Structural Inclusion in the pi-Calculus with Replication

Joost Engelfriet Tjalling Gelsema

Department of Computer Science, Leiden University,
P.O.Box 9512, 2300 RA Leiden, Netherlands
email:
engelfri@wi.leidenuniv.nl
gelsema@wi.leidenuniv.nl

Abstract

Three notions of structural inclusion between process terms of the π-calculus are considered, and proven to be decidable and to have axiomatizations that are sound and complete in the multiset semantics Mπ of the π-calculus. All three are strong simulation relations.

Introduction

This paper is a sequel to [2] which, in turn, is a sequel to [1]. The reader is therefore assumed to be familiar with the concepts and results of [1], and, more in particular, those of [2].

In [2], it was proven that in the π-calculus with replication, two processes are structurally congruent (for a natural extension of the structural congruence of [6]) if and only if they correspond to the same solution, i.e., the same multiset of the multiset transition system Mπ of [1]. Moreover, structural congruence was proven to be decidable. This paper is concerned with relations that express *inclusion* of solutions in Mπ and the corresponding structural inclusion relations on process terms of the π-calculus. That is, we want to define a good notion of one process being a 'substructure' of another process. We would like such a relation to satisfy some minimal constraints. Firstly, of course, it should correspond to an intuitively acceptable notion of substructure (where the 'structure' of a process term is modeled by its corresponding solution in Mπ). Secondly, it must have a natural axiomatization, similar to the case of structural congruence. Thirdly, since 'substructure' is a static notion, it must be decidable. Finally, it must preserve communications. Clearly, if a process contains the structure of another, it must be at least capable of the actions of the latter. Since the only actions we consider in Mπ are communications, a structural inclusion relation must be a *strong simulation* of communication actions (just as structural congruence was proven to be a *strong bisimulation* in [1]).

Structural inclusion of processes is more basic than structural congruence, since it expresses that a process is *at least* composed of the structure of another process, in much the same way as simulation is more basic than bisimulation. For a structural inclusion relation however, there does not appear to be one preferred candidate, but several, depending on different perspectives. For instance, it should be clear that we want a structural inclusion relation $\mathcal{R}$ to satisfy $P \mathcal{R} (P \mid Q)$ for any process term Q, since parallel composition is a typically structural operation that combines processes in the loosest imaginable way (and similarly we want that $P \mathcal{R}\ !P$). In fact, we require $\mathcal{R}$ to be compatible with parallel composition (and replication), and derive $P \mathcal{R} (P \mid Q)$ from $\mathbf{0} \mathcal{R} Q$ and the fact that $P \mid \mathbf{0}$ is structurally congruent with P. But now consider the case in which P and Q have a name x in common; let for instance $P = \overline{x}z.\mathbf{0}$ and $Q = x(y).\mathbf{0}$. On the one hand, it is plausible to infer $(\nu x)P \mathcal{R} (\nu x)(P \mid Q)$, i.e., letting $\mathcal{R}$ be compatible with restriction, since in our molecular view of processes $(\nu x)(P \mid Q)$ represents two molecules, viz. the one that $(\nu x)P$ represents and the one that $(\nu x)Q$ represents. On the other hand, $(\nu x)(P \mid Q)$ is an atomic process, in the sense that P and Q communicate through a "secret" name x (it is connected in the sense of [2]), and hence cannot be cut into non-trivial substructures. Thus, one can choose between letting $\mathcal{R}$ be compatible with restriction, or not; intuitively, this corresponds to allowing "secret" links to be broken, or not. Also, one can choose between letting $\mathcal{R}$ be compatible with the operation of guarding, or not. Since solutions in Mπ are of a recursive nature (each molecule of a solution guards a solution itself), it is natural to 'cut a solution' not only at the top level, but at arbitrary nesting depth. To illustrate this, let for instance $P = u(x).\overline{x}z.\mathbf{0}$ and $Q = u(x).(\overline{x}z.\mathbf{0} \mid x(y).\mathbf{0})$. In M$\pi$, these process terms correspond to $\{u(x).\{\overline{x}z.\varnothing\}\}$ and $\{u(x).\{\overline{x}z.\varnothing, x(y).\varnothing\}\}$, respectively. Although at the top level the two solutions are incomparible (in the sense that one is not a sub(multi-)set of the other), the solution that is guarded by $u(x)$ in the first is a sub(multi-)set of the one that is guarded by the same $u(x)$ in the second. So, in a sense, P is a *nested* substructure of Q. The choice of letting $\mathcal{R}$ be compatible with guarding or not can be made independently of the choice of letting $\mathcal{R}$ be compatible with restriction or not. Thus the combination of these choices results in four different relations. We will show that three of them satisfy the four minimal constraints discussed above (and hence deserve the predicate "natural"). The fourth relation (viz. the one that is compatible with guarding, but not with restriction), although shown to have a natural axiomatization and to be decidable, fails to be a simulation.

Another interesting property of a structural inclusion relation $\mathcal{R}$ is whether or not it captures structural congruence, or, to put it differently, whether or not $P \mathcal{R} Q$ and $Q \mathcal{R} P$ imply $P \equiv Q$ (this is the Cantor-Bernstein (CB) property of [3]). In the presence of infinite structures (by the replication operator $!$, process terms in general have an infinite structure) CB cannot be expected in the general case: it is shown that only one of the four structural inclusion relations satisfies CB (viz. the one that is neither compatible with restriction, nor with guarding).

Since in Mπ the semantics of a π-calculus process term is a multiset, the most natural inclusion relation to consider is based on *containment of multisets* $\subseteq$ (i.e., ordinary set-inclusion, respecting multiplicities), defining a process term P to be *multiset included* in another process term Q, if the solution corresponding to P is contained in the one of Q. These notions are presented in Section 2. In Section 6 we prove multiset inclusion of process terms to be decidable and to have a clear-cut axiomatization, called *structural inclusion* (of which the definition is already given in Section 2). A stronger version of multiset containment which is based on containment of connected components of solutions (cf. the first part of the third paragraph) is presented in Section 3, together with a proof of its decidability and the soundness and completeness of its axiomatization: *strong structural inclusion*. A third *nested containment* relation, based on containment of nested subsolutions (cf. the second part of the third paragraph) is considered in Section 4, together with a fourth, *strong nested containment* (that additionally respects connected components), as well as their axiomatizations: *nested structural inclusion* and *strong nested structural inclusion*, respectively. As mentioned earlier, strong nested containment is the "odd one out": based on its axiomatization (which is, after all, a natural combination of strong structural inclusion and nested structural inclusion), one might expect it to be as 'natural' a notion of substructure as the other three, but it is not. In Section 5, we present a normal form of process terms which extends the subconnected normal form of [2]. Section 6 contains the proof of decidability and soundness and completeness of the axiomatizations of all the inclusions. The normal form of Section 5 is used only for the case of strong nested structural inclusion. Finally, in Section 7 we show that three of the four inclusion relations are strong simulations.

This paper is another contribution to the theory of structure of process terms, initiated in [7, 4] (see also the flowgraphs in [5, 8]). We believe that the separation of structure and behaviour of process terms leads to a better understanding of both.

1 Basic Definitions

Since we use all of the terminology of [2], we refer to the Preliminaries and Sections 3, 4, and 5 of [2] for the basic definitions of the material in this paper. In particular, $\equiv$ denotes the structural congruence of [2], which extends the one of [6, 1]. Except in Section 7 (where we discuss simulation) we do not need to consider the behaviour of process terms and solutions (as formalized in the transition systems of the π-calculus and Mπ), since we are interested in structure only.

There is one small difference in notation: In this paper we let $\mathbb{N}_+ = \{1, 2, \ldots\}$ be the set of positive natural numbers, and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$ be the set of natural numbers (in [1, 2], these sets were denoted by $\mathbf{N}$ and $\mathbf{N} \cup \{0\}$, respectively). Also, the set of π-calculus names is now denoted by $\mathbf{N}$ (instead of N in [1, 2]). Recall that $\#I$ is the cardinality of I; if I is countably infinite, then $\#I = \omega$ (where ω stands for $\aleph_0$). For a function f and a set A, $f \restriction A$ denotes the restriction of f to A.

2 Structural Inclusion

In this section we define the usual multiset containment and we state some of its basic properties. Based on the containment of solutions, we induce a relation on process terms called multiset inclusion and we define its axiomatization: structural inclusion. The proof of soundness and completeness, together with the proof of decidability of the latter is postponed until Section 6. We refer to Section 3 of [2] for the basic properties of multisets.

For multisets S and T, S is *contained in* T, denoted $S \subseteq T$, if there exists a multiset U such that $S \cup U = T$. Note that this is equivalent to requiring $\phi_S(d) \leq \phi_T(d)$ for all $d \in D_S$ (where $n \leq \omega$ for all $n \in \mathbb{N}$); the former definition has the advantage of being independent of multiplicities. Note also that, due to multiplicity ω, the multiset U is not unique. It should, however, be clear that there is a minimal such U (with respect to $\subseteq$); this will be called the *subtraction of S from T*.

Multiset containment is a partial order: obviously $S \subseteq S$ (viz. by choosing $U = \varnothing$) which shows reflexivity. To show antisymmetry, let S and T be multisets over D with $S \subseteq T$ and $T \subseteq S$. Then for all $d \in D$ we have $\phi_S(d) \leq \phi_T(d)$ and $\phi_T(d) \leq \phi_S(d)$. Hence $\phi_S(d) = \phi_T(d)$ for all $d \in D$, and so $S = T$. To show transitivity, assume $S \cup U = V$ and $V \cup U' = T$. Then $S \subseteq T$ since $S \cup (U \cup U') = T$.

Below we state some other easy to prove properties of containment.

Lemma 2.1 *For all multisets S, T, S_i and T_i, $i \in I$, over D,*

(1) if $S_i \subseteq T_i$ for every $i \in I$, then $\bigcup_{i \in I} S_i \subseteq \bigcup_{i \in I} T_i$, and

(2) if $S \subseteq T$, then for every mapping $f : D \to E$, $f(S) \subseteq f(T)$.

Proof To show (1), let $S_i \cup U_i = T_i$ for all $i \in I$. Then $\bigcup_{i \in I} T_i = \bigcup_{i \in I}(S_i \cup U_i) = \bigcup_{i \in I} S_i \cup \bigcup_{i \in I} U_i$. To show (2), let $S \cup U = T$. Then $f(T) = f(S \cup U) = f(S) \cup f(U)$. $\square$

For solutions S and T, $S \subseteq T$ implies $\mathrm{new}(S) \subseteq \mathrm{new}(T)$. This is shown similar to the proof of Lemma 2.1(2), using ordinary set inclusion instead of containment: let $S \cup U = T$. Then $\mathrm{new}(T) = \mathrm{new}(S \cup U) = \mathrm{new}(S) \cup \mathrm{new}(U)$. Hence $\mathrm{new}(S) \subseteq \mathrm{new}(T)$.

Based on containment of multisets, we define multiset inclusion on processes (as in [1], where multiset congruence is defined based on equality of multisets).

Definition 2.2 For process terms P and Q, P is *multiset included* in Q, denoted $P \leq_m Q$, if there exist solutions S and T such that $P \Rightarrow S$, $Q \Rightarrow T$, and $S \subseteq T$.

Example 2.3 Let $R = \overline{x}z.\mathbf{0}$ and consider the process terms $P_1 = (\nu z)R$, $P_2 = (\nu z)(R \mid R)$, and $P_3 = (\nu z)R \mid (\nu z)R$. Then $P_1 \Rightarrow S_1 = \{\overline{x}n_1.\varnothing\}$, $P_2 \Rightarrow S_2 = \{\overline{x}n_1.\varnothing, \overline{x}n_1.\varnothing\}$, and $P_3 \Rightarrow S_3 = \{\overline{x}n_1.\varnothing, \overline{x}n_2.\varnothing\}$, for every $n_1, n_2 \in \text{New}$ with $n_1 \neq n_2$. Hence $P_1 \leq_m P_2$ and $P_1 \leq_m P_3$, since $S_1 \subseteq S_2$ and $S_1 \subseteq S_3$. Note that neither $P_2 \leq_m P_3$, nor $P_3 \leq_m P_2$. $\square$

Recall that the semantics of a process term P is unique upto taking copies: if $P \Rightarrow S$ and S' is a copy of S, then also $P \Rightarrow S'$ (see Lemma 2 of [2]). In the next lemma we show that multiset inclusion does not depend on taking copies; more precisely, if $P \leq_m Q$ and (S, T) is a pair of solutions corresponding to (P, Q) (i.e., $P \Rightarrow S$ and $Q \Rightarrow T$) such that $S \subseteq T$, then *for every* copy T' of T, a copy S' of S can be found such that $S' \subseteq T'$, and conversely.

Lemma 2.4 *For process terms P and Q, if $P \leq_m Q$ and $Q \Rightarrow T$, for a solution T, then there exists a solution S such that $P \Rightarrow S$ and $S \subseteq T$. Conversely, if $P \leq_m Q$ and $P \Rightarrow S$, for a solution S, then there exists a solution T such that $Q \Rightarrow T$ and $S \subseteq T$.*

Proof We will only prove the first statement, since it is similar to the proof of the second. Assume $P \leq_m Q$ and $Q \Rightarrow T$. By Definition 2.2 there exist S' and T' such that $P \Rightarrow S'$, $Q \Rightarrow T'$, and $S' \subseteq T'$. Thus, $\text{new}(S') \subseteq \text{new}(T')$. By Lemma 5 of [1] (Lemma 2 of [2]) there exists a bijection $f : \text{new}(T') \to \text{new}(T)$, such that $T = f(T')$. Let $S = f(S')$. Then $f \restriction \text{new}(S')$ is a bijection from $\text{new}(S')$ to $f(\text{new}(S')) = \text{new}(f(S')) = \text{new}(S)$. Hence, again by Lemma 5 of [1], $P \Rightarrow S$. Moreover, $S' \subseteq T'$ implies $S = f(S') \subseteq f(T') = T$ by Lemma 2.1(2). $\square$

Next, we give the axiomatization of multiset inclusion. The proof of correctness is postponed until Section 6.

Definition 2.5 *Structural inclusion*, denoted $\leq$, is the smallest relation on the set of process terms satisfying

$$\frac{}{\mathbf{0} \leq P}\ \textbf{MIN} \qquad \frac{P \equiv Q}{P \leq Q}\ \textbf{CGR} \qquad \frac{P \leq R \quad R \leq Q}{P \leq Q}\ \textbf{TRA}$$

$$\frac{P \leq Q}{P \mid R \leq Q \mid R}\ \textbf{CCOM} \qquad \frac{P \leq Q}{!P \leq\ !Q}\ \textbf{CREP} \qquad \frac{P \leq Q}{(\nu x)P \leq (\nu x)Q}\ \textbf{CRES}$$

The law **CGR** expresses that structurally congruent processes are structurally included in one another; it is obviously satisfied for any notion of substructure: if P and Q have the same structure, then, trivially, P is a substructure of Q. The law **TRA** expresses that $\leq$ is transitive, and hence a preorder (note that reflexivity is implied by **CGR**). By **MIN**, $\mathbf{0}$ is a minimal element of $\leq$ (and proven to be its least element 'modulo $\equiv$' in Lemma 6.3). By **CCOM**, **CREP**, and **CRES**, $\leq$ is compatible with parallel composition, replication, and restriction, respectively.

Structural inclusion should satisfy some intuitively valid properties. For instance, a process term P should be structurally included in a parallel composition of P with an arbitrary process term Q, i.e., $P \leq P \mid Q$ (which is seen immediately for $\leq_m$, by (S1) of the semantical relation $\Rightarrow$). To show this, note that we have $\mathbf{0} \leq Q$, by **MIN**. Hence, by **CCOM**, $\mathbf{0} \mid P \leq Q \mid P$. Now the left-hand side is structurally congruent to P, by laws (1.1) and (1.2) of structural congruence, and the right-hand side is structurally congruent to $P \mid Q$ by law (1.2), so we have $P \leq \mathbf{0} \mid P$, and $Q \mid P \leq P \mid Q$, by **CGR**. Hence $P \leq P \mid Q$, by **TRA**. As a special case, let $Q = \,!P$. Then we have $P \leq P \mid \,!P \equiv \,!P$, by law (3.1) of structural congruence. Hence $P \leq \,!P$, by **CGR** and **TRA**, respectively (the reader may verify that the last inclusion is also immediate for $\leq_m$, by (S4) of the semantical relation).

In the next example, the role of **CRES** in structural inclusion is discussed, cf. the third paragraph of the Introduction. Since allowing **CRES** in our structural inclusion relation amounts to the simplest form of multiset inclusion (to be proven in Section 6), its role is a valid one. Note also that $\mid$, $!$, and ν are precisely the "structural operations" of the π-calculus (cf. the Introduction of [2]). In another perspective however, the example clearly shows that a structural inclusion relation without **CRES** (to be defined in the next section) is also well motivated.

