should be empty before it occurs.[1]

[1] In the *a posteriori* approach [3], the second condition for enabledness is strengthened: for all $s \in S$, $[s \in {}^\circ U \Longrightarrow (M(s) = 0 \wedge s \notin U^\bullet)]$. Thus no inhibiting place of a transition in U is also an output place of any transition occurring in U.

If U is enabled at M, then it can be *executed*, which leads to the marking M' defined, for all $s \in S$, by:

$$M'(s) = M(s) - \sum_{t \in s^\bullet} U(t) + \sum_{t \in {}^\bullet s} U(t).$$

This means that the execution of U 'consumes' from each place s a token for each occurrence of a transition in U that has s as an input place, and 'produces' in each place s a token for each occurrence of a transition in U with s as an output place. If the execution of U leads from M to M' we write $M[U\rangle M'$. Note that the empty step $\underline{0}$ is enabled at every marking of N and that its execution has no effect: $M[\underline{0}\rangle M$ for all markings M of N.

A *step sequence* from a marking M to marking M' is a sequence $U_1 \ldots U_n$ of non-empty steps U_i, $1 \leq i \leq n$ with $n \geq 0$, such that

$$M = M_0 \, [U_1\rangle \, M_1 \, [U_2\rangle M_2 \, \cdots \, M_{n-1} \, [U_n\rangle \, M_n = M'$$

for some markings $M_1, \ldots, M_{n-1}$ of N. If τ is a step sequence from M to M' we write $M \, [\tau\rangle \, M'$ and M' is said to be *reachable* (in N) from M. Note that every marking is reachable from itself by the empty step sequence.
In case we want to make clear which (inhibitor) net we are dealing with, we may add a subscript N and write $[\cdot\rangle_N$ rather than $[\cdot\rangle$.

A *Place/Transition net with inhibitor arcs* (or PTI-net) is a tuple $N = (S, T, F, I, M_0)$, where $N' = (S, T, F, I)$ is its underlying inhibitor net, and M_0 is a marking of (S, T, F, I).[2] A *step sequence* of $N = (S, T, F, I, M_0)$ is a step sequence starting from M_0 in its underlying inhibitor net N'. The set of all step sequences of N is the set $steps(N) = \{\tau \mid \exists M : M_0[\tau\rangle_{N'} M\}$.

As the last point of this section, we look at the boundedness of places in N. A place $s \in S$ is *n-bounded* in N, where n is a positive integer, if $M(s) \leq n$ for every marking M reachable from M_0; it is *bounded* if it is n-bounded for some n, otherwise it is *unbounded*. N is *safe* if all of its places are 1-bounded. If s_1 is a bounded place of N, then $s_2 \in S$ is a *complement* place of s_1, if ${}^\bullet s_1 = s_2{}^\bullet$ and $s_1{}^\bullet = {}^\bullet s_2$. Then $bound(s_1) = M_0(s_1) + M_0(s_2)$ is a bound for both s_1 and s_2, and $bound(s_1) = M(s_1) + M(s_2)$, for every marking M reachable from M_0.

We call N a *PTBI-net* if all inhibiting places of all its transitions are bounded.

4 Processes

4.1 Occurrence nets

For safe P/T-nets and elementary net systems, *processes* can be used as a non-sequential representation of runs of the net (see, e.g., [2, 11, 14]). Processes are based on occurrence nets and may be viewed as (partial) acyclic unfoldings of the net. Each transition represents an occurrence of a transition in the original

[2] Note that I may be empty, in which case we are actually dealing with a P/T-net, and then N may also be specified in the form (S, T, F, M_0).

net, while each place corresponds to a token. Conflicts between transitions are resolved and thus places do not branch. An occurrence net defines a partial order on its transitions which in turn provides a partial order description of transition occurrences in the original net.

Definition 1. *A* (labelled) occurrence net *is a labelled net* $ON = (B, E, R, l)$ *such that:* $|{}^\bullet b| \leq 1 \geq |b^\bullet|$, *for every* $b \in B$*; the relation* $(R \circ R)|_{E \times E}$ *is acyclic; and* l *is a labelling function for* $B \cup E$*. The elements of* B *and* E *— the places and transitions of* ON *— are respectively called* conditions *and* events. □

The *minimal* and *maximal* conditions of ON are respectively $Min(ON) = \{b \in B \mid {}^\bullet b = \emptyset\}$ and $Max(ON) = \{b \in B \mid b^\bullet = \emptyset\}$. ON defines a set of step sequences which start from an implicit marking formed by $Min(ON)$ and lead to $Max(ON)$. (Note that the steps in these sequences are sets and that ON with initial marking $Min(ON)$ is safe.) Applying the labelling l to such step sequences yields the set $lsteps(ON) = \{l(\sigma) \mid Min(ON)[\sigma\rangle_{ON} Max(ON)\}$.
Since $(R \circ R)|_{E \times E}$ is acyclic, its transitive closure $\prec_{ON} = ((R \circ R)|_{E \times E})^+$ is irreflexive and we can associate with ON a labelled poset $po_{ON} = (E, \prec_{ON}, l|_E)$.

For EN-systems [14], the notion of occurrence nets provides a causality (partial order) semantics which can be defined in two different ways: (i) axiomatic, from the structure of the net; and (ii) operational, through unfolding based on step sequences. In both cases, the processes and hence also the associated partial orders are the same.

The above approach is not directly applicable to non-safe nets. For these, [6] and [1] propose to represent each of the multiple tokens in a place by a separate condition of an occurrence net. We now provide a rephrasing of the definitions of [1] for the case of general (possibly non-safe) finite P/T-nets.

