Recognizability equals Monadic Second-Order definability, for sets of graphs of bounded tree-width. *

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Abstract

For each integer k, we define in Monadic Second-Order logic a relation that associates with every hypergraph of treewidth at most k at least one of its tree-decompositions of width at most k. In "*The Monadic second-order logic of graphs, I : Recognizable sets of finite graphs*", Courcelle proves that every set of graphs is recognizable if it is definable in Monadic Second-Order logic and extends this result to a refinement of MSO logic, the Counting Monadic Second-Order logic. From all these results, it follows that Recognizability equals CMSO-definability for sets of graphs of bounded tree-width.

Introduction

A fundamental theorem by Büchi [2] states that a language of words is recognizable iff it is definable by some formula in *Monadic second order logic* (MSOL). This result is extended to finite ranked ordered trees by Doner [8], and to sets of finite unranked unordered trees by Courcelle [3]. This last result deals with an extension of MSOL, called *Counting monadic second-order logic* (CMSOL), that allows modular counting. These three results, relate an algebraic aspect, namely *Recognizability*, to a logical one.

For graphs (by graph, we mean a finite graph), similar relationship have been investigated. On the one hand, a graph can be viewed as a logical structure, hence we have a notion of a definable set of graphs. On the other

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hand, Bauderon and Courcelle [1] propose an algebraic structure over sets of graphs. The notion of a recognizable set of graphs follows, as an instance of the general notion of recognizability introduced by Mezei and Wright [13]. Courcelle [3] proves that every CMSO-definable set of graphs is recognizable, but not conversely.

However, in the same paper however, he conjectures that the converse holds for an interesting class of graphs. More precisely, Robertson and Seymour [14], in their study of minors, introduce the notions of *tree-width* and *tree-decomposition*. Courcelle [4] conjectures that:

Conjecture 1 If a set of graphs of bounded tree-width is recognizable, then it is CMSO-definable.

Such a conjecture has already been proved in several restricted cases. Courcelle [4] shows it holds for graphs of tree-width at most 2. Kaller [12] shows it holds for graphs of tree-width at most 3. Kabanets [11] shows it holds for graphs of path-width at most k. To address this conjecture, Courcelle [4] introduces the notion of a *MSO-definable* binary relation, a *MSO-transduction* for short, over *relational structures*. Let us recall that relational structures permit to code in a logical way different kinds of objects, like graphs, hypergraphs, tree-decompositions and algebraic terms defined on a finite signature. A MSO-transduction transforms a relational structure S into a relational one S' by defining S' inside S by means of MSO-formulas. Courcelle calls *strongly context-free* every set of graphs L that admits a MSO-transduction which transforms every graph $G \in L$ into an algebraic algebraic term of value G (for a more precise definition, see [4]). Courcelle shows in [4] that Conjecture 1 holds if the following one holds:

Conjecture 2 For every k, the set of graphs of tree-width $\leq k$ is strongly context-free.

The fact that Conjecture 2 is stronger than Conjecture 1 can be quickly explained as below by showing how the MSO-transduction of Conjecture2 "transports" the inclusion "Recognizability $\subseteq$ CMSO-definability" from the class of sets of terms into the class of sets of graphs of bounded tree-width. Let $k \geq 0$ and T_k be the set of algebraic terms presented in [4] that denote graphs of tree-width at most k. Let f_k be the homomorphism that associates with every term of T_k the graph it denotes and let g_k be the MSO-transduction induced by Conjecture 2. Recognizability is preserved by inverse homomorphism, and then by f_k^{-1}. Every recognizable subset of T_k is CMSO-definable (see [3]). Due to Courcelle [4], CMSO-definability is preserved by inverse MSO-transduction, and then by g_k^{-1}. Then, for every recognizable set L of graphs of bounded tree-width, the set $g_k^{-1}(f_k^{-1}(L))$ is CMSO-definable and is equal to L, if Conjecture 2 holds.