Example 2.6 Consider the two types of ball games R' and R depicted in Fig. 1. The first, R', is a two-player ball game; the second, R, is a three-player ball game. Initially, both in R' and R, the player of type P_1 is in possession of the ball. The rules of the game are simple: P_1 throws the ball x at the player of type P_2 (this is modeled by a communication of x via the shared link z; in R, P_1 can choose between either one of the two players of type P_2 to throw the ball to), whereafter P_2 can throw the ball at P_1 in return (or, in R, at the second player of type P_2). We model P_1 by $(\nu p)(\overline{z}x.\overline{p}.\mathbf{0} \mid P')$, and P_2 by $(\nu p)(z(y).\overline{z}y.\overline{p}.\mathbf{0} \mid P')$, where P' denotes the process term $!\,p.z(y).\overline{z}y.\overline{p}.\mathbf{0}$ and the guards p and $\overline{p}$ stand for $p(u)$ and $\overline{p}v$ for certain (irrelevant) names u and v. This internally organizes consecutive throwing and catching for each of the players individually. The process R' is modelled by $P_1 \mid P_2$, and R by $P_1 \mid P_2 \mid P_2$. The game R' proceeds as follows (where we have dropped the trailing $.\mathbf{0}$):

$$R' \quad = \quad P_1 \mid P_2$$

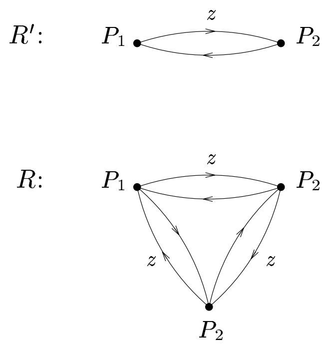

Figure 1: A two-player, and a three-player ball game

$$
\begin{aligned}
&= (\nu p)(\overline{z}x.\overline{p} \mid\, !\,p.z(y).\overline{z}y.\overline{p}) \mid (\nu p)(z(y).\overline{z}y.\overline{p} \mid P') \\
&\to (\nu p)(\overline{p} \mid\, !\,p.z(y).\overline{z}y.\overline{p}) \mid (\nu p)(\overline{z}x.\overline{p} \mid P') \\
&\equiv (\nu p)(\overline{p} \mid p.z(y).\overline{z}y.\overline{p} \mid\, !\,p.z(y).\overline{z}y.\overline{p}) \mid (\nu p)(\overline{z}x.\overline{p} \mid P') \\
&\to (\nu p)(z(y).\overline{z}y.\overline{p} \mid\, !\,p.z(y).\overline{z}y.\overline{p}) \mid (\nu p)(\overline{z}x.\overline{p} \mid P')
\end{aligned}
$$

so now (the former) P_2 is in possession of the ball and is ready to throw it at (the former) P_1, which is ready to receive it. It is easy to extend R' to the three-player game R: simply put another player of type P_2 in parallel with R'. In other words, we have $R' \leq R$. The action sequences for R are left to the reader.

Now suppose P_1 and P_2 have decided not to let any other player join their game. This is modeled by the process $(\nu z)R'$: no action can take place to the outside of $(\nu z)R'$ (placed in any context) via the link z. This means that whereas e.g. $R' \mid R$ models a five-player game (where a player of type P_1 in R' is also capable of throwing a ball at a player of type P_2 in R), the process $(\nu z)R' | (\nu z)R$ rather models a two-player game run in parallel with a three-player game. Note that we have $(\nu z)R' \leq (\nu z)R$ by **CRES**. Indeed, in $(\nu z)R$, P_1 and one of the two players of type P_2 can ignore the other player of type P_2, and not throw a ball at him. Then the process $(\nu z)R$ is just $(\nu z)R'$ with an added dummy. Hence, in this view it is natural to have $(\nu z)R' \leq (\nu z)R$. On the other hand, there is no way in which to restrict $(\nu z)R$ to $(\nu z)R'$ without violating the 'agreement of privacy' the players have in $(\nu z)R$. Even for a dummy P_2 in $(\nu z)R$, there exists *structurally* a bond between each of the players, and thus if the dummy P_2 is removed from $(\nu z)R$, then the "secret" links between the dummy P_2 and the other players in $(\nu z)R$ are broken (to use the terminology of [2]: $(\nu z)R$ is *connected*). This motivates an inclusion relation that respects the "secret" links of processes. $\square$

3 Strong Structural Inclusion

In this section, we define an inclusion relation on solutions that is stronger than containment. As for multiset inclusion, we base strong multiset inclusion, defined for processes, on this strong containment relation. Furthermore, we give an axiomatization, prove this axiomatization to be sound and complete, and show that strong multiset inclusion is decidable. Finally, we show that it, as the sole member of the inclusion relations defined in this paper, is antisymmetrical upto structural congruence, i.e., the intersection of this relation with its inverse yields structural congruence (using the terminology of [3], strong structural inclusion satisfies the Cantor-Bernstein property).

In the previous section, we suggested that a multiset inclusion relation that does not have the compatibility law for restriction, **CRES**, in its axiomatization, is at least as plausible as $\leq_m$. As we explained, the process term $(\nu z)R$ of Example 2.6 is connected (in the sense of [2, Section 4]) by the "secret" link z, and thus cannot be subdivided into smaller process terms without breaking that link. In this section we will treat such connected process terms as atomic. This means that for a solution T corresponding to an arbitrary process term Q, only those solutions S contained in T that respect the connected components of T (see [2, Section 4]) are allowed to correspond to a process term P that is structurally included in Q. Hence we restrict $\subseteq$ to pairs of solutions (S,T), such that S has no new names in common with its environment in T, i.e., S is disconnected from its subtraction U from T.

Definition 3.1 A solution S is *strongly contained* in a solution T, denoted $S \subseteq^{\mathrm{n}} T$ (the superscript n stands for *new-disjoint union*), if there exists a solution U such that $S \cup U = T$ and $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$.

Strong containment is a partial order: antisymmetry follows directly from antisymmetry of $\subseteq$. Also, $S \subseteq^{\mathrm{n}} S$ for every solution S, since for $U = \varnothing$, $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$, which shows reflexivity. To show transitivity of $\subseteq^{\mathrm{n}}$, let $S \cup U = V$ and $V \cup U' = T$, with $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$ and $\mathrm{new}(V) \cap \mathrm{new}(U') = \varnothing$. Now $S \cup (U \cup U') = T$. Furthermore, $\mathrm{new}(S) \cap \mathrm{new}(U \cup U') = (\mathrm{new}(S) \cap \mathrm{new}(U)) \cup (\mathrm{new}(S) \cap \mathrm{new}(U')) = \varnothing$, since $\mathrm{new}(S) \subseteq \mathrm{new}(V)$.

Properties similar to the ones of Lemma 2.1 hold for $\subseteq^{\mathrm{n}}$.

Lemma 3.2 *For all solutions S, T, S_i and T_i, $i \in I$,*

(1) if $S_i \subseteq^{\mathrm{n}} T_i$ for every $i \in I$, and the $\mathrm{new}(T_i)$ *are mutually disjoint, then $\bigcup_{i \in I} S_i \subseteq^{\mathrm{n}} \bigcup_{i \in I} T_i$,*

(2) if $S \subseteq^{\mathrm{n}} T$, then for every injection $f : \mathrm{new}(T) \to \mathrm{New}$, $f(S) \subseteq^{\mathrm{n}} f(T)$, and

(3) if $S \subseteq^{\mathrm{n}} T$, then for every mapping $f : \mathbf{N} \cup \mathbb{N}_+ \to \mathbf{N} \cup \mathbb{N}_+$, $f(S) \subseteq^{\mathrm{n}} f(T)$.

Proof To show (1), let $S_i \cup U_i = T_i$ with $\mathrm{new}(S_i) \cap \mathrm{new}(U_i) = \varnothing$. Note that by the proof of Lemma 2.1(1), it is sufficient to show that $\mathrm{new}(\bigcup_{i \in I} S_i) \cap \mathrm{new}(\bigcup_{i \in I} U_i) = \varnothing$. Since for every $i \in I$, $\mathrm{new}(S_i) \subseteq \mathrm{new}(T_i)$ and $\mathrm{new}(U_i) \subseteq \mathrm{new}(T_i)$, $\mathrm{new}(S_i)$ and $\mathrm{new}(U_j)$ are disjoint for $i \neq j$. Hence $\mathrm{new}(\bigcup_{i \in I} S_i) \cap \mathrm{new}(\bigcup_{i \in I} U_i) = (\bigcup_{i \in I} \mathrm{new}(S_i)) \cap (\bigcup_{i \in I} \mathrm{new}(U_i)) = \bigcup_{i \in I} (\mathrm{new}(S_i) \cap \mathrm{new}(U_i)) = \varnothing$. To show (2), let $S \cup U = T$ with $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$. By the proof of Lemma 2.1(2), it suffices to show that $\mathrm{new}(f(S)) \cap \mathrm{new}(f(U)) = \varnothing$. Since f is injective on $\mathrm{new}(T) = \mathrm{new}(S) \cup \mathrm{new}(U)$, we have $\mathrm{new}(f(S)) \cap \mathrm{new}(f(U)) = f(\mathrm{new}(S)) \cap f(\mathrm{new}(U)) = f(\mathrm{new}(S) \cap \mathrm{new}(U)) = \varnothing$. The proof of (3) follows immediately from the one of Lemma 2.1(2), since for any mapping $f : \mathbf{N} \cup \mathbb{N}_+ \to \mathbf{N} \cup \mathbb{N}_+$, we trivially have $\mathrm{new}(S) = \mathrm{new}(f(S))$ and $\mathrm{new}(U) = \mathrm{new}(f(U))$. $\square$

The reader may verify that the disconnectedness of the T_i, and the injectivity of f in Lemma 3.2, are both necessary conditions. For instance, if $S_i = \{\overline{x}n_i.\varnothing\}$, for $i \in \{1, 2\}$ and $n_i \in \mathrm{New}$ with $n_1 \neq n_2$, and $T_1 = T_2 = S_1 \cup S_2$, then $S_i \subseteq^{\mathrm{n}} T_i$, since the S_i are disconnected, but $S_1 \cup S_2 \not\subseteq^{\mathrm{n}} T_1 \cup T_2$, since obviously $S_1 \cup S_2$ is not disconnected from itself. Also, if f is a mapping such that $f(n_1) = f(n_2)$, then $f(S_1) \not\subseteq^{\mathrm{n}} f(T_1)$, since $f(S_1)$ is equal to its subtraction from $f(T_1)$.

As observed earlier, another way to view strong containment is to realize that, in fact, the inclusion relation does not operate on the level of molecules, but rather on the higher plane of connected components. Since in Definition 3.1, S and U are disconnected, the connected components of T are unaffected by $\subseteq^{\mathrm{n}}$, and hence S and U form a partition of the connected components of T rather than of its molecules. Thus, $S \subseteq^{\mathrm{n}} T$ iff S is the union of a number of connected components of T. This is formulated in the next lemma.

Lemma 3.3 *Let $T = \bigcup_{j \in J} T_j$ be a solution, where the T_j, $j \in J$, are the connected components of T. Then for every solution S, $S \subseteq^{\mathrm{n}} T$ if and only if there exists a subset J' of J such that $S = \bigcup_{j \in J'} T_j$.*

Proof (If) Obvious. (Only if) Let $S \cup U = \bigcup_{j \in J} T_j$ with $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$. By Lemma 9 of [2] there exist disjoint sets J_S and J_U such that $J = J_S \cup J_U$, $S = \bigcup_{j \in J_S} T_j$ and $U = \bigcup_{j \in J_U} T_j$. $\square$

Strong multiset inclusion of processes, defined next, is based on strong containment of the corresponding solutions (as multiset inclusion is based on containment).

Definition 3.4 A process term P is *strongly multiset included* in a process term Q, denoted $P \leq^{\mathrm{n}}_m Q$, if there exist solutions S and T such that $P \Rightarrow S$, $Q \Rightarrow T$, and $S \subseteq^{\mathrm{n}} T$.

Example 3.5 Consider the process terms and solutions of Example 2.3. Now $P_1 \not\leq^{\mathrm{n}}_m P_2$, since, for $U = \{\overline{x}n_1.\varnothing\}$, we have $S_1 \cup U = S_2$, but S_1 and U have the new name n_1 in common. However, $P_1 \leq^{\mathrm{n}}_m P_3$ does hold, because, for $U = \{\overline{x}n_2.\varnothing\}$, $S_1 \cup U = S_3$ and $\mathrm{new}(S_1) \cap \mathrm{new}(U) = \{n_1\} \cap \{n_2\} = \varnothing$. $\square$

In general, by (S1) of the semantical relation $\Rightarrow$, a process term P is strongly multiset included in the process term that is formed by a parallel composition of P with any other process term. So we have $P \leq^{\mathrm{n}}_m P \mid Q$ (as expected). In particular, $P \leq^{\mathrm{n}}_m P \mid\ !P \equiv_m\ !P$, and hence $P \leq^{\mathrm{n}}_m\ !P$.

Lemma 2.4 is also valid for strong multiset inclusion: in its proof, we can replace the occurrences of $S' \subseteq T'$ and $S \subseteq T$, by $S' \subseteq^{\mathrm{n}} T'$ and $S \subseteq^{\mathrm{n}} T$, respectively. The first replacement is valid because we assume $P \leq^{\mathrm{n}}_m Q$, and the second because $S' \subseteq^{\mathrm{n}} T'$ implies $S = f(S') \subseteq^{\mathrm{n}} f(T') = T$, by Lemma 3.2(2). For completeness sake, we state this version of Lemma 2.4.

Lemma 3.6 *For process terms P and Q, if $P \leq^{\mathrm{n}}_m Q$ and $Q \Rightarrow T$, for a solution T, then there exists a solution S such that $P \Rightarrow S$ and $S \subseteq^{\mathrm{n}} T$. Conversely, if $P \leq^{\mathrm{n}}_m Q$ and $P \Rightarrow S$, for a solution S, then there exists a solution T such that $Q \Rightarrow T$ and $S \subseteq^{\mathrm{n}} T$.*

Next, we give the axiomatization of strong multiset inclusion.

Definition 3.7 *Strong structural inclusion*, denoted $\leq^{\mathrm{n}}$, is the smallest relation on the set of process terms satisfying

$$\frac{}{\mathbf{0} \leq^{\mathrm{n}} P}\ \mathbf{MIN} \qquad \frac{P \equiv Q}{P \leq^{\mathrm{n}} Q}\ \mathbf{CGR} \qquad \frac{P \leq^{\mathrm{n}} R \quad R \leq^{\mathrm{n}} Q}{P \leq^{\mathrm{n}} Q}\ \mathbf{TRA}$$

$$\frac{P \leq^{\mathrm{n}} Q}{P \mid R \leq^{\mathrm{n}} Q \mid R}\ \mathbf{CCOM} \qquad \frac{P \leq^{\mathrm{n}} Q}{!P \leq^{\mathrm{n}}\ !Q}\ \mathbf{CREP}$$

Note that strong structural inclusion only differs from structural inclusion in the omission of law **CRES**, so strong structural inclusion is compatible with parallel composition and replication only. Note also that $P \leq^{\mathrm{n}} P \mid Q$ and $P \leq^{\mathrm{n}}\ !P$, as shown after Definition 2.5 (without using **CRES**).

We will use an equivalent notion of strong structural inclusion that matches better to the definition of strong multiset inclusion: by the next theorem, a process term is strong structurally included in another process term, if and only if, upto parallel composition, it is structurally equivalent to it. We will interchange this new notion with the original (in Definition 3.7) without explicit mentioning.

Theorem 3.8 *For process terms P and Q, $P \leq^{\mathrm{n}} Q$ if and only if there exists a process term R such that $P \mid R \equiv Q$.*

Proof (If) We have $P \leq^{\mathrm{n}} P|R \equiv Q$, and so $P \leq^{\mathrm{n}} Q$ by **CGR** and **TRA**. (Only if) We will show that the relation $\mathcal{R} = \{(P, Q) \mid \exists R' : P \mid R' \equiv Q\}$ satisfies the laws of strong structural inclusion in Definition 3.7. Since $\leq^{\mathrm{n}}$ is the smallest one satisfying these laws, clearly $P \leq^{\mathrm{n}} Q$ implies $(P, Q) \in \mathcal{R}$. Obviously $(\mathbf{0}, P) \in \mathcal{R}$ for every process term P, viz. by taking $R' = P$, so $\mathcal{R}$ satisfies

MIN. Also, $(P, Q) \in \mathcal{R}$ if $P \equiv Q$ (by taking $R' = \mathbf{0}$), which shows that **CGR** is satisfied. To show **TRA**, assume there exist R_1, R_2 such that $P \,|\, R_1 \equiv R$ and $R \,|\, R_2 \equiv Q$. Now take $R' = R_1 \,|\, R_2$. Then $P \,|\, R' = P \,|\, (R_1 \,|\, R_2) \equiv R \,|\, R_2 \equiv Q$, by the commutativity and associativity laws of structural congruence for parallel composition. Finally, assume $P \mid R' \equiv Q$. Then $(P \mid R) \mid R' \equiv Q \mid R$, and $!P \mid\, !R' \equiv\, !(P \mid R') \equiv\, !Q$ (by (3.2) of structural congruence) which proves that **CCOM** and **CREP** are satisfied, respectively. $\square$

Note that the process term R in Theorem 3.8 is not unique; for instance, we could derive $!P \leq^{\mathrm{n}}\, !P$ by the existence of the process terms $\mathbf{0}$, P, $P \mid P$ and, in fact, infinitely many others. Similarly, the solution U in Definition 3.1 is not unique, since $\bigcup_{i \in \mathbb{N}}\{g.S\} \subseteq^{\mathrm{n}} \bigcup_{i \in \mathbb{N}}\{g.S\}$ by the existence of $\bigcup_{1 \leq i < k}\{g.S\}$, for any $k \in \mathbb{N} \cup \{\omega\}$.

We now proceed with showing that strong multiset inclusion and strong structural inclusion are the same, i.e., soundness and completeness of the axiomatization of $\leq^{\mathrm{n}}_m$ in Definition 3.7. Much of the work in showing soundness of Definition 3.7 (i.e., $\leq^{\mathrm{n}}$ implies $\leq^{\mathrm{n}}_m$) has already been done in Theorem 3.8. In fact, by the definition of $\Rightarrow$, a parallel composition of two process terms can only correspond to a solution that is composed of two collections of connected components. This (and the free use of Theorem 3.8) is fundamental in the proof of soundness.

Lemma 3.9 *If* $P \leq^{\mathrm{n}} Q$ *then* $P \leq^{\mathrm{n}}_m Q$.