Definition 2. *Let* $N = (S, T, F, M_0)$ *be a P/T-net. A* process of N *is an occurrence net* $ON = (B, E, R, l)$ *such that the following conditions are satisfied:*

1. $l : B \cup E \to S \cup T$ *is such that* $l(B) \subseteq S$ *and* $l(E) \subseteq T$.
2. *For all* $s \in S$: $M_0(s) = |Min(ON) \cap l^{-1}(s)|$.
3. *For all* $s \in S$ *and* $e \in E$:
   (a) $|\{s\} \cap {}^\bullet l(e)| = |\{b \in l^{-1}(s) \mid (b, e) \in R\}|$
   (b) $|\{s\} \cap l(e)^\bullet| = |\{b \in l^{-1}(s) \mid (e, b) \in R\}|$.

We will use $on(N)$ *to denote the set of all processes of* N. □

The above is the axiomatic definition. Alternatively, we can start from a step sequence and construct a corresponding process.

Definition 3. *Let* $N = (S, T, F, M_0)$ *be a P/T-net and let* $\tau = U_1 \ldots U_n$ *be a step sequence of* N*. A* process generated by τ *is the last labelled net* N_n *in a series* $N_0, \ldots, N_n$ *with* $N_k = (B_k, E_k, R_k, l_k)$, *for* $0 \leq k \leq n$, *constructed thus.*

- *Step 0:* $N_0 = (B_0, E_0, R_0, l_0)$ *where*
  - $E_0 = R_0 = \emptyset$ *and* $B_0 = \{b_{s,i,0} \mid 1 \leq i \leq M_0(s)\}$.

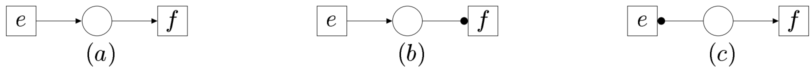

Fig. 2. (a,b) Two cases defining $e \prec_{aux} f$, and (c) one case defining $e \sqsubset_{aux} f$.

- $l_0 : B_0 \to S$ *is such that* $l(b_{s,i,0}) = s$, *for all* $b_{s,i,0} \in B_0$.

Let $Max_0 = B_0$.

– *Step* $m = k+1$*: Let* $N_k = (B_k, E_k, R_k, l_k)$. *Then* N_m *is defined thus:*

- $B_m = B_k \cup \{b_{s,t,i,m} \mid 1 \le i \le U_m(t) \wedge s \in t^\bullet\}$.
- $E_m = E_k \cup \{e_{t,i,m} \mid 1 \le i \le U_m(t)\}$. *Moreover, for each* $e_{t,i,m} \in E_m$ *and each* $s \in {}^\bullet t$ *we* choose[3] *a distinct* $\widehat{b}_{\langle s,t,i,m\rangle} \in Max_k \cap l^{-1}(s)$.
- $R_m = R_k \cup \left(\begin{array}{l} \{(\widehat{b}_{\langle s,t,i,m\rangle}, e_{t,i,m}) \mid e_{t,i,m} \in E_m \wedge s \in {}^\bullet t\} \cup \\ \{(e_{t,i,m}, b_{s,t,i,m}) \quad \mid e_{t,i,m} \in E_m \wedge s \in t^\bullet\} \end{array} \right)$.
- $l_m(b_{s,t,i,m}) = s$ *and* $l_m(e_{t,i,m}) = t$, *for all* $b_{s,t,i,m} \in B_m \setminus B_k$ *and* $e_{t,i,m} \in E_m \setminus E_k$. *Moreover,* $l_m(x) = l_k(x)$, *for all* $x \in B_k \cup E_k$.

Let $Max_m = \{b \in B_m \mid \neg\exists e \in E_m : (b,e) \in R_m\}$.

We will use $proc_\tau$ *to denote the set of all isomorphic copies*[4] *of all processes generated by* τ. □

4.2 Activator occurrence nets

The presence of inhibitor arcs makes the unfolding procedure more complicated, due to the fact that local information regarding the lack of tokens in a place cannot be explicitly represented in an occurrence net. In [8] this problem is solved by using complement places and representing inhibitor arcs by activator arcs connected to conditions representing complement places. The notion of an occurrence net is replaced by that of an activator occurrence net.

Definition 4. *A* (labelled) activator occurrence net *(ao-net) is a tuple* $AON = (B, E, R, Act, l)$ *such that:* $ON = (B, E, R, l)$ *is an occurrence net;* $Act \subseteq B \times E$ *are* activators *arcs; and the relational structure* $\mathcal{S}_{aux}(AON) = (E, \prec_{aux}, \sqsubset_{aux}) = (E, (R \circ R)|_{E \times E} \cup (R \circ Act), (Act^{-1} \circ R) \setminus id_E)$ *is* $\diamondsuit$*–acyclic.* □

In the diagrams, activator arcs have black dots as arrowheads; see, e.g., figure 4 where (b_2, e) is an activator arc. Figure 2 shows how $\prec_{aux}$ and $\sqsubset_{aux}$ are constructed from ordinary arcs and activator arcs.

Notice that the $\diamondsuit$-acyclicity of $\mathcal{S}_{aux}(AON)$ implies that $(R \circ R)|_{E\times E}$ is acyclic in the usual sense. Since $\mathcal{S}_{aux}(AON)$ is $\diamondsuit$–acyclic, we can associate with AON the labelled so-structure $\mathcal{S}(AON) = \mathcal{S}_{aux}(AON)^{\diamondsuit}$, see proposition 1. Figure 3 shows the labelled so-structures $\mathcal{S}(AON_i)$ for the ao-nets AON_i in figure 5.

[3] This is the only difference with the safe case, where there is only one candidate condition $\widehat{b}_{\langle s,t,i,m\rangle}$, and so the process associated with τ is unique (up to isomorphism).

[4] The construction of a process from step sequences in this and the next sections is based on concrete nodes which carry the labels. This provides us immediately with a fully specified representative of an isomorphism class which is both intuitive and useful in proofs.