Our main result is to prove Conjectures 1 and 2 (see Theorems 74 and 76). In fact, we will establish such results not uniquely for graphs but for hypergraphs. More precisely, we will consider *e-hypergraphs*, which are concrete unlabelled unoriented hypergraph with a distinguished hyperedge: its *source* hyperedge. In order to establish Theorem 74, we will consider a very similar object with the algebraic term: the *e-tree-decomposition*. Their set is denoted by $\mathcal{T}$. An e-tree-decomposition is a tree-decomposition of some e-hypergraph, called its *value*. Due to the fact that hyperedges are placed "into" the nodes of the e-tree-decomposition, the tree of the e-tree-decomposition is rooted and, then, any algebraic term can be viewed as an e-tree-decomposition in which we order the childrens of every node.

In this article, we study the power of the MSO logic, by showing that certain subsets of $\mathcal{T}$ are *MSO-parsable*. A subset $L \subseteq \mathcal{T}$ is MSO-parsable if the converse of the mapping that associates to every tree-decomposition of L its value is a MSO-transduction. In other words, if we can uniformly "MSO-define" in terms of the values of L the set L itself. Unfortunately, the subsets of $\mathcal{T}$ are not in general MSO-parsable: the set of all e-tree-decompositions having for value an *empty* e-hypergraph (with no vertex and no edge, except its source-edge) have a domain of size not uniformly bounded and, then, cannot be MSO-defined in terms of their values. It follows that for each k, $\mathcal{T}_k$ is not MSO-parsable, where $\mathcal{T}_k$ denotes the set of all e-tree-decompositions of width at most k (having in each node at most $k+1$ vertices). In order to obtain a comparable statment than in Conjecture 2, we call *equivalent* two e-tree-decompositions (resp. sets of) having same value (resp. sets of values). Under this formalism, we can present now our main technical result, established by Theorem 73:

1. For every k, $\mathcal{T}_k$ contains an equivalent MSO-parsable set.

To prove it, we introduce an algebra Π which produces subsets of $\mathcal{T}$ and that verifies two properties. These properties, expressed respectively by Theorems 72 and 65, are:

2. Each operation of Π preserves MSO-parsability.

3. For every k, Π produces an equivalent subset of $\mathcal{T}_k$.

The algebra Π is defined by six classes of n-ary operations on $\mathcal{P}(\mathcal{T})$. The first one is the nullary operation that associates the set of all *atomic*

e-tree-decompositions, that have a unique node. The second kind of operation is the MSO-transduction that intersects every subset of $\mathcal{T}$ with some given MSO-definable set. The third one is the MSO-transduction $+$, that "adds" a unique vertex to every e-tree-decomposition. Obviously, the three above operations preserve MSO-parsability. To define the fourth class of operations, we introduce an higher-order substitution $\otimes$ that associates with every couple $(u, v) \in \mathcal{P}(\mathcal{T})^2$ the set of all e-tree-decompositions obtained by refining some $X \in u$ thanks the set v, indeed by replacing simultaneously all the nodes of X by e-tree-decompositions of v. An important property of $\otimes$ is the fact that it preserves MSO-parsability, under an additional condition. Hence, we enrich Π with each operation of the form $(u, v) \mapsto u \otimes (v \cap Type_k)$ for some k, where $Type_k$ contains every e-tree-decomposition having at most k vertices incident with its source-edge. Theorem 71 states:

4. For each k, $(u, v) \mapsto u \otimes (v \cap Type_k)$ preserves MSO-parsability.

Concerning the fifth class of Π, let us just say that each of its operations is a restriction of the hyperedge-substitution introduced in [1] (see also Habel and Kreowski [10]) and is, in fact, a derived operation of $(u, v) \mapsto u \otimes (v \cap Type_k)$ for some k. The last operations of Π require the notion of an *internally connected* e-hypergraph G, that is such that every pair of elements of its domain is not separated by the extremities of its source-edge. In a natural way, we extend internal connectivity on $\mathcal{T}$ (its so defined subset is denoted $\mathcal{T}_{i.c}$) and introduce the notion of a *critical* edge in an e-hypergraph: an edge e is critical if it is needed to internally connect the extremities of the source-edge. That permits to consider a subset of $\mathcal{T}_{i.c}$: the set $\mathcal{T}_{i.c}^{\mathbf{nc}}$ of all *nowhere-critical* e-tree-decompositions. The last class of Π contains every nullary operation $\to \mathcal{T}_{i.c}^{\mathbf{nc}} \cap Rank_k$ for some $k \geq 0$, where $Rank_k$ denotes the set of all $X \in \mathcal{T}$ whose every arc and every edge has a degree bounded by k. Theorem 70 states:

5. For each k, $\mathcal{T}_{i.c}^{\mathbf{nc}} \cap Rank_k$ is MSO-parsable.

Now, let us present the power of Π, by showing how Π produces an equivalent subset of $\mathcal{T}_k$. The proof comports four steps. First one concerns the internally connected e-tree-decompositions, which are *linear*, indeed having a path-structure. This case is treated by using similar methods than for the internally connected case presented below, and this in a more simple way. Let us just say that this case requires the operation $+$ presented above. Note that this result is similar with (but different from) the result of Kabanets [11]. Second step concerns the internally connected e-tree-decompositions, which are *quasi linear*, indeed obtained by substituting a finite number of linear sets. Obviously, Π produce such sets from linear ones. Third step, the main difficult, concerns the internally connected e-tree-decompositions. To treat this case, we define the set $\mathcal{T}_{i.c}^{\mathbf{c}}$ that contains every internally connected *critical* e-tree-decomposition, that is critical "everywhere". The interest of a such set appears in the following equality:

6. $\mathcal{T}_{i.c} = \mathcal{T}_{i.c}^{\mathbf{nc}} \otimes \mathcal{T}_{i.c}^{\mathbf{c}}$.

The above equality, easy to prove (see Theorem 50), implies that, for each k, Π produced an equivalent subset of $\mathcal{T}_{i.c} \cap \mathcal{T}_k$, if Π produces an equivalent subset of $\mathcal{T}_{i.c}^{\mathbf{c}} \cap \mathcal{T}_k$. This last point is the object of Theorem 61 that states that for each k the set $\mathcal{T}_{i.c}^{\mathbf{c}} \cap \mathcal{T}_k$ is equivalent with a quasi linear subset of $\mathcal{T}_k$. As a consequence of a such result, the most difficult one of this paper, and of Lemma 44, it comes:

7. For every k, $\mathcal{T}_{i.c}^{\mathbf{c}} \cap \mathcal{T}_k$ is equivalent with a quasi linear subset of $\mathcal{T}_{i.c} \cap \mathcal{T}_k$.

Then, for each k, Π produces an equivalent subset of $\mathcal{T}_{i.c} \cap \mathcal{T}_k$. We jump easily the fourth step by using the fact that every e-tree-decomposition of $\mathcal{T}_k$ can be rewritten into an equivalent e-tree-decomposition of $\mathcal{T}_k$ that is internally connected, except, possibly, "at the root".

The paper is organized as follows.
The first section contains the necessarily definitions of hypergraphs, of e-hypergraphs, of operations over hypergraphs and e-ehypergraphs and the three notions of connectivity over e-hypergraphs: the internal connectivity and two auxiliar ones, which are 2-edge-connectivity and connectivity.
The second section contains the definition of tree-decompositions, of a few operations over e-tree-decompositions and the three notions of connectivity.
In Section 3, we define and study quasi-linear sets.
In Section 4, we introduce the notion of a nowhere-critical and of an (everywhere-)critical e-tree-decomposition. We establish the equality $\mathcal{T}_{i.c} = \mathcal{T}_{i.c}^{\mathbf{nc}} \otimes \mathcal{T}_{i.c}^{\mathbf{c}}$. We define two other notion of criticality associated with the two other notions of connectivity. That permits to study the linear and critical case and the critical case (see Theorems 60 and 61).
In Section 5, we recall briefly MSO logic, define the algebra Π and, by Theorem 65, show that Π produces, for each k, an equivalent subset of $\mathcal{T}_k$.
In Section 6, we recall MSO-transductions and establish that each operation of Π preserves MSO-parsability. It follows Theorems 70, 74, and 76.
In Appendix a (resp. b and c), we prove Theorem 60 (resp. 61, 70).