Proof Assume there exists a process term R with $P \,|\, R \equiv Q$. Then $P \,|\, R \equiv_m Q$ by Theorem 33 of [2]. Let T be a solution such that $Q \Rightarrow T$. Then $P \mid R \Rightarrow T$. Hence there exist solutions S and U such that $T = S \cup U$, $P \Rightarrow S$, $R \Rightarrow U$ and $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$. Thus $S \subseteq^{\mathrm{n}} T$, and so $P \leq^{\mathrm{n}}_m Q$, by Definition 3.4. $\square$

For proving completeness, i.e., the reverse of Lemma 3.9, we need to show that every process term that corresponds to a union of two disconnected solutions is a parallel composition. More accurately, if Q is a process term with $Q \Rightarrow T = S \cup U$, where $P \Rightarrow S$ and S and U have no new name in common, then we must show that there exists a process term R such that $P \,|\, R \equiv Q$ (and $R \Rightarrow U$). This is expressed in Lemma 28 of [2], but in order to use it, the copy-width of S and U must be bounded, i.e., must have their value in $\mathbb{N}$. We know however that the copy-width of S and T is bounded, i.e., $\mathrm{copy}(S), \mathrm{copy}(T) \in \mathbb{N}$, because S and T correspond to process terms, see Lemma 22 of [2]. Now let U_i, $i \in I$, be the connected components of U, and suppose that $\mathrm{copy}(U)$ is unbounded. This means that there exist U_i such that $\mathrm{mult}(U_i, U)$ is finite but arbitrary large. However, since there exists $k \in \mathbb{N}$ (viz. $k = \mathrm{copy}(T)$), such that for each i, $\mathrm{mult}(U_i, T) \leq k$ or $\mathrm{mult}(U_i, T) = \omega$, we must have for each i with $k < \mathrm{mult}(U_i, U) < \omega$ that $\mathrm{mult}(U_i, S) = \omega$. Hence if we cut off each of those U_i from U, then, obviously, the copy-width of the resulting solution is k. Moreover, we will show that the union of S and the resulting solution is a copy of $S \cup U$ and hence also corresponds to Q. This is the way in which the mapping cut of the following lemma operates. A similar technique was used in the proof of Lemma 15 of [2].

Lemma 3.10 *There is a mapping* $\mathrm{cut} : \mathrm{Sol} \times \mathbb{N} \to \mathrm{Sol}$ *such that for every solution* U *and every* $k \in \mathbb{N}$

(1) $\mathrm{cut}(U,k) \subseteq^{\mathrm{n}} U$*,*

(2) $\mathrm{copy}(\mathrm{cut}(U,k)) \leq k$*, and*

(3) for every solution S *with* $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$*,*
if $\mathrm{copy}(S \cup U) \leq k$*, then* $S \cup \mathrm{cut}(U,k)$ *is a copy of* $S \cup U$*.*

Proof Let $U = \bigcup_{i \in I} U_i$, where the U_i are the connected components of U, and let $J = \{i \in I \mid k < \mathrm{mult}(U_i, U) < \omega\}$. Then we define $\mathrm{cut}(U,k) = \bigcup_{i \in I-J} U_i$. Properties (1) and (2) are obvious. To show (3), let S be a solution with $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$ and $\mathrm{copy}(S \cup U) \leq k$. Then $S \cup U = (S \cup \mathrm{cut}(U,k)) \cup \bigcup_{i \in J} U_i$. By Lemma 14 of [2] it now suffices to prove that for all $i \in J$, $\mathrm{mult}(U_i, S \cup \mathrm{cut}(U,k)) = \omega$. Let $i \in J$, i.e., $k < \mathrm{mult}(U_i, U) < \omega$. By Lemma 12 of [2], $\mathrm{mult}(U_i, S \cup \mathrm{cut}(U,k)) = \mathrm{mult}(U_i, S) + \mathrm{mult}(U_i, \mathrm{cut}(U,k)) = \mathrm{mult}(U_i, S)$. Now suppose that $\mathrm{mult}(U_i, S) < \omega$. Then $k < \mathrm{mult}(U_i, S) + \mathrm{mult}(U_i, U) < \omega$. But, again by Lemma 12 of [2], $\mathrm{mult}(U_i, S) + \mathrm{mult}(U_i, U) = \mathrm{mult}(U_i, S \cup U)$. This implies that $\mathrm{copy}(S \cup U) > k$, contradicting the assumption. □

With the use of Lemma 28 of [2], completeness and decidability of $\leq^{\mathrm{n}}$ can now be shown easily.

Theorem 3.11 *For process terms* P *and* Q*,*

(1) $P \leq^{\mathrm{n}} Q$ *if and only if* $P \leq^{\mathrm{n}}_{m} Q$*, and*

(2) it is decidable whether or not $P \leq^{\mathrm{n}} Q$*.*

Proof Let $k = \max(\mathrm{copy}(P), \mathrm{copy}(Q))$. Observe that by Lemma 22 of [2] $k \neq \omega$. Let $\mathcal{Q}$ denote the finite set $\mathrm{comp}(Q,k)$, cf. Lemma 28 of [2]. We will show that the following statements are equivalent:

(i) $P \leq^{\mathrm{n}} Q$

(ii) $P \leq^{\mathrm{n}}_{m} Q$

(iii) there exists $(Q_1, Q_2) \in \mathcal{Q}$ such that $P \equiv Q_1$.

(i) $\Rightarrow$ (ii) is by Lemma 3.9.

(ii) $\Rightarrow$ (iii). Let $P \Rightarrow S$ and $Q \Rightarrow T$ such that $S \cup U = T$ with $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$. Thus $Q \Rightarrow S \cup U$. Note that $k = \max(\mathrm{copy}(S), \mathrm{copy}(T))$. Now take $U' = \mathrm{cut}(U,k)$. By Lemma 3.10, $U' \subseteq^{\mathrm{n}} U$, $\mathrm{copy}(U') \leq k$, and $S \cup U'$ is a copy of $S \cup U$. Hence $Q \Rightarrow S \cup U'$ and $\mathrm{new}(S) \cap \mathrm{new}(U') = \varnothing$. By Lemma 28(2) of [2] there exists $(Q_1, Q_2) \in \mathrm{comp}(Q, k)$ such that $Q_1 \Rightarrow S$ (and $Q_2 \Rightarrow U'$). Hence $P \equiv_m Q_1$ and so $P \equiv Q_1$ by Theorem 33 of [2].

(iii) $\Rightarrow$ (i). By Lemma 28(1) of [2], $Q \equiv Q_1 \mid Q_2 \equiv P \mid Q_2$. Hence $P \leq^{\mathrm{n}} Q$.

By the decidability of structural congruence (Theorem 34 of [2]), the decidability of (iii) now reduces to the computability of $\mathcal{Q}$. By the remark below Theorem 34 of [2], k is computable. Hence by Lemma 28 of [2], $\mathcal{Q}$ is computable.
$\square$

The remainder of this section is devoted to prove antisymmetry of $\leq^{\mathrm{n}}$ upto structural congruence, i.e., we show that $P \leq^{\mathrm{n}} Q$, and the reverse, $Q \leq^{\mathrm{n}} P$, imply that P and Q are structurally congruent. Note that by Theorem 33 of [2] and Theorem 3.11, it suffices to show antisymmetry of $\leq^{\mathrm{n}}_m$ upto $\equiv_m$. The proof of the latter is in three stages: first we show that for solutions S and T with $S \subseteq^{\mathrm{n}} T$, the number of copies of an arbitrary connected solution U in S is at most equal to the number of copies of U in T, i.e., $\mathrm{mult}(U, S) \leq \mathrm{mult}(U, T)$. This is a consequence of the fact that every connected component of S is a connected component of T. Actually, we will prove the above statement for the more general case in which $T' \subseteq^{\mathrm{n}} T$ and T' is any copy of S. Secondly, we use this to show that if $P \leq^{\mathrm{n}}_m Q$ and $Q \leq^{\mathrm{n}}_m P$, then for arbitrary connected U, $\mathrm{mult}(U, S) = \mathrm{mult}(U, T)$, where S corresponds to P, and T to Q, and finally this is shown to be the case only if S and T are copies. For the proof of the first part, we use that the copy-of relation distributes nicely over any subcollection of connected components in a solution. This is proven in Lemma 3.13. First we give an obvious property of taking copies of connected components.

Lemma 3.12 *Let $S = \bigcup_{i \in I} S_i$ and $T = \bigcup_{j \in J} T_j$ be solutions where the S_i, $i \in I$, and the T_j, $j \in J$, are the connected components of S and T, respectively. Then S is a copy of T if and only if there exists a bijection $\psi : I \to J$ such that S_i is a copy of $T_{\psi(i)}$, for all $i \in I$.*

Proof (If) By Lemma 3 of [2], $\bigcup_{i \in I} S_i$ is a copy of $\bigcup_{i \in I} T_{\psi(i)}$, which equals $\bigcup_{j \in J} T_j$ by property (a) of Section 3 of [2]. (Only if) This is similar to the proof of Lemma 13 of [2]. Let $f : \mathrm{new}(S) \to \mathrm{new}(T)$ be a bijection with $f(S) = T$. Then $f(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} f(S_i) = \bigcup_{j \in J} T_j$. By Lemma 7 of [2], $f(S_i)$ is connected for every $i \in I$. Hence by Lemma 10 of [2] there exists a bijection $\psi : I \to J$ such that $f(S_i) = T_{\psi(i)}$ for all $i \in I$, so S_i is a copy of $T_{\psi(i)}$. $\square$

Lemma 3.13 *Let $S = \bigcup_{i \in I} S_i$ and $T = \bigcup_{j \in J} T_j$ be solutions where the S_i, $i \in I$, and the T_j, $j \in J$, are the connected components of S and T, respectively. Then there exists a solution T' such that S is a copy of T' and $T' \subseteq^{\mathrm{n}} T$ if and only if there exists an injection $\psi : I \to J$ such that S_i is a copy of $T_{\psi(i)}$, for all $i \in I$.*

Proof (If) Let $T' = \bigcup_{i \in I} T_{\psi(i)}$ and apply Lemmas 3.12 and 3.3. (Only if) By Lemma 3.3, $T' = \bigcup_{j \in J'} T_j$ for some $J' \subseteq J$. Then, by Lemma 3.12, there exists a bijection $\psi : I \to J'$ such that S_i is a copy of $T_{\psi(i)}$ for all $i \in I$. $\square$

Now, everything is prepared to show antisymmetry upto $\equiv_m$.

Lemma 3.14 *For all process terms P and Q, if $P \leq^{\mathrm{n}}_m Q$ and $Q \leq^{\mathrm{n}}_m P$ then $P \equiv_m Q$.*

Proof First, we will prove the following two statements for all solutions S and T (the reader can easily verify that these statements are in fact valid in both directions):

(1) If S is a copy of T' and $T' \subseteq^{\mathrm{n}} T$, then for all connected solutions U, $\mathrm{mult}(U, S) \leq \mathrm{mult}(U, T)$.

(2) If, for all connected solutions U, $\mathrm{mult}(U, S) = \mathrm{mult}(U, T)$, then S is a copy of T.

Let $S = \bigcup_{i \in I} S_i$ and $T = \bigcup_{j \in J} T_j$ where the S_i, $i \in I$, and T_j, $j \in J$, are the connected components of S and T, respectively.

To prove (1), let $T' \subseteq^{\mathrm{n}} T$ and S a copy of T'. By Lemma 3.13, there exists an injection $\psi : I \to J$ such that S_i is a copy of $T_{\psi(i)}$ for all $i \in I$. Now let U be an arbitrary connected solution and let $I_U = \{i \in I \mid S_i \text{ is a copy of } U\}$ and $J_U = \{j \in J \mid T_j \text{ is a copy of } U\}$. Since $\psi(I_U) \subseteq J_U$, we have $\mathrm{mult}(U, S) = \#I_U = \#\psi(I_U) \leq \#J_U = \mathrm{mult}(U, T)$.

To prove (2), let $S = \bigcup_{k \in K} \bigcup_{i \in I_k} S_i$, where the I_k are disjoint sets with $I = \bigcup_{k \in K} I_k$, and

for all $k \in K$ and $i, i' \in I_k$, S_i is a copy of $S_{i'}$, and

for all $k, k' \in K$ with $k \neq k'$, S_i is not a copy of $S_{i'}$, where $i \in I_k$ and $i' \in I_{k'}$.

In other words, the I_k are the equivalence classes of the equivalence relation which holds between i and i' iff S_i is a copy of $S_{i'}$. Let $T = \bigcup_{l \in L} \bigcup_{j \in J_l} T_j$ with similar conditions. Then by assumption we have for all $k \in K$ and $i \in I_k$: $\#I_k = \mathrm{mult}(S_i, S) = \mathrm{mult}(S_i, T)$, and similarly for T_j. Hence there exists a bijection $\psi : K \to L$ such that for all $k \in K$, $\#I_k = \#J_{\psi(k)}$ and for all $i \in I_k$ and $j \in J_{\psi(k)}$ we have that S_i is a copy of T_j. Thus there exist bijections $\psi'_k : I_k \to J_{\psi(k)}$ such that for all $i \in I_k$, S_i is a copy of $T_{\psi'_k(i)}$. Let $\psi' = \bigcup_{k \in K} \psi'_k$. Then ψ' is a bijection from I to J such that for all $i \in I$, S_i is a copy $T_{\psi'(i)}$. By Lemma 3.12 we have that S is a copy of T.

Finally we use (1) and (2) to prove the statement of the lemma. Let $Q \Rightarrow T$. By two applications of Lemma 3.6, there exist solutions S and T' such that $P \Rightarrow S$, $Q \Rightarrow T'$, $S \subseteq^{\mathrm{n}} T$ and $T' \subseteq^{\mathrm{n}} S$. By Lemma 5 of [1], T' is a copy of T. Thus by (1), we have that for all connected solutions U, $\mathrm{mult}(U, S) \leq \mathrm{mult}(U, T)$ and the reverse: $\mathrm{mult}(U, T) \leq \mathrm{mult}(U, S)$. So $\mathrm{mult}(U, S) = \mathrm{mult}(U, T)$ for all connected solutions U. Hence by (2), S is a copy of T. Thus we have $P \equiv_m Q$. $\square$

Theorem 3.15 *For all process terms P and Q, $P \equiv Q$ if and only if $P \leq^{\mathrm{n}} Q$ and $Q \leq^{\mathrm{n}} P$.*

Proof Immediate from Lemma 3.14, Theorem 3.11, and Theorem 33 of [2]. □

A different approach to prove Theorem 3.15 is presented in [3]. It is shown there that Theorem 3.15 can be viewed as a special case of a more general 'Cantor-Bernstein-like' result. In [3], with regard to ordinary set inclusion, for arbitrary sets A and B with structured elements, it need not always be the case that $f(A) \subseteq B \subseteq A$ imply that A and B are isomorphic, where f is an injective mapping on the atomic objects, which the structured elements of A and B are composed of (for example, a graph can be seen as a collection of edges that are composed of vertices). However, it is indeed the case with regard to a stronger inclusion relation $\subseteq^{\nu}$, viz. one that respects the interrelationship of the structured elements (two elements are related if they share a common atomic object). Also, a means of computing an isomorphism from f is presented. Now we claim that it can be shown that solutions are such sets with structured elements and, for this particular instance, $\subseteq^{\nu}$ is strong containment ($\subseteq^{\mathrm{n}}$) and 'copy-of' is the correct notion of isomorphism. Furthermore, we claim that Lemma 3.14 is the instance, for this particular case, of the above result.

4 Nested Structural Inclusion

The structural inclusion relation (and the corresponding containment) defined in the previous section was motivated by excluding the compatiblility law **CRES** for restriction in its definition. In this section we study the two structural inclusion relations that result from involving a compatibility law **CGUA** for guards, in Definitions 2.5 and 3.7, respectively. Both correspond to a recursively defined containment relation on solutions (and are based on containment, and strong containment, respectively). Some preparatory work for Section 6 (where soundness and completeness for each of the four inclusion relations is proven) is done at the end of this section, stating some universal properties (i.e., properties that hold for each of the four types of containment).

The two containment relations on solutions in the previous sections were based on containment of ordinary multisets. They however completely disregard the recursive (or "nested") nature of solutions; only the top level molecules are taken into account. As we recall from [1], the set of solutions Sol is the smallest set $\mathcal{X}$ such that

> if $S_i \in \mathcal{X}$ and g_i is a schematic guard for every $i \in I$,
> then $\bigcup_{i \in I}\{g_i.S_i\} \in \mathcal{X}$.

So, by taking a subset of I at the *top* level of recursion, we can produce any solution that is contained in $S = \bigcup_{i \in I}\{g_i.S_i\}$ (and, as we saw in the previous section, by taking a special subset at the top level, we get a solution that is strongly contained in S). It is however completely natural to define a containment relation that allows taking a subset of I at *any* level of the recursion, i.e., that allows to take substructures of the nested solutions S_i too. Nested containment is based on this.

Definition 4.1 *Nested containment*, denoted $\subseteq^{\mathrm{g}}$, is the smallest relation on Sol such that

if $S \subseteq T$ and $S_i \subseteq^{\mathrm{g}} T_i$ for all $i \in I$,
then $S \cup \bigcup_{i \in I}\{g_i.S_i\} \subseteq^{\mathrm{g}} T \cup \bigcup_{i \in I}\{g_i.T_i\}$, (*)

where the g_i are schematic guards.

It can be shown that (*) is equivalent to the easier requirement below:

if $S_i \subseteq^{\mathrm{g}} T_i$ for all $i \in I$,
then $\bigcup_{i \in I}\{g_i.S_i\} \subseteq^{\mathrm{g}} T \cup \bigcup_{i \in I}\{g_i.T_i\}$.

However, we will use only (*) in our proofs since together with strong nested containment to be defined below, its use is more uniform.

Note that nested containment is strictly weaker than ordinary containment, i.e., if $S \subseteq T$, then $S \subseteq^{\mathrm{g}} T$, for all solutions S and T (take $I = \varnothing$ in Definition 4.1). Note also that this implies that $\subseteq^{\mathrm{g}}$ is reflexive; this is a consequence of reflexivity of $\subseteq$. We postpone the proof of transitivity of $\subseteq^{\mathrm{g}}$ until after the definition of strong nested containment (Theorem 4.13). By the next example, $\subseteq^{\mathrm{g}}$ is not antisymmetrical.

Example 4.2 Let

$$V = \{a.\{b.\varnothing, c.\varnothing\}, a.\{b.\varnothing, c.\varnothing\}, \ldots\} = \bigcup_{i \in \mathbb{N}}\{a.\{b.\varnothing, c.\varnothing\}\},$$

where a, b, and c are arbitrary schematic guards. Let

$$V' = \{a.\{b.\varnothing\}, a.\{b.\varnothing\}, \ldots\} = \bigcup_{i \in \mathbb{N}}\{a.\{b.\varnothing\}\},$$

and $W = V \cup V'$. Obviously, $V \subseteq^{\mathrm{g}} W$, since $V \subseteq W$ (i.e., take $S = V$, $T = W$, and $I = \varnothing$ in (*)). The reverse, $W \subseteq^{\mathrm{g}} V$, is also true: take $S_i = \{b.\varnothing\}$ and $T_i = \{b.\varnothing, c.\varnothing\}$ for every $i \in \mathbb{N}$. Since $S_i \subseteq T_i$, $S_i \subseteq^{\mathrm{g}} T_i$. Now $W = V \cup V' = V \cup \bigcup_{i \in \mathbb{N}}\{a.S_i\} \subseteq^{\mathrm{g}} V \cup \bigcup_{i \in \mathbb{N}}\{a.T_i\} = V \cup V = V$ (take $S = T = V$, $g_i = a$, and $I = \mathbb{N}$ in (*)). Hence $V \subseteq^{\mathrm{g}} W$ and $W \subseteq^{\mathrm{g}} V$, but $V \neq W$. $\square$

Thus $\subseteq^{\mathrm{g}}$ is not a partial order (as $\subseteq$ and $\subseteq^{\mathrm{n}}$ are) but rather a preorder (sometimes called quasi-order).