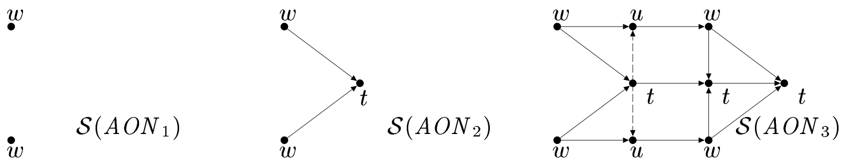

Fig. 3. Stratified order structures generated by ao-nets in figure 5.

Intuitively, an activator arc between a condition b and an event e means that the occurrence of e requires the holding of b, but the occurrence of e will not make b cease to hold. Formally, a step U of events is enabled at a marking M of AON if U is enabled in the underlying occurrence net ON at marking M and, furthermore, for all e in U and $b \in B$, $(b, e) \in Act$ implies that b is marked in M. The resulting marking M' is the same as the marking resulting from the execution of U in ON.[5] As before, we will write $M[\sigma\rangle_{AON} M'$ if executing a step sequence σ in AON leads from M to M'.

The *minimal* and *maximal* conditions of AON are respectively $Min(AON) = Min(ON)(= Min)$ and $Max(AON) = Max(ON)(= Max)$. The step sequences and the reachable markings of AON from the marking Min are also step sequences and reachable markings of ON with initial marking Min. Thus, in particular, (AON, Min) is safe, since (ON, Min) always is. As for occurrence nets, we consider those step sequences which lead from the minimal conditions to the maximal conditions. Applying the labelling l to such step sequences yields the set $lsteps(AON) = \{l(\sigma) \mid Min[\sigma\rangle_{AON} Max\}$. The following result states the correspondence between the (labelled) step sequences of an ao-net AON and the stratified extensions of its associated labelled so-structure $\mathcal{S}(AON)$.

Theorem 1. *$strat(\mathcal{S}(AON)) = \{po_\sigma \mid \sigma \in lsteps(AON)\}$.*

Proof. Let AON and ON be as in definition 4, and Min and Max be (safe) markings as above. It suffices to show the result assuming that l is the identity labelling for E.

Suppose that $Min[\sigma\rangle_{AON} Max$ and $\sigma = E_1 \dots E_n$. Then also $Min[\sigma\rangle_{ON} Max$. Thus, due to the standard properties of occurrence nets, each E_i is a set and E is the disjoint union of $E_1, \dots, E_n$. Moreover, there are sets of conditions $B_0, \dots, B_k$ of B (cuts of ON, see [1]) such that

$$Min = B_0[E_1\rangle_{ON} B_1 \dots B_{n-1}[E_n\rangle_{ON} B_n = Max \tag{2}$$

and, for every $b \in B$, there are $0 \le k_b \le l_b \le n$ such that

$$b \in B_i \text{ if and only if } k_b \le i \le l_b. \tag{3}$$

In the above, k_b is the index of the first cut B_i in the sequence $B_0, \dots, B_n$ in which condition b is marked, and l_b is the index of last such cut. Clearly,

$$Min = B_0[E_1\rangle_{AON} B_1 \dots B_{n-1}[E_n\rangle_{AON} B_n = Max \tag{4}$$

[5] Thus an activator arc does not interfere with normal arcs, unlike *read arcs*, [15, 3].

also holds. To show that $po_\sigma \in strat(\mathcal{S}(AON))$, it suffices to prove that if $e \in E_i$ and $f \in E_j$ then:

$$(\exists b \in B : (e,b) \in R \ \wedge \ (b,f) \in R \cup Act) \ \Rightarrow \ i < j. \tag{5}$$

$$(\exists b \in B : (b,e) \in Act \ \wedge \ (b,f) \in R) \ \Rightarrow \ i \leq j. \tag{6}$$

From (2,3,4) and $E = E_1 \uplus \ldots \uplus E_n$ and $|{}^\bullet b| \leq 1 \geq |b^\bullet|$ it follows that: $(e,b) \in R \Rightarrow i = k_b$; $(b,e) \in R \Rightarrow i{-}1 = l_b$; and $(b,e) \in Act \Rightarrow k_b \leq i{-}1 \leq l_b$. Thus (5) holds since $(e,b) \in R \wedge (b,f) \in R \cup Act$ implies $i = k_b$ and $l_b = j{-}1 \vee k_b \leq j{-}1$. And (6) holds since $(b,e) \in Act \wedge (b,f) \in R$ implies $i{-}1 \leq l_b = j{-}1$.

We have shown the ($\supseteq$) inclusion. To prove the reverse one, suppose that $po_\sigma \in strat(\mathcal{S}(AON))$ and $\sigma = E_1 \ldots E_n$ which means that $E = E_1 \uplus \ldots \uplus E_n$ and (5,6) hold. From (5) (without the $(b,f) \in Act$ part), it follows that $Min[\sigma\rangle_{ON} Max$. Hence there are $B_0, \ldots, B_n$ such that (2,3) hold. To show that $Min[\sigma\rangle_{AON} Max$ also holds, it suffices to observe that if $e \in E_i$ and $(b,e) \in Act$ then $b \in B_{i-1}$. Indeed, if this was not true, then $l_b < i{-}1$ or $k_b \geq i$. In the former case, there is $f \in E_{l_b+1}$ such that $(b,f) \in R$, a contradiction with (6). And, in the latter case, there is $f \in E_{k_b}$ such that $(f,b) \in R$, a contradiction with (5). □

The labelled step sequences of AON have a causality interpretation in terms of the partial order and the weak partial order provided by $\mathcal{S}(AON)$. In fact, a single partial order (as defined by an occurrence net) is insufficient, as it cannot fully express the relationship between simultaneous events (in a step) if they cannot be sequentialized. For example, in figure 4 we have that $\sigma_1 = \{e,f\}$ and $\sigma_2 = \{e\}\{f\}$ are step sequences leading from Min to Max, but $\{f\}\{e\}$ cannot be executed, despite the fact that e and f are independent as far as the usual partial ordering is concerned.