Notation

We denote by $[i, j]$ the set of integers $\{i, i+1, \ldots, j\}$ and by $[n]$ the interval $[1, n]$. Let A be a set. The cardinality of A is denoted by $\mathbf{card}(A)$, its powerset by $\mathcal{P}(A)$. The set of nonempty sequences of elements of A is denoted by A^+, and sequences are denoted by $(a_1, \ldots, a_n)$ with commas and parentheses. We use $:=$ for "equal by definition" i.e, for introducing new notations, and $:\Leftrightarrow$, similarly, for introducing logical conditions. A binary relation $R \subseteq A \times B$ is also called a *transduction*. The *domain of* R is $\mathbf{Dom}(R) := \{a \in A \mid (a, b) \in R\}$, and the *image of* R is $\mathbf{Im}(R) := \{b \in B \mid (a, b) \in R\}$. The composition of two relations $R \subseteq A \times B$, and $S \subseteq B \times C$ is denoted by $S \circ R \subseteq A \times C$. R is *functional* if $\mathbf{card}(\{b \mid (a, b) \in R\}) \leq 1$ for each $a \in \mathbf{Dom}(R)$. We identify functional relations $R \subseteq A \times B$ with partial functions $R : A \to B$. The restriction of a partial function f to a subset A' of $\mathbf{Dom}(f)$ is denoted by $f \restriction A'$. If two partial functions $f : A \to B$ and $g : A' \to B'$ coincide on $\mathbf{Dom}(f) \cap \mathbf{Dom}(g)$, we denote by $f \cup g$ their common extension into a partial function $A \cup A' \to B \cup B'$. By a mapping, we mean a total function.

1 Hypergraph

We deal with a certain class of concrete unoriented unlabeled hypergraphs, which we call simply "e-hypergraphs". Every e-hypergraph H is defined in a very simple way: it is a hypergraph, denoted by $\mathbf{G}_H$ with a distinguished edge: its "source-edge". We extend to such hypergraphs the operation of substitution defined by Bauderon and Courcelle [1] or by Habel and Kreowski [10]. We recall the notion of an internally connected e-hypergraph introduced in [4] that plays an important rôle in this articles and the both auxiliary notions of a connected e-hypergraph and of a 2-edge-connected e-hypergraph.

Definition 3 A *hypergraph* G is a sequence $(\mathbf{V}_G, \mathbf{E}_G, \mathbf{vert}_G)$, where $\mathbf{V}_G$ is the finite set of *vertices*, $\mathbf{E}_G$ is the finite set of *edges* and $\mathbf{vert}_G$ is a mapping $\mathbf{E}_G \to \mathcal{P}(\mathbf{V}_G)$ that associates with every edge of G its set of *extremities*. The sets $\mathbf{V}_G$ and $\mathbf{E}_G$ are supposed to be disjoint.

A vertex x and an edge e are *incident* if x is an extremity of e. Two distinct vertices (resp. edges) are *adjacent* if they are incident to the same edge (resp. vertex). A vertex is *isolated* if it is incident with no edge. The *degree* of an edge is the number of its extremities. A *graph* is a hypergraph, whose every edge has a degree 2. The *empty hypergraph* is the sequence $(\emptyset, \emptyset, \emptyset)$ denoted by $\emptyset$.

Definition 4 A hypergraph G is a *subhypergraph of a hypergraph* H or is *contained in* H, denoted by $G \subseteq H$, if $\mathbf{V}_G$ and $\mathbf{E}_G$ are subsets of $\mathbf{V}_H$ and $\mathbf{E}_H$, respectively, and if every edge d of G verifies: $\mathbf{vert}_G(d) = \mathbf{vert}_H(d)$. Let G and H be two hypergraphs such that $\mathbf{vert}_G(e) = \mathbf{vert}_H(e)$ for each $e \in \mathbf{E}_G \cap \mathbf{E}_H$. The *union of* G *and* H, denoted by $G \cup H$, is the minimal hypergraph that contains G and H. The *intersection of* G *and* H, denoted by $G \cap H$, is the maximal subhypergraph of both G and H.

Now we define the notion of a connected hypergraph. From a such definition, it follows that every hypergraph with no vertex is connected if and only if it contains a unique edge.