Lemma 4.3(1), (2) below is the analogue of Lemma 2.1(1), (2) for nested containment. A third basic property of nested containment is added, which, together with (1), expresses its recursive nature. Recall from [1] that a guard is a string of the form $\overline{x}y$ with $x, y \in \mathbf{N} \cup \mathrm{New}$, or $x(y)$ with $x \in \mathbf{N} \cup \mathrm{New}$ and $y \in \mathbf{N}$.

Lemma 4.3 *For all solutions S, T, S_i and T_i, $i \in I$,*

(1) if $S_i \subseteq^{\mathrm{g}} T_i$ for every $i \in I$, then $\bigcup_{i\in I} S_i \subseteq^{\mathrm{g}} \bigcup_{i\in I} T_i$,

(2) if $S \subseteq^{\mathrm{g}} T$, then for every mapping $f : \mathbf{N} \cup \mathrm{New} \cup \mathbb{N}_+ \to \mathbf{N} \cup \mathrm{New} \cup \mathbb{N}_+$, $f(S) \subseteq^{\mathrm{g}} f(T)$, and

(3) if $S \subseteq^{\mathrm{g}} T$, then for every guard g, $\{g.S\} \subseteq^{\mathrm{g}} \{g.T\}$.

Proof To show (1), let $S_i \subseteq^{\mathrm{g}} T_i$ for all $i \in I$. It follows from Definition 4.1 that S_i and T_i must be of the form $S_i = V_i \cup \bigcup_{j\in J_i}\{g_j.S'_j\}$ and $T_i = W_i \cup \bigcup_{j\in J_i}\{g_j.T'_j\}$, where $V_i \subseteq W_i$ and $S'_j \subseteq^{\mathrm{g}} T'_j$ for all $j \in J_i$; moreover, the J_i are mutually disjoint (which obviously may be assumed). Set $J = \bigcup_{i\in I} J_i$. Then $\bigcup_{i\in I} S_i = \bigcup_{i\in I} V_i \cup \bigcup_{j\in J}\{g_j.S'_j\} \subseteq^{\mathrm{g}} \bigcup_{i\in I} W_i \cup \bigcup_{j\in J}\{g_j.T'_j\} = \bigcup_{i\in I} T_i$ by Definition 4.1 (note that $\bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} W_i$ by Lemma 2.1(1)). We show (2) by induction on the definition of $\subseteq^{\mathrm{g}}$. Let $S = V \cup \bigcup_{i\in I}\{g_i.S_i\}$ and $T = W \cup \bigcup_{i\in I}\{g_i.T_i\}$, with $V \subseteq W$ and $S_i \subseteq^{\mathrm{g}} T_i$. Then, by induction, $f(S_i) \subseteq^{\mathrm{g}} f(T_i)$ for all $i \in I$. Hence $f(S) = f(V) \cup \bigcup_{i\in I}\{f(g_i).f(S_i)\} \subseteq^{\mathrm{g}} f(W) \cup \bigcup_{i\in I}\{f(g_i).f(T_i)\} = f(T)$ by Definition 4.1 (note that $f(V) \subseteq f(W)$ by Lemma 2.1(2)). Observe that for a schematic guard g, (3) is immediate from Definition 4.1 (take $S = T = \varnothing$ and I a singleton). This shows the case for $g = \overline{x}y$ (with $x, y \in \mathbf{N} \cup \mathrm{New}$), since then g is a schematic guard. In the other case, $g = x(y)$ (with $x \in \mathbf{N} \cup \mathrm{New}$ and $y \in \mathbf{N}$). Recall from [1] that $x(y).S$ is an abbreviation of $x(-).\mathrm{inc}(S)[1/y]$. Now let $S \subseteq^{\mathrm{g}} T$. Then by applying (2) twice, $\mathrm{inc}(S)[1/y] \subseteq^{\mathrm{g}} \mathrm{inc}(T)[1/y]$. Since $x(-)$ is a schematic guard, we have $\{x(y).S\} = \{x(-).\mathrm{inc}(S)[1/y]\} \subseteq^{\mathrm{g}} \{x(-).\mathrm{inc}(T)[1/y]\} = \{x(y).T\}$, by Definition 4.1. □

As expected, we base nested multiset inclusion on nested containment of solutions.

Definition 4.4 For process terms P and Q, P is *nested multiset included* in Q, denoted $P \leq^{\mathrm{g}}_m Q$, if there exist solutions S and T such that $P \Rightarrow S$, $Q \Rightarrow T$, and $S \subseteq^{\mathrm{g}} T$.

Example 4.5 Let $R = \overline{x}z.\mathbf{0}$ and consider the process terms $P_1 = g.R$, $P_2 = g.(R \mid R)$, and $P_3 = g.R \mid g.R$, where g is an arbitrary guard over $\mathbf{N}$. Then $P_1 \Rightarrow S_1 = \{g.\{\overline{x}z.\varnothing\}\}$, $P_2 \Rightarrow S_2 = \{g.\{\overline{x}z.\varnothing, \overline{x}z.\varnothing\}\}$, and $P_3 \Rightarrow S_3 = \{g.\{\overline{x}z.\varnothing\}, g.\{\overline{x}z.\varnothing\}\}$. Hence $P_1 \leq^{\mathrm{g}}_m P_2$ and $P_1 \leq^{\mathrm{g}}_m P_3$, since $S_1 \subseteq^{\mathrm{g}} S_2$ and $S_1 \subseteq^{\mathrm{g}} S_3$ (even $S_1 \subseteq S_3$), but neither $P_2 \leq^{\mathrm{g}}_m P_3$, nor $P_3 \leq^{\mathrm{g}}_m P_2$. Note that $R \leq^{\mathrm{g}}_m P_i$ iff $g = \overline{x}z$, for every $i \in \{1, 2, 3\}$. □

The axiomatization of nested multiset inclusion is given next. Note the addition of a compatibility law **CGUA** for guarded process terms.

Definition 4.6 *Nested structural inclusion*, denoted $\leq^{\mathrm{g}}$, is the smallest relation on the set of process terms satisfying

$$\frac{}{\mathbf{0} \leq^{\mathrm{g}} P}\ \textbf{MIN} \qquad \frac{P \equiv Q}{P \leq^{\mathrm{g}} Q}\ \textbf{CGR} \qquad \frac{P \leq^{\mathrm{g}} R \quad R \leq^{\mathrm{g}} Q}{P \leq^{\mathrm{g}} Q}\ \textbf{TRA}$$

$$\frac{P \leq^{\mathrm{g}} Q}{P \mid R \leq^{\mathrm{g}} Q \mid R}\ \textbf{CCOM} \qquad \frac{P \leq^{\mathrm{g}} Q}{!\,P \leq^{\mathrm{g}}\ !\,Q}\ \textbf{CREP} \qquad \frac{P \leq^{\mathrm{g}} Q}{(\nu x)P \leq^{\mathrm{g}} (\nu x)Q}\ \textbf{CRES}$$

$$\frac{P \leq^{\mathrm{g}} Q}{g.P \leq^{\mathrm{g}} g.Q}\ \textbf{CGUA}$$

Hence, nested structural inclusion is the relation obtained from the axioms of structural inclusion (see Definition 2.5), with the additional law **CGUA**. It is also natural to consider the set of axioms of strong structural inclusion with **CGUA**, which we call strong nested structural inclusion.

Definition 4.7 *Strong nested structural inclusion*, denoted $\leq^{\mathrm{ng}}$, is the smallest relation on the set of process terms satisfying

$$\frac{}{\mathbf{0} \leq^{\mathrm{ng}} P}\ \textbf{MIN} \qquad \frac{P \equiv Q}{P \leq^{\mathrm{ng}} Q}\ \textbf{CGR} \qquad \frac{P \leq^{\mathrm{ng}} R \quad R \leq^{\mathrm{ng}} Q}{P \leq^{\mathrm{ng}} Q}\ \textbf{TRA}$$

$$\frac{P \leq^{\mathrm{ng}} Q}{P \mid R \leq^{\mathrm{ng}} Q \mid R}\ \textbf{CCOM} \qquad \frac{P \leq^{\mathrm{ng}} Q}{!\,P \leq^{\mathrm{ng}}\ !\,Q}\ \textbf{CREP}$$

$$\frac{P \leq^{\mathrm{ng}} Q}{g.P \leq^{\mathrm{ng}} g.Q}\ \textbf{CGUA}$$

The reader may note that for strong nested structural inclusion the situation is reversed: we are given an axiomatization and hope to find a natural and intuitively acceptable notion of containment that corresponds to it. Basing the inductive scheme of Definition 4.1 on $\subseteq^{\mathrm{n}}$ rather than on $\subseteq$, we obtain the strong version of nested containment.

Definition 4.8 *Strong nested containment*, denoted $\subseteq^{\mathrm{ng}}$, is the smallest relation on Sol such that

if $S \subseteq^{\mathrm{n}} T$ and $S_i \subseteq^{\mathrm{ng}} T_i$ for all $i \in I$,
then $S \cup \bigcup_{i \in I}\{g_i.S_i\} \subseteq^{\mathrm{ng}} T \cup \bigcup_{i \in I}\{g_i.T_i\}$,

where the g_i are schematic guards with $\mathrm{new}(g_i) = \varnothing$, and the $\mathrm{new}(T_i)$ are mutually disjoint and disjoint with $\mathrm{new}(T)$.

And as before, we base strong nested multiset inclusion on strong nested containment.

Definition 4.9 For process terms P and Q, P is *strongly nested multiset included* in Q, denoted $P \leq_m^{\mathrm{ng}} Q$, if there exist solutions S and T such that $P \Rightarrow S$, $Q \Rightarrow T$, and $S \subseteq^{\mathrm{ng}} T$.

Requiring the disconnectedness of the T_i and T in Definition 4.8 is necessary, since at the top level of recursion, we want $\subseteq^{\mathrm{ng}}$ and $\subseteq^{\mathrm{n}}$ to be equal, i.e., for all solutions V and W, if $V \subseteq W$, then $V \subseteq^{\mathrm{ng}} W$ iff $V \subseteq^{\mathrm{n}} W$ (take $S = \varnothing$, $T = \{g_1.\{g.\varnothing\}\}$ with $\mathrm{new}(g_1) = \varnothing$ and $\mathrm{new}(g) \neq \varnothing$, $S_1 = T_1 = \{g.\{\varnothing\}\}$, and $I = \{1\}$ in Definition 4.8 to produce a counterexample). Intuitively, the two requirements in Definition 4.8 are needed to prohibit the existence of "secret" links between molecules $\{g_i.T_i\}$. The case in which a g_j contains a new name is then a pathological one: since we forbid the existence of any "secret" link from $\{g_j.T_j\}$ to its "environment" $T \cup \bigcup_{i \in I - \{j\}} \{g_i.T_i\}$, any new name occurring in g_j becomes superfluous (in the next section, we will regard such "top-secret" molecules $g_j.T_j$ as atomic). Observe that in Definition 4.8, also the $g_i.S_i$ are mutually disconnected and disconnected from S (this can be shown formally using Lemma 4.11, below).

By an argument similar to the one below Definition 4.1, the reader easily verifies that $\subseteq^{\mathrm{ng}}$ is strictly weaker than $\subseteq^{\mathrm{n}}$. In turn, $\subseteq^{\mathrm{g}}$ is strictly weaker than $\subseteq^{\mathrm{ng}}$, since $\subseteq^{\mathrm{n}}$ implies $\subseteq$. Note that $\subseteq^{\mathrm{ng}}$ is reflexive (by reflexivity of $\subseteq^{\mathrm{n}}$). By Example 4.2 (letting $\mathrm{new}(a) = \mathrm{new}(b) = \mathrm{new}(c) = \varnothing$), $\subseteq^{\mathrm{ng}}$ is not antisymmetrical. Transitivity of $\subseteq^{\mathrm{ng}}$ is shown in Theorem 4.13. Thus, as $\subseteq^{\mathrm{g}}$, $\subseteq^{\mathrm{ng}}$ is a preorder.

At this point we have completed the inclusion diagram of Fig. 2 below. Note that $\subseteq$ and $\subseteq^{\mathrm{ng}}$ are incomparable: in Example 2.3, $S_1 \subseteq S_2$, but $S_1 \not\subseteq^{\mathrm{ng}} S_2$. On the other hand, in the above example, $\{g_1.\varnothing\} \subseteq^{\mathrm{ng}} T$, but $\{g_1.\varnothing\} \not\subseteq T$.

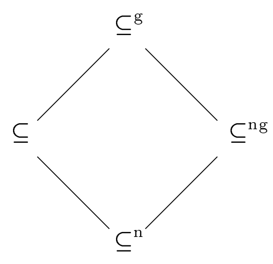

Figure 2: Inclusion diagram for $\subseteq^{\mathrm{n}}$, $\subseteq$, $\subseteq^{\mathrm{ng}}$ and $\subseteq^{\mathrm{g}}$

The reader may object to Definition 4.8 of strong nested containment in arguing that it has too many side conditions to be a natural and intuitively acceptable notion of containment. Indeed, since, unlike the other three notions of containment, it was 'constructed to fit its axiomatization', it does not seem to satisfy the first of the four minimal constraints of the Introduction. Unfortunately, it does not preserve communications either, or, to put it differently, it is not a strong simulation (as will be shown in Example 7.5 of Section 7).

We restate Lemma 4.3 for strong nested containment. Note that in (1), the T_i are assumed disconnected, and in (2), the mapping f is injective (as in Lemma 3.2). Also note that the guard in (4) does not contain any new names.

Lemma 4.10 *For all solutions S, T, S_i and T_i, $i \in I$,*

(1) if $S_i \subseteq^{\mathrm{ng}} T_i$ for every $i \in I$, and the $\mathrm{new}(T_i)$ are mutually disjoint, then $\bigcup_{i\in I} S_i \subseteq^{\mathrm{ng}} \bigcup_{i\in I} T_i$,

(2) if $S \subseteq^{\mathrm{ng}} T$, then for every injection $f : \mathrm{new}(T) \to \mathrm{New}$, $f(S) \subseteq^{\mathrm{ng}} f(T)$,

(3) if $S \subseteq^{\mathrm{ng}} T$, then for every mapping $f : \mathbf{N} \cup \mathbb{N}_+ \to \mathbf{N} \cup \mathbb{N}_+$, $f(S) \subseteq^{\mathrm{ng}} f(T)$, and

(4) if $S \subseteq^{\mathrm{ng}} T$, then for every guard g with $\mathrm{new}(g) = \varnothing$, $\{g.S\} \subseteq^{\mathrm{ng}} \{g.T\}$.

Proof We use the same proof scheme as in the proof of Lemma 4.3(1) to show (1). Now $\bigcup_{i\in I} S_i \subseteq^{\mathrm{ng}} \bigcup_{i\in I} T_i$, since $\bigcup_{i\in I} V_i \subseteq^{\mathrm{n}} \bigcup_{i\in I} W_i$, by Lemma 3.2(1). The other conditions hold by assumption. We show (2) by induction on the definition of $\subseteq^{\mathrm{ng}}$, cf. the proof of Lemma 4.3(2). Let $S = V \cup \bigcup_{i\in I}\{g_i.S_i\}$ and $T = W \cup \bigcup_{i\in I}\{g_i.T_i\}$, where $V \subseteq^{\mathrm{n}} W$, $S_i \subseteq^{\mathrm{ng}} T_i$, the $\mathrm{new}(T_i)$ are mutually disjoint and disjoint with $\mathrm{new}(W)$, and $\mathrm{new}(g_i) = \varnothing$. Then, by induction, $f(S_i) \subseteq^{\mathrm{ng}} f(T_i)$ for all $i \in I$. Hence $f(S) = f(V) \cup \bigcup_{i\in I}\{f(g_i).f(S_i)\} \subseteq^{\mathrm{ng}} f(W) \cup \bigcup_{i\in I}\{f(g_i).f(T_i)\} = f(T)$, since $f(V) \subseteq^{\mathrm{n}} f(W)$ by Lemma 3.2(2), and the $f(g_i)$ do not contain any names from New (because, in fact, $f(g_i) = g_i$). Moreover, the $\mathrm{new}(f(T_i))$ are mutually disjoint, since for every $i_1, i_2 \in I$ with $i_1 \neq i_2$, $\mathrm{new}(f(T_{i_1})) \cap \mathrm{new}(f(T_{i_2})) = f(\mathrm{new}(T_{i_1})) \cap f(\mathrm{new}(T_{i_2})) = f(\mathrm{new}(T_{i_1}) \cap \mathrm{new}(T_{i_2})) = \varnothing$. By a similar argument, the $\mathrm{new}(f(T_i))$ are disjoint with $\mathrm{new}(f(W))$. By an inductive proof similar to that of (2), (3) can be shown using Lemma 3.2(3). The proof of (4) is similar to the proof of Lemma 4.3(3), using Definition 4.8 and (3) (note that inc and $[1/y]$ are both mappings $\mathbf{N} \cup \mathbb{N}_+ \to \mathbf{N} \cup \mathbb{N}_+$). $\square$

In the proof of soundness and completeness of each of the four inclusion relations that we have defined, we need some properties that hold for each of the four corresponding containment relations (we will give soundness and completeness proofs for all four simultaneously in Section 6). The first was already used in Section 2 and 3, for $\subseteq$ and $\subseteq^{\mathrm{n}}$, respectively, and we will use it hereafter without explicit mentioning. As a convenient way to unify such properties into single lemmas (instead of four for each containment relation), we use the letter $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$ in roman font, and the meta inclusion relation $\subseteq^{\mathrm{x}}$ to index $\subseteq^{\mathrm{n}}$, $\subseteq$, $\subseteq^{\mathrm{ng}}$, and $\subseteq^{\mathrm{g}}$, respectively.