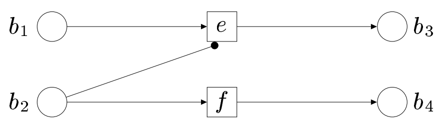

Fig. 4. An activator occurrence net where $Min = \{b_1, b_2\}$ and $Max = \{b_3, b_4\}$.

In the next section, we will combine the approaches of [1] and [8] in order to obtain a causal semantics for PTI-nets in case the inhibiting places have known bounds. The treatment of unbounded inhibiting places will require a further extension of occurrence nets.

5 The bounded case

In this section $N = (S,T,F,M_0,I)$ is a fixed PTBI-net and $N' = (S,T,F,M_0)$ is its underlying P/T-net. We assume here that every inhibiting place $s \in S$ has a unique complement place $s^{cpl} \in S$ with $M_0(s) + M_0(s^{cpl}) = bound(s)$ where $bound(s) > 0$ is a bound of s in N. The processes of N are defined as follows.

Definition 5. *An* activator process of N *is an ao-net* $AON = (B, E, R, Act, l)$ *such that* $ON = (B, E, R, l) \in on(N')$ *and, for all* $s \in S$ *and* $e \in E$*:*

$$|\{s\} \cap {}^{\circ}l(e)| \cdot bound(s) = |\{b \in l^{-1}(s^{cpl}) \mid (b, e) \in Act\}|. \tag{7}$$

We will use $aon(N)$ *to denote the set of all activator processes of* N*.* □

Figure 5 shows an example of a PTBI-net and its three activator processes.

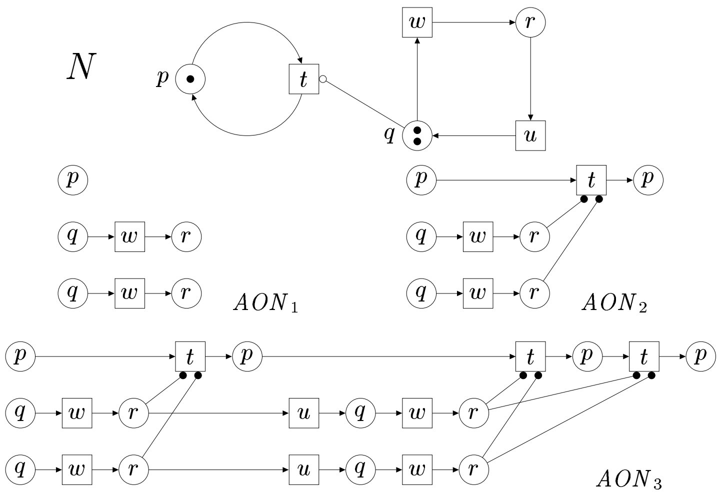

Fig. 5. Three activator processes AON_i of a PTBI-net N.

The first result we show states that an activator process of a P/T-net describes a set of valid step sequences of the original net.

Lemma 1. *If* $AON \in aon(N)$*, then* $lsteps(AON) \subseteq steps(N)$*.*

Proof. Let AON and ON be as in definition 5, and $\tau \in lsteps(AON)$. Then, by theorem 1, there is a step sequence $\sigma = E_1 \dots E_n$ such that $\tau = l(\sigma)$ and $Min(AON)[\sigma\rangle_{AON} Max(AON)$ and $E = E_1 \uplus \dots \uplus E_n$ and (5,6) in the proof of theorem 1 hold. Since $ON \in on(N')$, we have, by the standard theory [1], that $\tau \in steps(N')$. Moreover, there are sets of conditions $B_0, \dots, B_n$ such that (2,3) in the proof of theorem 1 hold and:

$$M_0 = l(B_0)[l(E_1)\rangle_{N'} l(B_1) \dots [l(E_n)\rangle_{N'} l(B_n)$$

Thus, to prove $\tau \in steps(N)$, it suffices to show that if $e \in E_i$ and $s \in {}^{\circ}l(e)$, then $l(B_{i-1})(s) = 0$. The latter is equivalent to $l(B_{i-1})(s^{cpl}) = bound(s)$. If this does not hold then, by (7), there is $b \in B$ such that $(b, e) \in Act$ and $l_b < i - 1$ or $i \leq k_b$. We then obtain a contradiction with (5,6), similarly as in the last part of the proof of theorem 1. □

Definition 5 can be made operational through the following net unfolding which takes a step sequence and constructs an ao-net corresponding to it.

Definition 6. *Let* $\tau = U_1 \ldots U_n$ *be a step sequence of* N*. An* activator process *generated by* τ *is the last labelled net* N_n *with activator arcs in a series* $N_0, \ldots, N_n$ *with* $N_k = (B_k, E_k, R_k, Act_k, l_k)$*, for* $0 \leq k \leq n$*, constructed thus.*

- *Step 0:* $N_0 = (B_0, E_0, R_0, Act_0, l_0)$ *where* $Act_0 = \emptyset$*, and all other components are as in Step 0 of definition 3, including* Max_0*.*
- *Step* $m = k+1$*: Let* $N_k = (B_k, E_k, R_k, Act_k, l_k)$*. Then* N_m *is defined thus:*
  - B_m, E_m, R_m*,* l_m *and* Max_m *are as in Step m of definition 3.*
  - $Act_m = Act_k \cup \{(b,e) \in Max_k \times (E_m \setminus E_k) \mid (l_m(b)^{cpl}, l(e)) \in I\}$.