Lemma 4.11 *Let $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$. If $S \subseteq^{\mathrm{x}} T$, then $\mathrm{new}(S) \subseteq \mathrm{new}(T)$.*

Proof Recall that the case $\mathrm{x} = \lambda$ was shown in Section 2. Note that by the inclusion diagram in Fig. 2, it suffices to show $\mathrm{new}(S) \subseteq \mathrm{new}(T)$ for $S \subseteq^{\mathrm{g}} T$; we prove this by induction on the definition of $\subseteq^{\mathrm{g}}$. Let $S = V \cup \bigcup_{i \in I}\{g_i.S_i\} \subseteq^{\mathrm{g}} W \cup \bigcup_{i \in I}\{g_i.T_i\}$ with $V \subseteq W$ and $S_i \subseteq^{\mathrm{g}} T_i$ for all $i \in I$. Note that $\mathrm{new}(V) \subseteq \mathrm{new}(W)$ by the comment above. By induction, $\mathrm{new}(S_i) \subseteq \mathrm{new}(T_i)$. Then

$$\begin{aligned}
\mathrm{new}(S) &= \mathrm{new}(V) \cup \mathrm{new}(\bigcup_{i \in I}\{g_i.S_i\}) \\
&= \mathrm{new}(V) \cup \bigcup_{i \in I} \mathrm{new}(\{g_i.S_i\}) \\
&\subseteq \mathrm{new}(W) \cup \bigcup_{i \in I} \mathrm{new}(\{g_i.T_i\}) \\
&= \mathrm{new}(T).
\end{aligned}$$

□

We now prove that $\subseteq^{\mathrm{g}}$ and $\subseteq^{\mathrm{ng}}$ are transitive (and hence they are preorder relations). To show this, we need the next lemma, which expresses that the composition of $\subseteq^{\mathrm{g}}$ and $\subseteq$ (in any order) yields $\subseteq^{\mathrm{g}}$, and the composition of $\subseteq^{\mathrm{ng}}$ and $\subseteq^{\mathrm{n}}$ (in any order) yields $\subseteq^{\mathrm{ng}}$.

Lemma 4.12 *Let* $\mathrm{x} \in \{\lambda, \mathrm{n}\}$. *For all solutions* S, U, *and* T,

(1) if $S \subseteq^{\mathrm{xg}} U \subseteq^{\mathrm{x}} T$, *then* $S \subseteq^{\mathrm{xg}} T$, *and*

(2) if $S \subseteq^{\mathrm{x}} U \subseteq^{\mathrm{xg}} T$, *then* $S \subseteq^{\mathrm{xg}} T$.

Proof We consider the two cases for x:

(i) ($\mathrm{x} = \lambda$). To prove (1), let $S = V \cup \bigcup_{i \in I}\{g_i.S_i\} \subseteq^{\mathrm{g}} W \cup \bigcup_{i \in I}\{g_i.U_i\} = U$, with $V \subseteq W$ and $S_i \subseteq^{\mathrm{g}} U_i$ for all $i \in I$, and let $U \cup U' = T$. Then $S \subseteq^{\mathrm{g}} T$, since $V \subseteq W \cup U'$. To prove (2), let $S \cup S' = U = V \cup \bigcup_{i \in I}\{g_i.U_i\} \subseteq^{\mathrm{g}} W \cup \bigcup_{i \in I}\{g_i.T_i\} = T$, with $V \subseteq W$ and $U_i \subseteq^{\mathrm{g}} T_i$ for all $i \in I$. By Lemma 6 of [2], there exist solutions S_1 and S_2 such that $S = S_1 \cup S_2$, $S_1 \subseteq V$, and $S_2 \subseteq \bigcup_{i \in I}\{g_i.U_i\}$. By Lemma 5 of [2], $S_2 = \bigcup_{i \in I'}\{g_i.U_i\}$ for a subset I' of I. Hence $S = S_1 \cup \bigcup_{i \in I'}\{g_i.U_i\} \subseteq^{\mathrm{g}} (W \cup \bigcup_{i \in I - I'}\{g_i.T_i\}) \cup \bigcup_{i \in I'}\{g_i.T_i\} = T$, since obviously $S_1 \subseteq V \subseteq W \subseteq W \cup \bigcup_{i \in I - I'}\{g_i.T_i\}$.

(ii) ($\mathrm{x} = \mathrm{n}$). We use the same proof-scheme as in (i). To prove (1), let $S = V \cup \bigcup_{i \in I}\{g_i.S_i\} \subseteq^{\mathrm{ng}} W \cup \bigcup_{i \in I}\{g_i.U_i\} = U$, with $V \subseteq^{\mathrm{n}} W$, the $\mathrm{new}(U_i)$ are mutually disjoint and disjoint with $\mathrm{new}(W)$, and $S_i \subseteq^{\mathrm{ng}} U_i$ for all $i \in I$. Let $U \cup U' = T$ with $\mathrm{new}(U) \cap \mathrm{new}(U') = \varnothing$. Note that this implies that $\mathrm{new}(W) \cap \mathrm{new}(U') = \varnothing$, since U contains W. Now $S \subseteq^{\mathrm{g}} T$, since $V \subseteq^{\mathrm{n}} W \cup U'$. To prove (2), let $S \cup S' = U = V \cup \bigcup_{i \in I}\{g_i.U_i\} \subseteq^{\mathrm{ng}} W \cup \bigcup_{i \in I}\{g_i.T_i\} = T$, with $\mathrm{new}(S) \cap \mathrm{new}(S') = \varnothing$, $V \subseteq^{\mathrm{n}} W$, $\mathrm{new}(g_i) = \varnothing$, the $\mathrm{new}(T_i)$ are mutually disjoint and disjoint with $\mathrm{new}(W)$, and $U_i \subseteq^{\mathrm{ng}} T_i$ for all $i \in I$. Similar to the proof of (i), $S = S_1 \cup S_2$, $S_1 \subseteq^{\mathrm{n}} V$, and $S_2 \subseteq^{\mathrm{n}} \bigcup_{i\in I}\{g_i.U_i\}$ (since $\mathrm{new}(S)$ is disjoint with $\mathrm{new}(S')$). Hence, as in (i), $S_2 = \bigcup_{i\in I'}\{g_i.U_i\}$ for a subset I' of I. Thus $S = S_1 \cup \bigcup_{i\in I'}\{g_i.U_i\} \subseteq^{\mathrm{ng}} (W \cup \bigcup_{i\in I-I'}\{g_i.T_i\}) \cup \bigcup_{i\in I'}\{g_i.T_i\} = T$, since $S_1 \subseteq^{\mathrm{n}} V \subseteq^{\mathrm{n}} W \subseteq^{\mathrm{n}} W \cup \bigcup_{i\in I-I'}\{g_i.T_i\}$.

□

Theorem 4.13 *Let* $\mathrm{x} \in \{\mathrm{g}, \mathrm{ng}\}$. $\subseteq^{\mathrm{x}}$ *is transitive.*

Proof We consider the two cases for x:

(i) ($\mathrm{x} = \mathrm{g}$). We prove by induction on the definition of $S \subseteq^{\mathrm{g}} U$: if $S \subseteq^{\mathrm{g}} U \subseteq^{\mathrm{g}} T$, then $S \subseteq^{\mathrm{g}} T$. Let $S = V \cup \bigcup_{i\in I}\{g_i.S_i\} \subseteq^{\mathrm{g}} X \cup \bigcup_{i\in I}\{g_i.X_i\} = U$, where $S_i \subseteq^{\mathrm{g}} X_i$ for all $i \in I$, and $V \subseteq X$. Let also $U = Y \cup \bigcup_{j\in J}\{h_j.Y_j\} \subseteq^{\mathrm{g}} W \cup \bigcup_{j\in J}\{h_j.T_j\} = T$ with similar conditions. By Lemma 6 of [2], there exist solutions $Z_{1,1}, Z_{1,2}, Z_{2,1}, Z_{2,2}$, with $X = Z_{1,1} \cup Z_{1,2}$, $Y = Z_{1,1} \cup Z_{2,1}$, $\bigcup_{i\in I}\{g_i.X_i\} = Z_{2,1} \cup Z_{2,2}$, and $\bigcup_{j\in J}\{h_j.Y_j\} = Z_{1,2} \cup Z_{2,2}$. Hence by Lemma 5 of [2], there exist partitions I_1, I_2, and J_1, J_2 of I and J respectively, such that $\bigcup_{i\in I_1}\{g_i.X_i\} = Z_{2,1}$, $\bigcup_{j\in J_1}\{h_j.Y_j\} = Z_{1,2}$, and $\bigcup_{i\in I_2}\{g_i.X_i\} = \bigcup_{j\in J_2}\{h_j.Y_j\} = Z_{2,2}$. By Lemma 4 of [2], this implies that $g_i.X_i = h_{\psi(i)}.Y_{\psi(i)}$, for all $i \in I_2$, for some bijection $\psi : I_2 \to J_2$. Note that $S_i \subseteq^{\mathrm{g}} X_i = Y_{\psi(i)} \subseteq^{\mathrm{g}} T_{\psi(i)}$ for all $i \in I_2$, and so $S_i \subseteq^{\mathrm{g}} T_{\psi(i)}$, by induction. Now the reader can verify that

$$\begin{array}{rcl}
S & = & V \cup \bigcup\limits_{i\in I}\{g_i.S_i\} \\
& \subseteq & Z_{1,1} \cup Z_{1,2} \cup \bigcup\limits_{i\in I}\{g_i.S_i\} \\
& = & Z_{1,1} \cup \bigcup\limits_{j\in J_1}\{h_j.Y_j\} \cup \bigcup\limits_{i\in I_1}\{g_i.S_i\} \cup \bigcup\limits_{i\in I_2}\{g_i.S_i\} \\
& \subseteq^{\mathrm{g}} & Z_{1,1} \cup \bigcup\limits_{j\in J_1}\{h_j.T_j\} \cup \bigcup\limits_{i\in I_1}\{g_i.X_i\} \cup \bigcup\limits_{i\in I_2}\{h_{\psi(i)}.T_{\psi(i)}\} \\
& = & Z_{1,1} \cup Z_{2,1} \cup \bigcup\limits_{j\in J_1}\{h_j.T_j\} \cup \bigcup\limits_{j\in J_2}\{h_j.T_j\} \\
& \subseteq & W \cup \bigcup\limits_{j\in J}\{h_j.T_j\} = T,
\end{array}$$

so we have $S \subseteq^{\mathrm{g}} T$, by Lemma 4.12.

(ii) ($\mathrm{x} = \mathrm{ng}$). We use the same proof-scheme as in (i), only with the following conditions: $S_i \subseteq^{\mathrm{ng}} X_i$ for all $i \in I$, $V \subseteq^{\mathrm{n}} X$, $\mathrm{new}(g_i) = \varnothing$, and the $\mathrm{new}(X_i)$, $\mathrm{new}(X)$ are mutually disjoint. Similar conditions are assumed for the Y, Y_j, W, and T_j. Now, $\mathrm{new}(Z_{1,1})$ and $\mathrm{new}(Z_{2,1})$ are disjoint (because $\mathrm{new}(X) \cap \mathrm{new}(\bigcup_{i\in I}\{g_i.X_i\}) = \varnothing$), and $\mathrm{new}(Z_{1,1} \cup Z_{2,1})$ is disjoint with $\mathrm{new}(\bigcup_{j\in J}\{h_j.T_j\})$ (because the latter is disjoint with $\mathrm{new}(W)$, and $Z_{1,1}\cup Z_{2,1}=Y\subseteq W$). Also, by induction we may conclude that $S_i\subseteq^{\mathrm{ng}} T_{\psi(i)}$ for all $i\in I_2$. Hence

$$\begin{aligned} S &\subseteq^{\mathrm{n}} Z_{1,1}\cup Z_{1,2}\cup\bigcup_{i\in I}\{g_i.S_i\} \\ &\subseteq^{\mathrm{ng}} Z_{1,1}\cup\bigcup_{j\in J_1}\{h_j.T_j\}\cup\bigcup_{i\in I_1}\{g_i.X_i\}\cup\bigcup_{i\in I_2}\{h_{\psi(i)}.T_{\psi(i)}\} \\ &\subseteq^{\mathrm{n}} W\cup\bigcup_{j\in J}\{h_j.T_j\}=T, \end{aligned}$$

so $S\subseteq^{\mathrm{ng}} T$, by Lemma 4.12.

□

We conclude this section with a theorem that characterizes the subdivision of two unions of families of solutions, of which the one is contained in the other (and the largest family is disconnected). Lemma 6 of [2] states a similar result for multiset equality.

Theorem 4.14 *Let T_j, $j\in J$, be solutions such that the* $\mathrm{new}(T_j)$ *are mutually disjoint. For every* $\mathrm{x}\in\{\mathrm{n},\lambda,\mathrm{ng},\mathrm{g}\}$,

> $\bigcup_{i\in I} S_i\subseteq^{\mathrm{x}}\bigcup_{j\in J} T_j$ *if and only if there exist solutions $U_{i,j}$ such that $S_i=\bigcup_{j\in J} U_{i,j}$ and $\bigcup_{i\in I} U_{i,j}\subseteq^{\mathrm{x}} T_j$, for every $i\in I$ and $j\in J$.*

If the $\mathrm{new}(S_i)$ *are mutually disjoint, then the* $\mathrm{new}(U_{i,j})$ *are mutually disjoint.*

Proof The if-parts for each $\mathrm{x}\in\{\mathrm{n},\lambda,\mathrm{ng},\mathrm{g}\}$ can be easily derived from Lemmas 3.2(1), 2.1(1), 4.10(1), and 4.3(1), respectively. We show the only-if parts by considering the four cases for x:

(i) ($\mathrm{x}=\lambda$). This is a special case of Lemma 6 of [2]: Let $(\bigcup_{i\in I} S_i)\cup S'=\bigcup_{j\in J} T_j$. By Lemma 6 of [2], there exist $U_{i,j}$ and U_j such that $S_i=\bigcup_{j\in J} U_{i,j}$ for every $i\in I$, $S'=\bigcup_{j\in J} U_j$, and $T_j=(\bigcup_{i\in I} U_{i,j})\cup U_j$ for every $j\in J$. Hence $\bigcup_{i\in I} U_{i,j}\subseteq T_j$ for every $j\in J$.

(ii) ($\mathrm{x}=\mathrm{n}$). We use the same proof-scheme as in (i), with the additional condition that $\mathrm{new}(S')$ is disjoint with $\mathrm{new}(\bigcup_{i\in I} S_i)$. Now $\mathrm{new}(U_j)\cap\mathrm{new}(U_{i,j})=\varnothing$, for all $j\in J$ and $i\in I$, and hence $\bigcup_{i\in I} U_{i,j}\subseteq^{\mathrm{n}} T_j$.

(iii) ($\mathrm{x}=\mathrm{g}$). Let $\bigcup_{i\in I} S_i=V\cup\bigcup_{k\in K}\{g_k.V_k\}$ and let $\bigcup_{j\in J} T_j=W\cup\bigcup_{k\in K}\{g_k.W_k\}$ such that $V\subseteq W$ and $V_k\subseteq^{\mathrm{g}} W_k$ for all $k\in K$. By Lemma 6 and Lemma 5 of [2] respectively, there exist V_i' and mutually disjoint sets K_i, such that $K=\bigcup_{i\in I} K_i$, $V=\bigcup_{i\in I} V_i'$, and $S_i=V_i'\cup\bigcup_{k\in K_i}\{g_k.V_k\}$ for every $i\in I$. Let $X_i=V_i'\cup\bigcup_{k\in K_i}\{g_k.W_k\}$. Clearly $\bigcup_{i\in I} X_i\subseteq\bigcup_{j\in J} T_j$. By (i) there exist $Y_{i,j}$ such that $X_i=\bigcup_{j\in J} Y_{i,j}$ and $\bigcup_{i\in I} Y_{i,j}\subseteq T_j$

for all $i \in I$ and $j \in J$. Thus, $\bigcup_{j\in J} Y_{i,j} = V_i' \cup \bigcup_{k\in K_i}\{g_k.W_k\}$ for all $i \in I$. Hence by Lemma 6 and Lemma 5 of [2] respectively, there exist $V_{i,j}'$ and mutually disjoint sets $K_{i,j}$ such that $K_i = \bigcup_{j\in J} K_{i,j}$, $V_i' = \bigcup_{j\in J} V_{i,j}'$ for all $i \in I$, and $Y_{i,j} = V_{i,j}' \cup \bigcup_{k\in K_{i,j}}\{g_k.W_k\}$ for all $i \in I$ and $j \in J$. Let $U_{i,j} = V_{i,j}' \cup \bigcup_{k\in K_{i,j}}\{g_k.V_k\}$. Then $\bigcup_{j\in J} U_{i,j} = \bigcup_{j\in J} V_{i,j}' \cup \bigcup_{j\in J}(\bigcup_{k\in K_{i,j}}\{g_k.V_k\}) = S_i$, and $\bigcup_{i\in I} U_{i,j} \subseteq^{\mathrm{g}} \bigcup_{i\in I} Y_{i,j} \subseteq T_j$ by Definition 4.1 and Lemma 4.3(1). Hence $\bigcup_{i\in I} U_{i,j} \subseteq^{\mathrm{g}} T_j$ by Theorem 4.13.

(iv) (x = ng). We use the same proof-scheme as in (iii), with the additional conditions that $V_k \subseteq^{\mathrm{ng}} W_k$ for all $k \in K$, $V \subseteq^{\mathrm{n}} W$, the new($W$) and new($W_k$) are disjoint, and new($g_k$) $= \varnothing$. Now $\bigcup_{i\in I} X_i \subseteq^{\mathrm{n}} \bigcup_{j\in J} T_j$, since $\bigcup_{i\in I} V_i' \subseteq^{\mathrm{n}} W$ and new(W) is disjoint with new($\bigcup_{k\in K}\{g_k.W_k\}$). By (ii) there exist $Y_{i,j}$ such that $X_i = \bigcup_{j\in J} Y_{i,j}$ and $\bigcup_{i\in I} Y_{i,j} \subseteq^{\mathrm{n}} T_j$. Now $\bigcup_{i\in I} U_{i,j} \subseteq^{\mathrm{ng}} \bigcup_{i\in I} Y_{i,j} \subseteq^{\mathrm{n}} T_j$ by Definition 4.8 and Lemma 4.10(1).

Furthermore, in each case the new($U_{i,j}$) are easily shown to be mutually disjoint. Consider U_{i_1,j_1} and U_{i_2,j_2} with $i_1 \neq i_2$ or $j_1 \neq j_2$. Since $S_i = \bigcup_{j\in J} U_{i,j}$ and the new(S_i) are assumed to be mutually disjoint, $i_1 \neq i_2$ implies new(U_{i_1,j_1}) $\cap$ new(U_{i_2,j_2}) $= \varnothing$. Moreover, since $\bigcup_{i\in I} U_{i,j} \subseteq^{\mathrm{x}} T_j$, by Lemma 4.11 we have new($U_{i,j}$) $\subseteq$ new(T_j) for every $i \in I$ and $j \in J$. Hence, since the new(T_j) are mutually disjoint, $j_1 \neq j_2$ implies new(U_{i_1,j_1}) $\cap$ new(U_{i_2,j_2}) $= \varnothing$. $\square$

5 Topconnected Process Terms

After defining strong nested multiset containment in the previous section, we expressed the need for a new kind of atomicity of solutions. In this section, we look at solutions that are both connected and *top-secret*, the latter meaning that we exclude singleton solutions $\{g.S\}$ with new(g) $= \varnothing$. We show that this gives rise to a normal form on processes that is stronger than the normal form of subconnected processes in Lemma 18 of [2].

Intuitively, the normal form of subconnected process terms was devised to guarantee that restrictions and replications appearing in such terms, were nested as deeply as possible. Constructing a subconnected process term equivalent to an arbitrary other process term, this (among others) gave a direction to structural congruence law (2.3), using it 'from left to right', but law (2.4) was not considered. In this section we show that for the latter as well, there exists a natural direction, viz. also from left to right, nesting restrictions even more deeply. As an example, let $P = (\nu x)(g.\mathbf{0} \mid g.(\overline{x}z.\mathbf{0} \mid x(y).\mathbf{0} \mid g.\mathbf{0}))$, where g is a guard not containing x. Using law (2.3) of structural congruence from left to right, $P \equiv g.\mathbf{0} \mid (\nu x)g.(\overline{x}z.\mathbf{0} \mid x(y).\mathbf{0} \mid g.\mathbf{0})$, the latter process term being subconnected (as opposed to P). However, x can 'break through the guard g', using (2.4) and obtaining the process term $g.\mathbf{0} \mid g.(\nu x)(\overline{x}z.\mathbf{0} \mid x(y).\mathbf{0} \mid g.\mathbf{0})$ equivalent to P. Finally, once again by (2.3), this process term is equivalent to the subconnected $g.\mathbf{0} \mid g.((\nu x)(\overline{x}z.\mathbf{0} \mid x(y).\mathbf{0}) \mid g.\mathbf{0})$, of which the restriction (νx) cannot be moved inwards any further. This normal form is defined below.

Definition 5.1 A solution S is *top-secret*, if $S = \{g.S'\}$ implies that $\mathrm{new}(g) \neq \varnothing$. A process term P is top-secret, if $P \Rightarrow S$ and S is top-secret. A process term P is *topconnected*, if P is subconnected and each subterm $(\nu x)Q$ of P is top-secret.

Observe that every non-singleton solution (i.e., a solution $S \neq \{g.S'\}$) is top-secret.

We use the next lemma as one of the cases (the most difficult one, to be exact) in the inductive proof of Lemma 5.3, in which we show that Definition 5.1 indeed defines a normal form on process terms. The reader may note that it is the lemma below that gives direction to law (2.4) of structural congruence.

Lemma 5.2 *For every topconnected process term P and every $x \in \mathbf{N}$, a topconnected process term P' can be computed such that $(\nu x)P \equiv P'$.*

Proof The proof is by induction on the number of guards in P. Note that by the definition of a subconnected process term, $P = \mathbf{0}$ if P does not contain any guards; so let $P' = \mathbf{0}$ in this case (cf. structural law (2.2)).

Now assume, using structural congruence laws (1.2) and (1.3) only, that $P \equiv Q_1 \mid \cdots \mid Q_l \mid R_1 \mid \cdots \mid R_k$, where $x \notin \mathrm{fn}(Q_j)$, $x \in \mathrm{fn}(R_i)$, and the Q_j and R_i are not parallel compositions. If $k \neq 1$, or $k = 1$ and R_1 is not a guarded process term, then let $P' = Q_1 \mid \cdots \mid Q_l \mid (\nu x)(R_1 \mid \cdots \mid R_k) \equiv (\nu x)P$, using structural congruence law (2.3). Note that $(\nu x)(R_1 \mid \cdots \mid R_k)$ is connected by Lemma 17 of [2]. In both cases it is also top-secret: let $R_i \Rightarrow V_i$ for every $1 \leq i \leq k$. Note that by assumption, and Lemma 20 of [2], $V_i \neq \varnothing$. Now $(\nu x)(R_1 \mid \cdots \mid R_k) \Rightarrow V = \bigcup_{1 \leq i \leq k} V_i[n/x]$ for an appropriate $n \in \mathrm{New}$. If $k \neq 1$, then V contains at least two molecules and hence is top-secret. In the other case, $k = 1$ and R_1 is a replication $!R$ or a restriction $(\nu y)R$. Since R is non-zero, $(\nu x)!R$ corresponds to a solution with infinitely many molecules and hence is top-secret. Furthermore, since we assumed P to be topconnected, $(\nu y)R$ is top-secret, which means that $\mathrm{new}(g) \neq \varnothing$, if $(\nu y)R \Rightarrow \{g.U\}$. Hence $\mathrm{new}(g[n/x]) \neq \varnothing$, if $(\nu x)(\nu y)R \Rightarrow \{g.U\}[n/x] = \{g[n/x].U[n/x]\}$, so $(\nu x)(\nu y)R$ is top-secret. Hence, in both cases, P' is topconnected.

It remains to consider the case that $k = 1$ and R_1 is a guarded process term $g.R$. We consider three cases: first assume x occurs free in g. Then $(\nu x)R_1$ is topconnected, and hence the above P' will do in this case also. Next, assume x occurs bound in g. Now we can α-convert $g.R$ to $\tilde{g}.\tilde{R}$, where x does not occur in $\tilde{g}$. Moreover, as the reader can easily check, $\tilde{g}.\tilde{R}$ is topconnected. The proof now proceeds as in the last case: assume x does not occur in g. Then, by an application of structural congruence law (2.4), $(\nu x)g.R \equiv g.(\nu x)R$. By induction, there exists topconnected R' such that $(\nu x)R \equiv R'$. Hence $(\nu x)R_1 \equiv g.R'$, and obviously, $g.R'$ is topconnected. Now let $P' = Q_1 \mid \cdots \mid Q_m \mid g.R'$. $\square$

Lemma 5.3 *For a process term P, a topconnected process term P' can be computed such that $P \equiv P'$.*

Proof We compute P' by induction on the syntactical structure of P. We claim that for $P = \mathbf{0}$, $P = P_1 \mid P_2$, $P = g.P_1$ and $P = \,!\,P_1$, the proof is similar to the proof of Lemma 18 of [2]. Let $P = (\nu x)P_1$. By induction, a topconnected P_1' has been computed such that $P_1 \equiv P_1'$. Hence $P \equiv (\nu x)P_1'$. By Lemma 5.2, a topconnected P' can be computed such that $(\nu x)P_1' \equiv P'$. Hence $P \equiv P'$. □

6 Soundness and Completeness

We finally turn to the proofs of soundness and completeness of each of the four inclusion relations in this paper. Simultaneously we show their decidability. These results were already shown for strong structural inclusion in Section 3, by proofs easier than those in this section. However, for uniformity reasons, we decided to include them.

As we prove the above results for each of the four simultaneously, they must have certain properties in common. Indeed, for the containment relations, some of those were already stated in Section 4. The next lemma is a generalization of Lemmas 2.4 and 3.6 for $\leq^{\mathrm{x}}$, for every $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$, and is proven similarly. Note that we use the meta relation $\leq^{\mathrm{x}}$ to index $\leq^{\mathrm{n}}$, $\leq$, $\leq^{\mathrm{ng}}$, and $\leq^{\mathrm{g}}$, for each $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$, respectively. In combination with $\subseteq^{\mathrm{x}}$ and $\leq^{\mathrm{x}}$, we use $\leq^{\mathrm{x}}_m$ to index each of the four corresponding multiset inclusion relations. Hence in subsequent proofs it is understood that when, for example, $\leq^{\mathrm{x}}$ is under consideration for $\mathrm{x} = \mathrm{ng}$, we assume $\subseteq^{\mathrm{x}}$ to denote $\subseteq^{\mathrm{ng}}$, and $\leq^{\mathrm{x}}_m$ to denote $\leq^{\mathrm{ng}}_m$.

Lemma 6.1 *Let $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$. For process terms P and Q, if $P \leq^{\mathrm{x}}_m Q$ and $Q \Rightarrow T$, for a solution T, then there exists a solution S such that $P \Rightarrow S$ and $S \subseteq^{\mathrm{x}} T$. Conversely, if $P \leq^{\mathrm{x}}_m Q$ and $P \Rightarrow S$, for a solution S, then there exists a solution T such that $Q \Rightarrow T$ and $S \subseteq^{\mathrm{x}} T$.*

Proof Similar to the proof of Lemma 2.4, using Lemma 4.11, and Lemmas 3.2(2), 2.1(2), 4.10(2), and 4.3(2), respectively. □

Next, we show that $\leq^{\mathrm{n}}$, $\leq$, $\leq^{\mathrm{g}}$, and $\leq^{\mathrm{ng}}$ are sound with respect to their multiset counterparts $\leq^{\mathrm{n}}_m$, $\leq_m$, $\leq^{\mathrm{g}}_m$, and $\leq^{\mathrm{ng}}_m$, respectively. It turns out that the proof mainly relies on the basic properties of the corresponding multiset containment relations as stated in Lemmas 3.2, 2.1, 4.10, and 4.3, respectively.

Lemma 6.2 *Let $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$. For all process terms P and Q, if $P \leq^{\mathrm{x}} Q$, then $P \leq^{\mathrm{x}}_m Q$.*

Proof We will prove that each $\leq^{\mathrm{x}}_m$ satisfies the corresponding laws of Definitions 3.7, 2.5, 4.7, and 4.6, respectively. Since for each x, $\leq^{\mathrm{x}}$ is the smallest relation satisfying the laws in these definitions, clearly $P \leq^{\mathrm{x}} Q$ implies $P \leq^{\mathrm{x}}_m Q$.

Now each $\leq_m^{\mathrm{x}}$ satisfies **MIN**, since $\varnothing \subseteq^{\mathrm{n}} S$ for every solution S. The proof of **CGR** relies on reflexivity of each $\subseteq^{\mathrm{x}}$, and uses Theorem 33 of [2]. To prove that $\leq_m^{\mathrm{x}}$ satisfies **TRA**, assume $P \leq_m^{\mathrm{x}} R$ and $R \leq_m^{\mathrm{x}} Q$. Let V and T be solutions with $R \Rightarrow V$, $Q \Rightarrow T$, and $V \subseteq^{\mathrm{x}} T$. By Lemma 6.1 there exists a solution S with $P \Rightarrow S$ and $S \subseteq^{\mathrm{x}} V$. Since each $\subseteq^{\mathrm{x}}$ is transitive (see Theorem 4.13 for transitivity of $\subseteq^{\mathrm{g}}$ and $\subseteq^{\mathrm{ng}}$), we conclude $S \subseteq^{\mathrm{x}} T$ and hence $P \leq_m^{\mathrm{x}} Q$. To prove that $\leq_m^{\mathrm{x}}$ satisfies **CCOM**, assume $P \leq_m^{\mathrm{x}} Q$. Let T and V be solutions with $Q \mid R \Rightarrow T \cup V$, $Q \Rightarrow T$, $R \Rightarrow V$, and $\mathrm{new}(T) \cap \mathrm{new}(V) = \varnothing$. By Lemma 6.1 there is a solution S with $P \Rightarrow S$ and $S \subseteq^{\mathrm{x}} T$. Since $\mathrm{new}(S) \cap \mathrm{new}(V) = \varnothing$ by Lemma 4.11, $P \mid R \Rightarrow S \cup V$. Moreover, by Lemmas 3.2(1), 2.1(1), 4.10(1), and 4.3(1), we have $S \cup V \subseteq^{\mathrm{x}} T \cup V$, and thus $P \mid R \leq_m^{\mathrm{x}} Q \mid R$. The proof of **CREP** is similar. We show that $\leq_m^{\mathrm{x}}$ satisfies **CRES** for $\mathrm{x} \in \{\lambda, \mathrm{g}\}$. Let $P \leq_m^{\mathrm{x}} Q$. Then $S \subseteq^{\mathrm{x}} T$, where $P \Rightarrow S$ and $Q \Rightarrow T$. Hence $(\nu x)P \Rightarrow S[n/x]$ and $(\nu x)Q \Rightarrow T[n/x]$, for some $n \in \mathrm{New} - (\mathrm{new}(S) \cup \mathrm{new}(T))$. By Lemmas 2.1(2) and 4.3(2) respectively, we have $S[n/x] \subseteq^{\mathrm{x}} T[n/x]$. Hence $(\nu x)P \leq_m^{\mathrm{x}} (\nu x)Q$. Finally, $\leq_m^{\mathrm{x}}$ satisfies **CGUA**, for $\mathrm{x} \in \{\mathrm{ng}, \mathrm{g}\}$. This is proven similarly by Lemmas 4.10(4) and 4.3(3), respectively. $\square$

The proof of completeness of each structural inclusion relation, i.e., whether $P \leq_m^{\mathrm{x}} Q$ implies $P \leq^{\mathrm{x}} Q$ for each $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$, and the proof of their respective decidability is based on the proof method of decidability and completeness of $\equiv$ in [2]. As in this method, these two results will be proven simultaneously. Leaving technical details aside for now, we show that for given P,

(1) *(Base)* for each combination of Q and x listed in Fig. 3, $P \leq_m^{\mathrm{x}} Q$ if and only if $P \leq^{\mathrm{x}} Q$, and it is decidable whether or not $P \leq^{\mathrm{x}} Q$,

Q	$\mathrm{x} \in$
$\mathbf{0}$	$\{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$
$g.Q'$	$\{\mathrm{n}, \lambda\}$
$(\nu x)Q'$	$\{\mathrm{n}, \mathrm{ng}\}$

Figure 3: Base case: $P \leq_m^{\mathrm{x}} Q \iff P \leq^{\mathrm{x}} Q$

(2i) *(Induction)* for each combination of Q and x listed in Fig. 4, a finite set $\mathcal{D}(P, Q)$ can be constructed such that

(a) if $P \leq_m^{\mathrm{x}} Q$, then $P' \leq_m^{\mathrm{x}} Q'$ for some $P' \in \mathcal{D}(P, Q)$, and

(b) if there exists $P' \in \mathcal{D}(P, Q)$ such that $P' \leq^{\mathrm{x}} Q'$, then $P \leq^{\mathrm{x}} Q$,

and

Q	$\mathrm{x} \in$
$g.Q'$	$\{\mathrm{ng}, \mathrm{g}\}$
$(\nu x)Q'$	$\{\lambda, \mathrm{g}\}$

Figure 4: Induction case: $P \leq_m^{\mathrm{x}} Q \Longrightarrow (\exists P' \in \mathcal{D}(P,Q) \ : \ P' \leq_m^{\mathrm{x}} Q')$, and $(\exists P' \in \mathcal{D}(P,Q) \ : \ P' \leq^{\mathrm{x}} Q') \Longrightarrow P \leq^{\mathrm{x}} Q$

(2ii) *(Induction)* for $Q = Q_1 \,|\, Q_2$, $!\,Q'$ and $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$, a finite set $\mathcal{D}(P,Q)$ can be constructed, such that

$$(\mathrm{a}) \begin{cases} \text{if } P \leq_m^{\mathrm{x}} Q_1 \,|\, Q_2, \text{ then } P_i \leq_m^{\mathrm{x}} Q_i \text{ for some } (P_1, P_2) \in \mathcal{D}(P, Q_1 \,|\, Q_2), \\ \text{if } P \leq_m^{\mathrm{x}} \,!\,Q', \text{ then } P' \leq_m^{\mathrm{x}} Q' \textit{ for all } P' \in \mathcal{D}(P, !\,Q'), \text{ and} \end{cases}$$

$$(\mathrm{b}) \begin{cases} \text{if } P_i \leq^{\mathrm{x}} Q_i \text{ for some } (P_1, P_2) \in \mathcal{D}(P, Q_1 \,|\, Q_2), \text{ then } P \leq^{\mathrm{x}} Q_1 \,|\, Q_2, \\ \text{if } P' \leq^{\mathrm{x}} Q' \textit{ for all } P' \in \mathcal{D}(P, !\,Q'), \text{ then } P \leq^{\mathrm{x}} \,!\,Q'. \end{cases}$$

The claim that $P \leq_m^{\mathrm{x}} Q$ implies $P \leq^{\mathrm{x}} Q$ can now be deduced from an obvious inductive proof on the structure of Q, in which each of the statements in (a) is combined with its counterpart in (b). Using Lemma 6.2, we then have $\leq_m^{\mathrm{x}} = \leq^{\mathrm{x}}$, for each $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$. Together with the proof of the decidability of $P \leq^{\mathrm{x}} Q$ in (1), the above construction clearly decides whether $P \leq^{\mathrm{x}} Q$ in the general case.

The set $\mathcal{D}(P,Q)$ denotes the set $\mathrm{gua}(P,g)$, $\mathrm{res}(P,x)$, $\mathrm{comp}(P, \mathrm{copy}(P))$ or $\mathrm{rep}(P)$ of [2, Lemmas 26, 24, 28, 30], depending on the form of Q, i.e., whether Q is a guarded process term, a restriction, a parallel composition, or a replication, respectively. Note that $\mathrm{copy}(P) \in \mathbb{N}$, by Lemma 22 of [2], so the set $\mathrm{comp}(P, \mathrm{copy}(P))$ exists. The proofs of (1), (2i) and (2ii), respectively are formed by the next seven lemmas.