We will use $proc^{ao}_\tau$ *to denote the set of all isomorphic copies of all activator processes generated by* τ*.* □

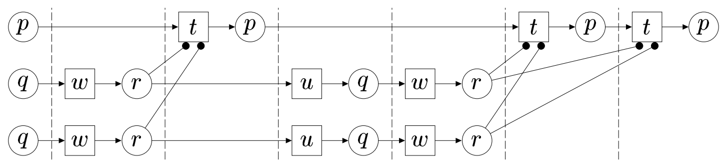

Fig. 6. Deriving an activator process in $proc^{ao}_\tau$ for $\tau = \{w, w\}\{t\}\{u, u\}\{w, w\}\{t\}\{t\}$.

Figure 6 illustrates the construction of an activator process for the PTBI-net in figure 5. The vertical lines indicate the stages (from left to right) in which the net has been derived. Notice that it is an activator process of N in figure 5 as it is isomorphic to AON_3 shown there. The next results states that this is not a mere chance, since every unfolding of a PTBI-net satisfies the axiomatic definition of an activator process.

Lemma 2. *For* τ *and* N_n *in definition 6,* $N_n \in aon(N)$ *and* $\tau \in lsteps(N_n)$*.*

Proof. Assume the notation from definition 6. That $ON = (B_n, E_n, R_n, l_n) \in proc_\tau$ for N' follows directly from the definitions and thus, by the standard results for P/T-nets [1], $ON \in on(N')$. Moreover, the construction is such that, for $k = 1, \ldots, n$, $M_0 = l(Max_0)[U_1 \ldots U_k\rangle_{N'} l(Max_k)$ and so also $M_0 = l(Max_0)[U_1 \ldots U_k\rangle_N l(Max_k)$. Thus, if $e \in E_k \setminus E_{k-1}$ and $s \in {}^\circ l(e)$, then we have $l(Max_{k-1})(s) = 0$ and so $l(Max_{k-1})(s^{cpl}) = bound(s)$. Hence

$$|\{b \in l^{-1}(s^{cpl}) \mid (b,e) \in Act_k\}| = |\{b \in Max_{k-1} \mid l(b) = s^{cpl}\}| = bound(s).$$

As a result, (7) is satisfied. To complete the proof of $N_n \in aon(N)$, we still need to show that $\mathcal{S}_{aux}(N_n)$ is $\diamondsuit$-acyclic. This, however, follows from an easy observation that the conditions (5,6) from the proof of theorem 1 (suitably reinterpreted by setting each E_i to be the set of events added in step i of the construction described in definition 6), hold here by construction.

That $\tau \in \mathit{lsteps}(N_n)$ follows immediately from the construction of N_n and a simple inductive argument. □

Corollary 1. *If* $\tau \in \mathit{steps}(N)$ *and* $AON \in \mathit{proc}^{ao}_{\tau}$*, then* $\tau \in \mathit{lsteps}(AON)$. □

Similarly as it is the case for processes of ordinary P/T-nets, the axiomatic and operational definitions of processes of a PTBI-net coincide.

Theorem 2. $\mathit{aon}(N) = \bigcup_{\tau \in \mathit{steps}(N)} \mathit{proc}^{ao}_{\tau}$.

Proof. The ($\supseteq$) inclusion follows from lemma 2. To show the reverse one, we take AON and ON as in definition 5. Then, by $\mathit{strat}(\mathcal{S}(AON)) \neq \emptyset$ which always holds [7], there is at least one τ such that $\mathit{po}_{\tau} \in \mathit{strat}(\mathcal{S}(AON))$. By lemma 1 and theorem 1, $\tau \in \mathit{steps}(N)$ and so $\tau \in \mathit{steps}(N')$. Thus, by the standard properties of processes of P/T-nets, there is a way in which the construction described in definition 3 generates a net $N_n = (B_n, E_n, R_n, l_n)$ which is isomorphic to ON. One can then re-run the construction of ON, adding at each stage the sets Act_k, as prescribed in definition 6. This is a deterministic procedure which results in an activator net which is isomorphic to AON. In proving the latter, one takes advantage of theorem 1, which guarantees that $\tau \in \mathit{lsteps}(AON)$. □

We now can establish that activator processes of a PTBI-net generate exactly the same step sequences as the original net.

Theorem 3. $\mathit{steps}(N) = \bigcup_{AON \in \mathit{aon}(N)} \mathit{lsteps}(AON)$.

Proof. The ($\supseteq$) inclusion has been proved in lemma 1. The reverse inclusion follows from corollary 1 and theorem 2. □

The last result can be re-stated in terms of labelled stratified posets and thus shows that the activator processes of a PTBI-net correctly describe causality in the runs of the net.

Corollary 2. $\{\mathit{po}_{\tau} \mid \tau \in \mathit{steps}(N)\} = \bigcup_{AON \in \mathit{aon}(N)} \mathit{strat}(\mathcal{S}(AON))$.

Proof. Follows from theorems 1 and 3. □

6 Unboundedness

In this section, we deal with PTI-nets whose inhibiting places can be unbounded. Thus we cannot use complement places to represent the emptiness of places, and therefore need to introduce another device. It will be provided by z-places that will play a role similar to that of the complement places in activator process. However, z-places will represent *logical conditions* rather than tokens (*resources*), and will admit branching. Let $N = (S, T, F, M_0, I)$ be a PTI-net fixed throughout this section.