Lemma 6.3 *Let* $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$. *For a process term* P*,*

(1) $P \leq_m^{\mathrm{x}} \mathbf{0}$ *if and only if* $P \leq^{\mathrm{x}} \mathbf{0}$*, and*

(2) it is decidable, whether or not $P \leq^{\mathrm{x}} \mathbf{0}$.

Proof Both (1) and (2) follow from Theorems 33 and 34 of [2] and the following two observations:

(i) $P \leq_m^{\mathrm{x}} \mathbf{0}$ if and only if $P \equiv_m \mathbf{0}$, and

(ii) $P \leq^{\mathrm{x}} \mathbf{0}$ if and only if $P \equiv \mathbf{0}$.

By Theorem 33 of [2] and Lemma 6.2, it suffices to prove the only-if part of (i) and the if-part of (ii). To show the only-if part of (i), we prove that $S = \varnothing$ if $S \subseteq^{\mathrm{x}} \varnothing$, where $P \Rightarrow S$. By the inclusion diagram in Fig. 2, it suffices to show that $S \subseteq^{\mathrm{g}} \varnothing$ implies $S = \varnothing$. This follows directly from Definition 4.1 (with $I = \varnothing$ and $T = \varnothing$). The if-part of (ii) follows directly from **CGR**. $\square$

Lemma 6.4 *Let* $\mathrm{x} \in \{\mathrm{n}, \lambda\}$*. For process terms* P *and* Q*,*

(1) $P \leq_m^{\mathrm{x}} g.Q$ *if and only if* $P \leq^{\mathrm{x}} g.Q$*, and*

(2) it is decidable, whether or not $P \leq^{\mathrm{x}} g.Q$*.*

Proof Both (1) and (2) are consequences of Theorems 33 and 34 of [2] and the following two observations:

(i) $P \leq_m^{\mathrm{x}} g.Q$ if and only if $P \equiv_m \mathbf{0}$ or $P \equiv_m g.Q$, and

(ii) $P \leq^{\mathrm{x}} g.Q$ if and only if $P \equiv \mathbf{0}$ or $P \equiv g.Q$.

By Theorem 33 of [2] and Lemma 6.2, it suffices to prove the only-if part of (i) and the if-part of (ii). In fact, if $P \leq_m^{\mathrm{x}} g.Q$, then there exist solutions S and T such that $P \Rightarrow S$, $Q \Rightarrow T$ and $S \subseteq \{g.T\}$ (see Fig. 2). Hence either $S = \varnothing$ or $S = \{g.T\}$. Consequently, $P \equiv_m \mathbf{0}$ or $P \equiv_m g.Q$. To show the if-part of (ii), note that by **CGR**, we have $P \leq^{\mathrm{x}} \mathbf{0}$ or $P \leq^{\mathrm{x}} g.Q$. Now $\mathbf{0} \leq^{\mathrm{x}} g.Q$ by **MIN**, and hence $P \leq^{\mathrm{x}} g.Q$ by **TRA**. □

Lemma 6.5 *Let* $(\nu x)Q$ *be a topconnected process term. Let* $\mathrm{x} \in \{\mathrm{n}, \mathrm{ng}\}$*. For a process term* P*,*

(1) $P \leq_m^{\mathrm{x}} (\nu x)Q$ *if and only if* $P \leq^{\mathrm{x}} (\nu x)Q$*, and*

(2) it is decidable, whether or not $P \leq^{\mathrm{x}} (\nu x)Q$*.*

Proof Both (1) and (2) are consequences of Theorems 33 and 34 of [2] and of the following two observations:

(i) $P \leq_m^{\mathrm{x}} (\nu x)Q$ if and only if $P \equiv_m \mathbf{0}$ or $P \equiv_m (\nu x)Q$, and

(ii) $P \leq^{\mathrm{x}} (\nu x)Q$ if and only if $P \equiv \mathbf{0}$ or $P \equiv (\nu x)Q$.

By Theorem 33 of [2] and Lemma 6.2, it suffices to prove the only-if part of (i) and the if-part of (ii). Let $P \leq_m^{\mathrm{x}} (\nu x)Q$. Then there exist solutions S and T such that $P \Rightarrow S$, $(\nu x)Q \Rightarrow T$ and $S \subseteq^{\mathrm{ng}} T$ (see Fig. 2). Hence $S = V \cup \bigcup_{i \in I}\{g_i.S_i\}$ and $T = W \cup \bigcup_{i \in I}\{g_i.T_i\}$ where the $\mathrm{new}(T_i)$ are mutually disjoint and disjoint with $\mathrm{new}(W)$, $V \subseteq^{\mathrm{n}} W$, and $\mathrm{new}(g_i) = \varnothing$. Note that the $\mathrm{new}(\{g_i.T_i\})$ are mutually disjoint and disjoint with $\mathrm{new}(W)$, since $\mathrm{new}(g_i) = \varnothing$. Hence, since T is connected, by Lemma 8 of [2], $T = W$ or $T = \{g_j.T_j\}$ for some $j \in I$. The last case however contradicts the fact that T is top-secret, so $I = \varnothing$, $T = W$, and $S = V$. Since W is connected, there do not exist nonempty solutions W_1 and W_2 with $W = W_1 \cup W_2$ and disjoint $\mathrm{new}(W_i)$. Hence $V = W$ or $V = \varnothing$, so we conclude $P \equiv_m (\nu x)Q$ or $P \equiv_m \mathbf{0}$. The if-part of (ii) follows directly from **CGR**, **MIN**, and **TRA**, as in the proof of Lemma 6.4. □

Lemma 6.6 *Let* $\mathrm{x} \in \{\mathrm{ng}, \mathrm{g}\}$*. For process terms* P *and* Q*,*

(1) if $P \leq_m^{\mathrm{x}} g.Q$*, then* $P \equiv_m \mathbf{0}$ *or there exists* $P' \in \mathrm{gua}(P, g)$ *such that* $P' \leq_m^{\mathrm{x}} Q$*, and*

(2) if $P \equiv \mathbf{0}$ or there exists $P' \in \mathrm{gua}(P,g)$ such that $P' \leq^{\mathrm{x}} Q$, then $P \leq^{\mathrm{x}} g.Q$.

Proof To show (1), let $P \Rightarrow S$ and $Q \Rightarrow T$, such that $S \subseteq^{\mathrm{x}} \{g.T\}$. By Definition 4.1 and Definition 4.8, either $S = \{g.S'\}$ with $S' \subseteq^{\mathrm{x}} T$, or $S = \varnothing$. In the last case, $P \equiv_m \mathbf{0}$. In the first case, by Lemma 26(2) of [2], there exists $P' \in \mathrm{gua}(P,g)$ such that $P' \Rightarrow S'$. Hence $P' \leq^{\mathrm{x}}_m Q$.

To show (2), observe first that $\mathbf{0} \leq^{\mathrm{x}} g.Q$ and hence $P \leq^{\mathrm{x}} g.Q$, if $P \equiv \mathbf{0}$. Next, assume there exists $P' \in \mathrm{gua}(P,g)$ such that $P' \leq^{\mathrm{x}} Q$. By Lemma 26(1) of [2], $P \equiv g.P'$ and hence $P \leq^{\mathrm{x}} g.Q$, by **CGR** and **CGUA**. □

Lemma 6.7 *Let $\mathrm{x} \in \{\lambda, \mathrm{g}\}$. For process terms P and Q,*

(1) if $P \leq^{\mathrm{x}}_m (\nu x)Q$, then there exists $P' \in \mathrm{res}(P,x)$ such that $P' \leq^{\mathrm{x}}_m Q$, and

(2) if there exists $P' \in \mathrm{res}(P,x)$ such that $P' \leq^{\mathrm{x}} Q$, then $P \leq^{\mathrm{x}} (\nu x)Q$.

Proof To prove (1), let $P \leq^{\mathrm{x}}_m (\nu x)Q$. Take T such that $Q \Rightarrow T$. Then $(\nu x)Q \Rightarrow T[n/x]$, with $n \in \mathrm{New} - \mathrm{new}(T)$. By Lemma 6.1 there exists S such that $P \Rightarrow S$ and $S \subseteq^{\mathrm{x}} T[n/x]$. Let $S' = S[x/n]$. Since $x \notin \mathrm{fn}(T[n/x])$, $x \notin \mathrm{fn}(S)$, so $S'[n/x] = S$ and thus $P \Rightarrow S'[n/x]$ with $n \notin \mathrm{new}(S')$. Hence by Lemma 24(2) of [2], there exists $P' \in \mathrm{res}(P,x)$ such that $P' \Rightarrow S'$. Finally $S' \subseteq^{\mathrm{x}} T[n/x][x/n] = T$ by Lemma 2.1(2) and Lemma 4.3(2), and hence $P' \leq^{\mathrm{x}}_m Q$.

The proof of (2) is immediate from Lemma 24(1) of [2], for then $P \equiv (\nu x)P' \leq^{\mathrm{x}} (\nu x)Q$, by **CRES**. □

Lemma 6.8 *Let $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$. For process terms P, Q_1 and Q_2,*

(1) if $P \leq^{\mathrm{x}}_m Q_1 \mid Q_2$, then there exists $(P_1,P_2) \in \mathrm{comp}(P, \mathrm{copy}(P))$ such that $P_1 \leq^{\mathrm{x}}_m Q_1$ and $P_2 \leq^{\mathrm{x}}_m Q_2$, and

(2) if there exists $(P_1,P_2) \in \mathrm{comp}(P, \mathrm{copy}(P))$ such that $P_1 \leq^{\mathrm{x}} Q_1$ and $P_2 \leq^{\mathrm{x}} Q_2$, then $P \leq^{\mathrm{x}} Q_1 \mid Q_2$.

Proof To show (1), let $P \Rightarrow S$ and $Q_i \Rightarrow T_i$ such that the $\mathrm{new}(T_i)$ are disjoint and $S \subseteq^{\mathrm{x}} T_1 \cup T_2$. By Theorem 4.14, there exist solutions S_1 and S_2, such that $S = S_1 \cup S_2$, $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$, and $S_i \subseteq^{\mathrm{x}} T_i$. Hence $P \Rightarrow S_1 \cup S_2$. Let $k = \mathrm{copy}(P) = \mathrm{copy}(S_1 \cup S_2)$ and take $S_1' = \mathrm{cut}(S_1, k)$ and $S_2' = \mathrm{cut}(S_2,k)$, as defined in Lemma 3.10. By Lemma 3.10(3), $S_1 \cup S_2'$ is a copy of $S_1 \cup S_2$. By Lemma 13 of [2], $\mathrm{copy}(S_1 \cup S_2') = \mathrm{copy}(S_1 \cup S_2) = k$. By Lemma 3.10(1) we have $S_2' \subseteq^{\mathrm{n}} S_2$, so $\mathrm{new}(S_1) \cap \mathrm{new}(S_2') = \varnothing$, by Lemma 4.11. Hence similarly $S_1' \cup S_2'$ is a copy of $S_1 \cup S_2$, and so $P \Rightarrow S_1' \cup S_2'$. Moreover, $\mathrm{new}(S_1') \cap \mathrm{new}(S_2') = \varnothing$ and, by Lemma 3.10(2), $\mathrm{copy}(S_i') \leq k$. Hence, by Lemma 28(2) of [2], there exists $(P_1,P_2) \in \mathrm{comp}(P,k)$ such that $P_1 \Rightarrow S_1'$ and $P_2 \Rightarrow S_2'$. Since by Lemma 3.10(1), $S_i' \subseteq^{\mathrm{n}} S_i \subseteq^{\mathrm{x}} T_i$, we have $P_i \leq^{\mathrm{x}}_m Q_i$.

To prove (2), assume there exists $(P_1,P_2) \in \mathrm{comp}(P, \mathrm{copy}(P))$ such that $P_1 \leq^{\mathrm{x}} Q_1$ and $P_2 \leq^{\mathrm{x}} Q_2$. Thus $P_1 \mid P_2 \leq^{\mathrm{x}} Q_1 \mid Q_2$ by **CCOM**. Since $P \equiv P_1 \mid P_2$ by Lemma 28(1) of [2], we have $P \leq^{\mathrm{x}} Q_1 \mid Q_2$ by **CGR** and **TRA**. □

Lemma 6.9 *Let* $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$*. For a process term* R*,*

(1) if $P \leq_m^{\mathrm{x}} \,!R$*, then* $P' \leq_m^{\mathrm{x}} R$ *for all* $P' \in \mathrm{rep}(P)$*, and*

(2) if $P' \leq^{\mathrm{x}} R$ *for all* $P' \in \mathrm{rep}(P)$*, then* $P \leq^{\mathrm{x}} \,!R$*.*

Proof Note that we may restrict ourselves to subconnected P, as in the proof of Lemma 30 of [2]. We will prove the above statements by induction on the structure of P. Assume $!R \Rightarrow \bigcup_{k\in\mathbb{N}} U_k$ with $R \Rightarrow U_k$ and mutually disjoint $\mathrm{new}(U_k)$ in the remainder of this proof.

The cases $P = \mathbf{0}$, $P = (\nu x)Q$ and $P = g.Q$ are treated in one stroke. Recall from the proof of Lemma 30 of [2] that in this case, $\mathrm{rep}(P) = \{P\}$. To prove (2), observe that $R \leq^{\mathrm{x}} \,!R$ (as proved after Definition 2.5) and hence $P \leq^{\mathrm{x}} R \leq^{\mathrm{x}} \,!R$. To show (1), assume $P \leq_m^{\mathrm{x}} \,!R$. Then by Lemma 6.1 there exists S, such that $P \Rightarrow S$ and $S \subseteq^{\mathrm{x}} \bigcup_{k\in\mathbb{N}} U_k$. Since P is subconnected, S is connected. By Theorem 4.14, there exist V_k, $k \in \mathbb{N}$, such that $S = \bigcup_{k\in\mathbb{N}} V_k$ and $V_k \subseteq^{\mathrm{x}} U_k$ for all $k \in \mathbb{N}$. Also, the $\mathrm{new}(V_k)$ are mutually disjoint. By Lemma 8 of [2], there exists $j \in \mathbb{N}$ such that $V_j = S$. So $S \subseteq^{\mathrm{x}} U_j$, and $P \leq_m^{\mathrm{x}} R$ consequently.

Let $P = Q_1|Q_2$. Now $\mathrm{rep}(P) = \mathrm{rep}(Q_1) \cup \mathrm{rep}(Q_2)$. To show (1), note that we may conclude $Q_i \leq_m^{\mathrm{x}} \,!R$ from $Q_1 \mid Q_2 \leq_m^{\mathrm{x}} \,!R$, because clearly $Q_i \leq_m^{\mathrm{n}} Q_1 \mid Q_2$ and $\leq_m^{\mathrm{x}}$ is transitive (as shown in the proof of Lemma 6.2). By induction $P' \leq_m^{\mathrm{x}} R$ for all $P' \in \mathrm{rep}(Q_i)$, and hence $P' \leq_m^{\mathrm{x}} R$ for all $P' \in \mathrm{rep}(P)$. To show (2), assume $P' \leq^{\mathrm{x}} R$ for all $P' \in \mathrm{rep}(P)$. By induction $Q_i \leq^{\mathrm{x}} \,!R$. Hence $Q_1 \mid Q_2 \leq^{\mathrm{x}} \,!R \mid \,!R \equiv \,!R$ by **CCOM** and structural congruence law (3.5).

Finally, let $P = \,!Q$. By definition, $\mathrm{rep}(P) = \mathrm{rep}(Q)$. To prove (1), let $!Q \leq_m^{\mathrm{x}} \,!R$. Since, clearly, $Q \leq_m^{\mathrm{n}} \,!Q$, this implies that $Q \leq_m^{\mathrm{x}} \,!R$. By induction, $P' \leq_m^{\mathrm{x}} R$ for all $P' \in \mathrm{rep}(Q)$. Since $\mathrm{rep}(P) = \mathrm{rep}(Q)$, this proves (1). To show (2), assume that $P' \leq^{\mathrm{x}} R$ for all $P' \in \mathrm{rep}(Q)$. By induction, $Q \leq^{\mathrm{x}} \,!R$, and hence $!Q \leq^{\mathrm{x}} \,!!R \equiv \,!R$ by **CREP** and structural congruence law (3.3). □

Next, we show the two main results of this paper: completeness and decidability of each of the four structural inclusion relations.

Theorem 6.10 *Let* $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$*.*
For process terms P *and* Q*,* $P \leq_m^{\mathrm{x}} Q$ *if and only if* $P \leq^{\mathrm{x}} Q$*.*

Proof The if-part is by Lemma 6.2. To prove the only-if part, assume $P \leq_m^{\mathrm{x}} Q$. Let Q' be a topconnected process term such that $Q \equiv Q'$ (by Lemma 5.3, such a process term indeed exists). Note that by Definition 5.1, this means that each subterm of Q' is topconnected (this is required in order to use Lemma 6.5). By Theorem 33 of [2] we have $P \leq_m^{\mathrm{x}} Q'$, and hence it suffices to show that $P \leq^{\mathrm{x}} Q'$. This is done by induction on the syntactical structure of Q', using Lemmas 6.3, 6.4, and 6.5 for the base cases (see Fig. 3), and Lemmas 6.6, 6.7, 6.8, and 6.9 in the induction steps (see Fig. 4). □

Theorem 6.11 *Let* $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{ng}, \mathrm{g}\}$*.*
It is decidable for process terms P *and* Q*, whether or not* $P \leq^{\mathrm{x}} Q$*.*

Proof Suppose $P \leq^{\mathrm{x}} Q$ is to be decided. We may assume that Q is topconnected because Lemma 5.3 is effective, cf. the proof of Theorem 6.10. The decidability of $P \leq^{\mathrm{x}} Q$ for the base cases (see Fig. 3 for the base-combinations of Q and x), is proven in Lemmas 6.3, 6.4, and 6.5. The non-base cases are by Lemmas 6.6, 6.7, 6.8, and 6.9 (depending on the form of Q): using Theorem 6.10, we obtain $P \leq^{\mathrm{x}} Q$ iff $f(P_1 \leq^{\mathrm{x}} Q_1, \ldots, P_n \leq^{\mathrm{x}} Q_n)$, where f is a boolean function of n arguments, the Q_i are direct subterms of Q, and the P_i and Q_i can be effectively computed from P and Q, using the fact that the finite sets $\mathrm{gua}(P, g)$, $\mathrm{res}(P, x)$, $\mathrm{comp}(P, \mathrm{copy}(P))$, and $\mathrm{rep}(P)$ can be computed (see Lemmas 26, 24, 28, and 30 of [2]). Observe that the computation of $\mathrm{copy}(P)$ is guaranteed by the remark below Theorem 34 in [2]. Thus, as in [2] for $\equiv$, the truth value of $P \leq^{\mathrm{x}} Q$ can be computed by a recursive boolean function procedure with arguments P and Q. Since, in its body, the second argument of each recursive call is a proper subterm of Q, this procedure always halts. $\Box$

The next counterexamples show that strong structural inclusion is in fact the only relation defined in this paper that is antisymmetrical upto structural congruence, i.e., $P \leq^{\mathrm{x}} Q$ and $Q \leq^{\mathrm{x}} P$ implies $P \equiv Q$, only holds for $\mathrm{x} = \mathrm{n}$ (Theorem 3.15). The first is a counterexample for $x = \lambda$, the second for $x \in \{\mathrm{ng}, \mathrm{g}\}$.