Definition 7. *A (labelled)* z-activator occurrence net *(zao-net) is a tuple* $AON^z = (B, E, R, Act, l)$ *such that:* $ON = (B^n, E, R', l|_{B^n \cup E})$ *is an occurrence net, where* $B^n = B \setminus B^z$ *and* $B^z = \{b \in B \mid (b, e) \in Act\}$*;* $R \subseteq (B \times E) \cup (E \times B)$ *and* $R' = R|_{(B^n \times E) \cup (E \times B^n)}$*;* $Act \subseteq B^z \times E$ *is a set of activator arcs;* l *is a labelling function for* $B \cup E$*; and the relational structure* $\mathcal{S}_{aux}(AON^z) = (E, \prec_{aux}, \sqsubset_{aux}) = (E, (R \circ R)|_{E \times E} \cup (R \circ Act), (Act^{-1} \circ R) \setminus id_E)$ *is* $\diamondsuit$*-acyclic.* □

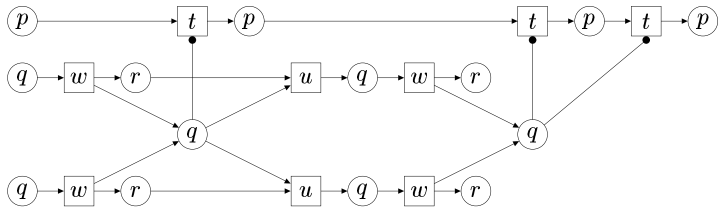

Fig. 7. A z-activator occurrence net.

Figure 7 shows an example of a zao-net. The semantics of a zao-net can be understood in two ways. First, we can take the underlying order structure $\mathcal{S}(AON^z) = \mathcal{S}_{aux}(AON^z)^{\diamondsuit}$, as we did for ao-nets, and derive all stratified order extensions, or step sequences corresponding to these. The alternative view, involving step sequences executed from the initial marking, $Min(AON^z) = Min(ON)$, to the final marking, $Max(AON^z) = Max(ON)$, is not directly applicable since z-conditions allow branching. However, it is possible to replace the z-conditions by sets of ordinary conditions for each pair of pre- and post-event of a given z-condition, as described below.

Definition 8. *Let* $AON^z = (B, E, R, Act, l)$ *be as in definition 7, and:*

- $B' = B^n \cup B''$ *where*
  $B'' = \{b_{x,y} \mid b \in B^z \wedge (x \in {}^\bullet b \vee x = \emptyset = {}^\bullet b) \wedge (y \in b^\bullet \vee y = \emptyset = b^\bullet)\}$.
- $R' = R|_{(B^n \times E) \cup (E \times B^n)} \cup \{(e, b_{e,y}) \mid e \in E\} \cup \{(b_{x,e}, e) \mid e \in E\}$.
- $Act' = \{(b_{x,y}, e) \mid (b, e) \in Act\}$.
- $l'|_{B^n \cup E} = l|_{B^n \cup E}$ *and* $l'(b_{x,y}) = l(b)$*, for all* $b \in B''$.

We then call $\zeta(AON^z) = (B', E, R', Act', l')$ *the* z-pruning *of* AON^z. □

It is not difficult to see that the z-pruning of AON^z is an ao-net. It is used to give the activator arcs an operational semantics which corresponds to the intuition behind the z-conditions. We define the (labelled) step sequences of AON^z by $lsteps(AON^z) = lsteps(\zeta(AON^z))$. Observe that $\mathcal{S}(\zeta(AON^z)) = \mathcal{S}(AON^z)$. Consequently, $strat(\mathcal{S}(AON^z)) = \{po_\sigma \mid \sigma \in lsteps(AON^z)\}$, by theorem 1. We now give an axiomatisation of the notion of process for the PTI-net N.

Definition 9. *A* z-activator process of N *is a zao-net* $AON^z = (B, E, R, Act, l)$ *such that:*

1. $l : B \cup E \to S \cup T$ *is such that* $l(B) \subseteq S$ *and* $l(E) \subseteq T$.
2. *For all* $s \in S$*:* $M_0(s) = |Min(AON^z) \cap l^{-1}(s) \cap B^n|$.
3. *For all* $s \in S$ *and* $e \in E$*:*
   (a) $|\{s\} \cap {}^\bullet l(e)| = |\{b \in l^{-1}(s) \cap B^n \mid (b, e) \in R\}|$.
   (b) $|\{s\} \cap l(e)^\bullet| = |\{b \in l^{-1}(s) \cap B^n \mid (e, b) \in R\}|$.
   (c) $|\{s\} \cap {}^\circ l(e)| = |\{b \in l^{-1}(s) \mid (b, e) \in Act\}|$.
4. *For all* $b^z \in B^z$ *and* $e \in E$*:*
   (a) *If* $(b^z, e) \in R$*, then* $(l(e), l(b^z)) \in F$.
   (b) *If* $(b^z, e) \in R^*$ *and* $(l(e), l(b^z)) \in F$*, then there is a unique* $b \in B^z$ *such that* $l(b) = l(b^z)$ *and* $(b, e) \in R$ *and* $(b^z, b) \in R^*$.
   (c) *If* $(e, b^z) \in R$*, then* $(l(b^z), l(e)) \in F$.
   (d) *If* $(e, b^z) \in R^*$ *and* $(l(b^z), l(e)) \in F$*, then there is a unique* $b \in B^z$ *such that* $l(b) = l(b^z)$ *and* $(e, b) \in R$ *and* $(b, b^z) \in R^*$.
5. *For all* $b^z \in B^z$ *and* $b \in B$*, if* $l(b) = l(b^z)$*, then* $(b^z, b) \in R^*$ *or* $(b, b^z) \in R^*$.

We will use $aon^z(N)$ *to denote the set of z-activator processes of* N. □

Note the absence of place bounds in the above definition. Instead, we have an explicit 'record' of the fact that a place was empty in the form of a z-condition. By points 4(a) and 4(c) above, if a z-condition b^z is input (output) to an event e, then the inhibiting place $l(b^z)$ of N is output (input) to the transition $l(e)$. Requirement 4(b) prescribes that whenever transition $l(e)$ adds a token to the inhibiting place $l(b^z)$, only the most recent record b of $l(b^z)$ being empty in the past of the occurrence e of $l(e)$ is input to e. Similarly, 4(d) stipulates that whenever transition $l(e)$ removes a token from the inhibiting place $l(b^z)$, while sometime in the future of this occurrence $l(b^z)$ is successfully tested for emptiness, the occurrence e of $l(e)$ is only connected to the earliest future record b of $l(b^z)$ being empty. Note that by definition 9(5), all records of the emptiness of an inhibiting place are linearly ordered by R^*. Moreover, according to R^* an inhibiting place is never recorded to be empty while it contains a token.

Figure 7 shows a z-activator processes for the net shown in figure 5. It corresponds to AON_3 in figure 5 in the sense that they generate isomorphic labelled so-structures. The last definition is also illustrated for a non-PTBI-net, in figure 8. We finally define an unfolding procedure for PTI-nets.

Definition 10. *Let* $\tau = U_1 \dots U_n$ *be a step sequence of* N*. A* z-activator process generated by τ *is the last labelled net* N_n *with activator arcs in a series* $N_0, \dots, N_n$ *with* $N_k = (B_k, E_k, R_k, Act_k, l_k)$*, for* $0 \le k \le n$*, constructed thus:*

- *Step 0:* $N_0 = (B_0, E_0, R_0, Act_0, l_0)$ *where:*
  - $E_0 = R_0 = Act_0 = B_0^z = \emptyset$.
  - $B_0 = B_0^n = \{b_{s,i,0} \mid 1 \le i \le M_0(s)\}$.
  - $l_0 : B_0 \to S$ *is such that* $l(b_{s,i,0}) = s$*, for all* $b_{s,i,0} \in B_0$.

  Let $Max_0 = B_0$.
- *Step* $m = k + 1$*: Let* $N_k = (B_k, E_k, R_k, Act_k, l_k)$*. Then* N_m *is defined thus:*
  - $B_m^n = B_k^n \cup \{b_{s,t,i,m} \mid 1 \le i \le U_m(t) \wedge s \in t^\bullet\}$ *and*
    $B_m^z = B_k^z \cup \{b_{s,m} \mid \exists t \in U_m : s \in {}^\circ t \setminus l_k(Max_k \cap B_k^z)\}$.

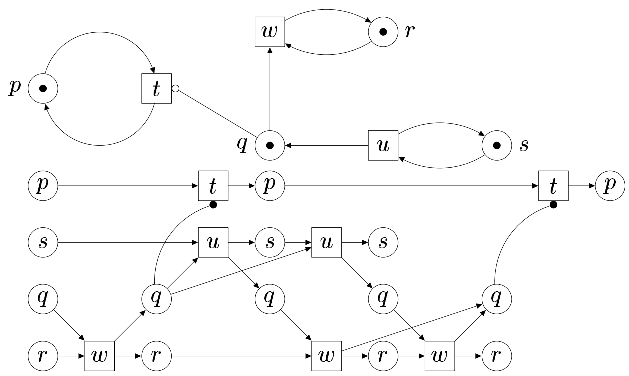

Fig. 8. A z-activator process of a PTI-net which is not a PTBI-net.

- $E_m = E_k \cup \{e_{t,i,m} \mid 1 \leq i \leq U_m(t)\}$.
  Moreover, for each $e_{t,i,m} \in E_m$ *and for each* $s \in {}^\bullet t$ *we choose a distinct* $\widehat{b}_{\langle s,t,i,m\rangle} \in Max_k \cap B^n_k \cap l^{-1}(s)$.
- $l_m(b_{s,t,i,m}) = s$ *and* $l_m(b_{s,m}) = s$ *and* $l_m(e_{t,i,m}) = t$,
  for all $b_{s,t,i,m} \in B^n_m \setminus B^n_k$ *and* $b_{s,m} \in B^z_m \setminus B^z_k$ *and* $e_{t,i,m} \in E_m \setminus E_k$.
  $l_m(x) = l_k(x)$, *for all* $x \in B_k \cup E_k$.
- $R_m = R_k \cup \begin{pmatrix} \{(\widehat{b}_{\langle s,t,i,m\rangle}, e_{t,i,m}) \mid e_{t,i,m} \in E_m \wedge s \in {}^\bullet t\} \cup \\ \{(e_{t,i,m}, b_{s,t,i,m}) \quad \mid e_{t,i,m} \in E_m \wedge s \in t^\bullet\} \end{pmatrix} \cup R'_m \cup R''_m$
  where
  $$R'_m = \left\{ (e, b_{s,m}) \in E_k \times (B^z_m \setminus B^z_k) \,\middle|\, \begin{array}{c} (s, l_k(e)) \in F \wedge \neg\exists b' \in B^z_k : \\ l_k(b') = s \;\wedge\; (e, b') \in R_k \end{array} \right\}$$
  $$R''_m = \left\{ (b_{s,i}, e) \in B^z_m \times (E_m \setminus E_k) \,\middle|\, \begin{array}{c} (l_m(e), s) \in F \wedge \\ \forall b_{s,j} \in B^z_m : j \leq i \end{array} \right\}.$$
- $Act_m = Act_k \cup \{(b,e) \in (Max_m \cap B^z_m) \times (E_m \setminus E_k) \mid (l_m(b), l_m(e)) \in I\}$,
  where $Max_m = \{b \in B_m \mid \neg\exists e \in E_m : (b,e) \in R_m\}$.

We will use $proc^{zao}_\tau$ *to denote the set of all isomorphic copies of all z-activator processes generated by* τ. □

The above definition is illustrated for the PTBI-net of figure 5 and its step sequence $\tau = \{w, w\}\{t\}\{u, u\}\{w, w\}\{t\}\{t\}$. As before, figure 9 shows stages in which the nodes and connections were generated.