Example 6.12 Let $P = \,!(\nu z)R$ and $Q = P \,|\, !(\nu z)R'$, where it is assumed that $R' \leq R$, and $z \in \mathrm{fn}(R')$. Clearly $P \leq Q$. To show the reverse, $Q \leq P$, observe that by **CCOM** we have

$$!(\nu z)R \,|\, !(\nu z)R' \leq \,!(\nu z)R \,|\, !(\nu z)R, \qquad (*)$$

since $!(\nu z)R' \leq \,!(\nu z)R$, by **CRES** and **CREP**, respectively. Now the left-hand side of (*) is Q, and the right-hand side is structurally congruent to P, by law (3.5) of structural congruence. But in general, $P \equiv Q$ does not hold. To see this, we turn our example into an intuitively more clear one: let R and R' be the process terms of Example 2.6. We can think of the process term P above as a model for a beach with an infinite number of three-player ball games (note that it is not possible to accidentally throw a ball at a neighbouring group of players, by the restriction on z), whereas Q models a beach which, in addition, has an infinite number of two-player ball games. It is now easy to see that $P \not\equiv Q$, since in a solution S (where $P \Rightarrow S$) all molecules consisting of a sequence of three guards, i.e., those molecules corresponding to $z(y).\overline{z}y.\overline{p}.\mathbf{0}$, can be paired by a new name n they share (one that corresponds to z). Thus, for each molecule $n(y).\{\overline{n}y.\{\overline{m}_1 v.\varnothing\}\}$ in S there exists exactly one other molecule of the form $n(y).\{\overline{n}y.\{\overline{m}_2 v.\varnothing\}\}$ in S (where $\overline{m}_1 v$ and $\overline{m}_2 v$ correspond to the guard p), whereas in T (where $Q \Rightarrow T$) those molecules need not be paired. $\Box$

Example 6.13 Let $P = \,!a.(b.\mathbf{0} \,|\, c.\mathbf{0})$ (where a, b, and c are arbitrary guards over $\mathbf{N}$), and let $Q = P \,|\, !a.b.\mathbf{0}$. Now $P \leq^{\mathrm{x}} Q$ is obvious. To show the reverse for $\mathrm{x} \in \{\mathrm{ng}, \mathrm{g}\}$, observe that $b.\mathbf{0} \leq^{\mathrm{x}} b.\mathbf{0} \,|\, c.\mathbf{0}$ and so $!\,a.b.\mathbf{0} \leq^{\mathrm{x}} \,!\,a.(b.\mathbf{0} \,|\, c.\mathbf{0}) = P$, by **CGUA** and **CREP**, respectively. Hence $Q = P \mid \,!\,a.b.\mathbf{0} \leq^{\mathrm{x}} P \mid P \equiv P$, by **CCOM** and law (3.5) of structural congruence (note that P is a replication). Yet P is not structurally congruent to Q, since $P \Rightarrow V$ and $Q \Rightarrow W$, for V and W of Example 4.2. □

7 Simulation

We conclude this paper with some words on simulation. We prove that, for the transition system Mπ, three of the four containment relations are strong simulations. This implies that the corresponding multiset inclusion relations (and hence, by the results of the previous section, the corresponding structural inclusion relations) are strong simulations on process terms. By a counterexample it will be shown that the fourth, viz. $\subseteq^{\mathrm{ng}}$, is not a simulation; nor is $\leq^{\mathrm{ng}}$ a simulation on process terms.

We recall from [1] that for arbitrary solutions S, S', and S'', transitions in Mπ are of the form

$$\{x(-).S, \overline{x}z.S'\} \cup S'' \to \mathrm{dec}(S[z/1]) \cup S' \cup S'',$$

where $x, z \in \mathbf{N} \cup \mathrm{New}$.

Definition 7.1 A relation $\mathcal{S} \subseteq \mathrm{Sol} \times \mathrm{Sol}$ is a *strong simulation*, if $(S, T) \in \mathcal{S}$ implies that

if $S \to S'$, then there exists T' such that $T \to T'$ and $(S', T') \in \mathcal{S}$.

$\mathcal{S}$ is a *strong bisimulation*, if both $\mathcal{S}$ and $\mathcal{S}^{-1}$ are strong simulations.

Theorem 7.2 *Let* $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{g}\}$. *The relation* $\subseteq^{\mathrm{x}}$ *is a strong simulation.*

Proof We consider the three cases for x:

(i) ($\mathrm{x} = \lambda$). Let $S \cup U = T$ with $S \to S'$, and let $T' = S' \cup U$. Recall from [1] that by the "chemical law" we have that $S \to S'$ implies $S \cup U \to T'$. Hence $T \to T'$.

(ii) ($\mathrm{x} = \mathrm{n}$). We use the same proof-scheme as in (i) with the restriction $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$. Since by Lemma 2 of [1], $\mathrm{new}(S') \subseteq \mathrm{new}(S)$, we have $S' \subseteq^{\mathrm{n}} T'$.

(iii) ($\mathrm{x} = \mathrm{g}$). Let $S = U \cup \bigcup_{i \in I}\{g_i.S_i\}$ and $T = V \cup \bigcup_{i \in I}\{g_i.T_i\}$, with $U \subseteq V$ and $S_i \subseteq^{\mathrm{g}} T_i$ for all $i \in I$. Assume $S \to S'$. Recall from [1, Section 4] that this means that $S = \{x(-).X, \overline{x}z.X'\} \cup X''$ and $S' = \mathrm{dec}(X[z/1]) \cup X' \cup X''$, for solutions X, X', and X'', and $\{x, z\} \subseteq \mathbf{N} \cup \mathrm{New}$. By Lemma 6 of [2], we consider four cases:

(1) For some $j, k \in I$, $g_j.S_j = x(-).X$ and $g_k.S_k = \overline{x}z.X'$. Now $S' = U \cup \mathrm{dec}(S_j[z/1]) \cup S_k \cup \bigcup_{i \in I-\{j,k\}}\{g_i.S_i\}$. Let $T' = V \cup \mathrm{dec}(T_j[z/1]) \cup T_k \cup \bigcup_{i \in I-\{j,k\}}\{g_i.T_i\}$. Observe that $T \to T'$. Note also that $\mathrm{dec}(S_j[z/1]) \subseteq^{\mathrm{g}} \mathrm{dec}(T_j[z/1])$, by Lemma 4.3(2). Furthermore, since by assumption, $S_k \subseteq^{\mathrm{g}} T_k$, we have $S' \subseteq^{\mathrm{g}} T'$ by Lemma 4.3(1).

(2) For some $j \in I$, $g_j.S_j = x(-).X$, and for some solution W, $U = \{\overline{x}z.X'\} \cup W$. Since $U \subseteq V$, there exists Z with $U \cup Z = V$. Hence $V = \{\overline{x}z.X'\} \cup W \cup Z$. Now we have $S' = \mathrm{dec}(S_j[z/1]) \cup X' \cup W \cup \bigcup_{i \in I-\{j\}}\{g_i.S_i\}$. Let $T' = \mathrm{dec}(T_j[z/1]) \cup X' \cup W \cup Z \cup \bigcup_{i \in I-\{j\}}\{g_i.T_i\}$. By arguments similar to the above case, we have $T \to T'$ and $S' \subseteq^{\mathrm{g}} T'$.

(3) For some $k \in I$, $g_k.S_k = \overline{x}z.X'$, and for some solution W, $U = \{x(-).X\} \cup W$. Again, let $U \cup Z = V$. Hence $V = \{x(-).X\} \cup W \cup Z$. Now $S' = \mathrm{dec}(X[z/1]) \cup W \cup S_k \cup \bigcup_{i \in I-\{k\}}\{g_i.S_i\}$. Let $T' = \mathrm{dec}(X[z/1]) \cup W \cup Z \cup T_k \cup \bigcup_{i \in I-\{k\}}\{g_i.T_i\}$. By arguments similar to the above cases, we have $T \to T'$ and $S' \subseteq^{\mathrm{g}} T'$.

(4) For some solution W, $U = \{x(-).X, \overline{x}z.X'\} \cup W$ (so $X'' = W \cup \bigcup_{i \in I}\{g_i.S_i\}$). Hence $U \to \mathrm{dec}(X[z/1]) \cup X' \cup W = U'$. Since $U \subseteq V$, by (i) there exists V' with $V \to V'$ and $U' \subseteq V'$. Let $T' = V' \cup \bigcup_{i \in I}\{g_i.T_i\}$. By the chemical law, we have $S \to U' \cup \bigcup_{i \in I}\{g_i.S_i\} = S'$ and $T \to T'$. Moreover, since $U' \subseteq V'$, we have $S' \subseteq^{\mathrm{g}} T'$.

□

We can define simulation and bisimulation for the transition system of the π-calculus (of [6]) similar to Definition 7.1. Since the semantical relation $\Rightarrow$ is a strong bisimulation between the transition systems of the π-calculus and Mπ (see result (A) of [1]), by Theorem 7.2 and Theorem 6.10 we also have that the structural inclusion relations corresponding to the three containment relations of Theorem 7.2 are strong simulations in this sense. This is because, by definition, $\leq^{\mathrm{x}}_m$ is the composition of $\Rightarrow$, $\subseteq^{\mathrm{x}}$, and $\Rightarrow^{-1}$, and strong simulations are closed under composition.

Theorem 7.3 *Let* $\mathrm{x} \in \{\mathrm{n}, \lambda, \mathrm{g}\}$. *The relation* $\leq^{\mathrm{x}}$ *is a strong simulation on process terms.*

Example 7.4 Let R' and R be the two-player and three-player ball game of Example 2.6. Recall that $R' \leq R$. Obviously R is capable of the same action sequence as R':

$$\begin{array}{rcl} R & = & P_1 \mid P_2 \mid P_2 \\ & = & (\nu p)(\overline{z}x.\overline{p} \mid P') \mid (\nu p)(z(y).\overline{z}y.\overline{p} \mid P') \mid P_2 \\ & \to & (\nu p)(\overline{p} \mid P') \mid (\nu p)(\overline{z}x.\overline{p} \mid P') \mid P_2 \\ & \to & (\nu p)(z(y).\overline{z}y.\overline{p} \mid P') \mid (\nu p)(\overline{z}x.\overline{p} \mid P') \mid P_2, \end{array}$$

letting the second player of type P_2 act as a dummy. Similarly, $(\nu z)R$ can simulate the actions of $(\nu z)R'$. $\square$

We conclude this section with an example, showing that $\subseteq^{\rm ng}$ and $\leq^{\rm ng}$ fail to be strong simulations.

Example 7.5 Let
$$S = \{\overline{x}n.\varnothing, x(-).\{\overline{1}u.\{\overline{v}u.\varnothing\}\}\}$$
and
$$T = \{\overline{x}n.\varnothing, x(-).\{\overline{1}u.\{\overline{v}u.\varnothing, \overline{w}u.\varnothing\}\}\},$$
where $n \in \text{New}$ and $x, u, v, w \in \mathbf{N}$. Note that $P \Rightarrow S$ and $Q \Rightarrow T$ for process terms
$$P = (\nu y)\overline{x}y.\mathbf{0} \mid x(z).\overline{z}u.\overline{v}u.\mathbf{0}$$
and
$$Q = (\nu y)\overline{x}y.\mathbf{0} \mid x(z).\overline{z}u.(\overline{v}u.\mathbf{0} \mid \overline{w}u.\mathbf{0}).$$
Now $S \subseteq^{\rm ng} T$, since $\{\overline{x}n.\varnothing\} \subseteq^{\rm n} \{\overline{x}n.\varnothing\}$ and
$$\{x(-).\{\overline{1}u.\{\overline{v}u.\varnothing\}\}\} \subseteq^{\rm ng} \{x(-).\{\overline{1}u.\{\overline{v}u.\varnothing, \overline{w}u.\varnothing\}\}\}.$$
Both S and T are only capable of a communication via the link x, and hence $S \to \{\overline{n}u.\{\overline{v}u.\varnothing\}\} = S'$ and $T \to \{\overline{n}u.\{\overline{v}u.\varnothing, \overline{w}u.\varnothing\}\} = T'$. Note that by communicating the new name n, both S' and T' become 'top-secret' (in general, the communication of a new name models 'scope extrusion' in the π-calculus). Thus $S' \not\subseteq^{\rm ng} T'$, since both the molecule in S' and the molecule in T' is now guarded by $\overline{n}u$, which contains the new name n. This means that strong nested structural inclusion ($\leq^{\rm ng}$) of process terms also fails to be a strong simulation. In fact, it is easy to see that $P \leq^{\rm ng} Q$ (by the completeness of $\leq^{\rm ng}$ this is even immediate from $S \subseteq^{\rm ng} T$). Now let
$$P \to P' = (\nu y)\overline{y}u.\overline{v}u.\mathbf{0} \Rightarrow S',$$
and suppose $Q \to Q'$ for some Q'. Since $\Rightarrow$ is a strong bisimulation, there exists a T'' such that $T \to T''$ and $Q' \Rightarrow T''$. Since T' is the unique solution such that $T \to T'$, we have $T'' = T'$. Hence $P' \not\leq^{\rm ng}_m Q'$, and so $P' \not\leq^{\rm ng} Q'$, by the soundness of $\leq^{\rm ng}$.

$\square$

8 Conclusion

In this paper we presented three inclusion relations on process terms of the π-calculus, based on three different containment relations for solutions in Mπ. Each of them expresses 'substructure' of a π-calculus process term in a different way, but all are 'natural' notions of substructure, and all were proven to be decidable, to have natural axiomatizations, and to preserve communications. Resuming their features, the first, $\leq_m$, is based on ordinary multiset inclusion $\subseteq$. The second (and strongest of the three), $\leq^{\mathrm{n}}_m$, is based on inclusion of connected components of solutions $\subseteq^{\mathrm{n}}$. Both $\subseteq$ and $\subseteq^{\mathrm{n}}$ are partial orders. The third (and weakest of the three), $\leq^{\mathrm{g}}_m$, is based on $\subseteq^{\mathrm{g}}$ (a preorder), which respects the 'nested' nature of solutions. Of the three, only $\leq^{\mathrm{n}}_m$ has the additional property that $\leq^{\mathrm{n}}_m \cap (\leq^{\mathrm{n}}_m)^{-1} = \equiv$, and only $\leq^{\mathrm{g}}_m$ is compatible with all the operations of the π-calculus. Via a combination of the axiomatizations of $\leq^{\mathrm{n}}_m$ and $\leq^{\mathrm{g}}_m$, a fourth relation $\leq^{\mathrm{ng}}_m$ was introduced, that fails to preserve communications, however. It is open whether there exist natural axiomatizations of $\leq^{\mathrm{x}}_m$ that do not use **CGR**. This will be the interest of future research.

Appendix

An overview of the congruence relations considered in [1, 2], and of all the inclusion relations defined in this paper, is listed below; for completeness sake, we include (multiset) equality in the list.

$=$	(multiset) equality	
$\equiv_m$	multiset congruence	[1, page 79]
$\equiv$	structural congruence	[1, page 81]
$\subseteq$	containment	page 4
$\leq_m$	multiset inclusion	Definition 2.2
$\leq$	structural inclusion	Definition 2.5
$\subseteq^{\mathrm{n}}$	strong containment	Definition 3.1
$\leq^{\mathrm{n}}_m$	strong multiset inclusion	Definition 3.4
$\leq^{\mathrm{n}}$	strong structural inclusion	Definition 3.7
$\subseteq^{\mathrm{g}}$	nested containment	Definition 4.1
$\leq^{\mathrm{g}}_m$	nested multiset inclusion	Definition 4.4
$\leq^{\mathrm{g}}$	nested structural inclusion	Definition 4.6
$\subseteq^{\mathrm{ng}}$	strong nested containment	Definition 4.8
$\leq^{\mathrm{ng}}_m$	strong nested multiset inclusion	Definition 4.9
$\leq^{\mathrm{ng}}$	strong nested structural inclusion	Definition 4.7

References

[1] J. Engelfriet, A multiset semantics for the pi-calculus with replication, *The-*

oret. Comput. Sci. **153** (1996) 65-94.

[2] J. Engelfriet and T.E. Gelsema, Multisets and structural congruence of the pi-calculus with replication, Tech. Report 95-02, Department of Computer Science, Leiden University, January 1995. To appear in *Theoret. Comput. Sci.*

[3] J. Engelfriet and T.E. Gelsema, Axioms for Generalized Graphs, illustrated by a Cantor-Bernstein Proposition, revised version of "A Cantor-Bernstein Result for Structured Objects", Tech. Report 96-29, Department of Computer Science, Leiden University, October 1996.

[4] R. Milner, Flowgraphs and flow algebras, *Journ. of the ACM* **26** (1979) 794–818.

[5] R. Milner, *Communication and Concurrency*, (Prentice-Hall, Englewood Cliffs, NJ, 1989).

[6] R. Milner, Functions as processes, *Math. Struct. in Comput. Sci.* **2** (1992) 119–141.

[7] G. Milne and R. Milner, Concurrent processes and their syntax, *Journ. of the ACM* **26** (1979) 302–321.

[8] R. Milner, J. Parrow and D. Walker, A calculus of mobile processes, *Inform. and Comput.* **100** (1992) 1–77.