The z-conditions are generated 'on-demand', when it is necessary to 'legitimise' transition occurrences. In general, this excludes undesirable orderings between events. For consider the net N in figure 5 and its step sequence $\tau = \{w, w\}\{u, u\}$. If we were to add a z-condition each time q becomes empty, then we would generate an occurrence net as shown in figure 10. Intuitively, such a net would introduce artificial causal relationships between some of the event occurrences.

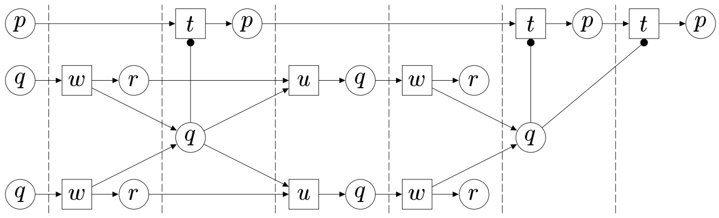

Fig. 9. A z-activator process generated for a step sequence of a PTBI-net.

As in the case of processes of P/T-nets and PTBI-nets, the axiomatic and operational definitions coincide. Moreover, we have the desired consistency between step sequences of a PTI-net and its zao-processes.

Theorem 4. *The following are satisfied.*

1. $aon^z(N) = \bigcup_{\tau \in steps(N)} proc_\tau^{zao}$.
2. $steps(N) = \bigcup_{AON^z \in aon^z(N)} lsteps(AON^z)$.
3. $\{po_\tau \mid \tau \in steps(N)\} = \bigcup_{AON^z \in aon^z(N)} strat(\mathcal{S}(AON^z))$. □

The proofs of the various parts of theorem 4 follow those of similar results presented in the previous section. A main change is that we no longer can use complement places to establish the emptiness of an inhibiting place, and instead need to refer to the corresponding z-conditions.

It can be seen that both the process semantics and the causal semantics for PTI-nets developed in this section are consistent with those developed for PTBI-nets in the previous section. The latter, in turn generalises the semantics of P/T-nets [1] and elementary net systems with inhibitor arcs from [8].

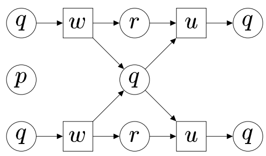

Fig. 10. Generating z-conditions may not be desirable.

7 Concluding remarks

The basic contribution of this paper is a proposal for a process semantics for P/T-nets with inhibitor arcs while assuming an a priori operational semantics. This contrasts with the approach of [3] where transitions can occur in a step if and only if they can occur in either order. First we generalised the existing process notions for ordinary P/T-nets ([6, 1, 14]) and for safe nets with inhibitor arcs ([8]) to the case of P/T-nets with bounded and complemented inhibiting places. In order to obtain a process semantics for general PTI-nets, z-activator occurrence nets were introduced. Given the processes, their associated stratified order structures provide a specification of the net behaviours in terms of causality and weak causality. Thus the results in this paper form a basis for a further investigation of the abstract causal relations within the behaviours of a PTI-net. There are at least two potential applications of these results: first, they can be useful in the development of model checking algorithms for PTI-nets based on unfoldings; second, they can be used as a basis for obtaining a causality semantics for P/T-nets with priorities, extending the results obtained for the elementary net systems with priorities in [10]. Finally, the approach presented in this paper can easily be generalised to nets with weighted arcs; an extension to weighted inhibitor arcs is a matter for future research.

Acknowledgment We are grateful to the referees for their constructive criticism and many suggestions to improve our presentation.

References

1. E. Best and R. Devillers: Sequential and Concurrent Behaviour in Petri Net Theory. *Theoretical Computer Science* 55 (1988) 87-136.
2. E. Best and C. Fernández: *Nonsequential Processes. A Petri Net View.* EATCS Monographs on Theoretical Computer Science, Springer-Verlag (1988).
3. N. Busi and G. M. Pinna: Process Semantics for Place/Transition Nets with Inhibitor and Read Arcs. *Fundamenta Informaticae* 40 (1999) 165-197.
4. R. Fräisse: *Theory of Relations.* North Holland (1986).
5. H. Gaifman and V. Pratt: Partial Order Models of Concurrency and the Computation of Function. Proc. of *LICS'87*, IEEE Computer Society Press (1987) 72-85.
6. U. Goltz and W. Reisig: The Non-sequential Behaviour of Petri Nets. *Information and Control* 57 (1983) 125-147.
7. R. Janicki and M. Koutny: Structure of Concurrency. *Theoretical Computer Science* 112 (1993) 5-52.
8. R. Janicki and M. Koutny: Semantics of Inhibitor Nets. *Information and Computation* 123 (1995) 1-16.
9. R. Janicki and M. Koutny: Order Structures and Generalisations of Szpilrajn's Theorem. *Acta Informatica* 34 (1997) 367-388.
10. R. Janicki and M. Koutny: On Causality Semantics of Nets with Priorities. *Fundamenta Informaticae* 38 (1999) 1-33.
11. M. Nielsen, G. Rozenberg and P. S. Thiagarajan: Behavioural Notions for Elementary Net Systems. *Distributed Computing* 4 (1990) 45-57.
12. J. L. Peterson: *Petri Net Theory and the Modeling of Systems.* Prentice Hall (1981).
13. E. Szpilrajn: Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae* 16 (1930) 386-389.
14. G. Rozenberg and J. Engelfriet: Elementary Net Systems. In: *Advances in Petri Nets. Lectures on Petri Nets I: Basic Models*, W. Reisig and G. Rozenberg (Eds.). Springer-Verlag, Lecture Notes in Computer Science 1491 (1998) 12-121.
15. W. Vogler: Partial Order Semantics and Read Arcs. Proc. of *MFCS'97*, I. Privara and P. Ruzicka (Eds.). Springer-Verlag, Lecture Notes in Computer Science 1295 (1997) 508-517.