Definition 5 Let G be a hypergraph. A *path* of G is a nonempty sequence $p = (o_1, \ldots, o_m) \in (\mathbf{V}_G \cup \mathbf{E}_G)^+$ for some $m \geq 1$, with o_i and o_{i+1} incident for every $i \in [m-1]$. The *initial* (resp. *terminal*) *extremity* of p is o_1 (resp. o_m). An *internal vertex of* p is a vertex of the subsequence, eventually empty, $(o_2, \ldots, o_{m-1})$. The path p is *elementary* if every two edges of the form o_i and o_j with $1 \leq i < j \leq m$ are distinct. A path is a *cycle* having as extremities two identical vertices. Let p and q two paths of the form respectively $(o_1, \ldots, o_m)$ and $(u_1, \ldots, u_p)$ with $o_m = u_1$, the *concatenation of* p *and* q is the path $(o_1, \ldots, o_m, u_2, \ldots, u_p)$. A hypergraph G is *connected* if it is nonempty, and if every two elements of $\mathbf{V}_G \cup \mathbf{E}_G$ are the extremities of some path of G. A *connected component of* G is a maximal subhypergraph of G that is connected. A connected hypergraph G is 2-*edge-connected* if it contains at least one vertex, and if every two vertices of G are the extremities of two edge-disjoint paths of G.

We need three useful operations on hypergraphs that enable us to define from a hypergraph G and a set a new hypergraph that is necessarily a subhypergraph of G (operations "$\backslash$" and "$\restriction$") or not (operation $\backslash\backslash$).

Notation 6 Let G be a hypergraph and D a set. We denote by $G \backslash D$ (resp. $G \restriction D$) the maximal (resp. minimal) subhypergraph of G that does not contain (resp. contains) any element of D (as edge or as vertex). If D is a singleton $\{d\}$, $G \backslash D$ (resp. $G \restriction D$) is denoted by $G \backslash d$ (resp. $G \restriction d$). We denote by $G \backslash\backslash D$ the hypergraph $(\mathbf{V}_G - D, \mathbf{E}_G, f)$ where f associates with every edge d of G the set $\mathbf{vert}_G(d) - D$.

For instance, if e designs some edge of some hypergraph G, $G \upharpoonright e$ denotes a hypergraph with one edge and with vertices the extremities of e in G and $G \upharpoonright \mathbf{vert}_G(e)$ denotes the discrete one with vertices the extremities of e. It comes: $(G \upharpoonright e)\backslash e = G \upharpoonright \mathbf{vert}_G(e)$.

The next result, extension over hypergraphs of a classical result on graphs, that can be found in [16], will be admitted:

Lemma 7 *Every hypergraph G is 2-edge-connected if and only if G is connected and if $G\backslash d$ is connected, for every edge $d \in \mathbf{E}_G$.*

Let us define e-hypergraph.

Definition 8 An *e-hypergraph H* is a sequence $(\mathbf{e}_H, \mathbf{V}_H, \mathbf{E}_H, \mathbf{vert}_H)$ where $(\mathbf{V}_H, \mathbf{E}_H, \mathbf{vert}_H)$ is a hypergraph and $\mathbf{e}_H$ an edge of $\mathbf{E}_H$, the *source-edge* of H. A *source* (resp. *internal vertex*) of H is any vertex of $\mathbf{vert}_H(\mathbf{e}_H)$ (resp. $\mathbf{V}_H - \mathbf{vert}_H(\mathbf{e}_H)$). In order to simplify, $(\mathbf{V}_H, \mathbf{E}_H, \mathbf{vert}_H)$ is denoted by $\mathbf{G}_H$ and H shall be identified with the pair $(\mathbf{e}_H, \mathbf{G}_H)$. $\mathcal{G}$ denotes the set of all e-hypergraphs.

The *type* of H is the degree of $\mathbf{e}_H$. The *rank* of H is the maximal degree of all of its edges. A *subhypergraph* (resp. *connected component*) of H is a subhypergraph (resp. connected component) of $\mathbf{G}_H\backslash\mathbf{e}_H$ (that is not an e-hypergraph!).

Now, we recall the notion of an internally connected e-hypergraph due to Courcelle [4]. We extend the notions of a connected hypergraph and of a 2-edge-connected hypergraph to e-hypergraphs. These extensions are made in two different way: either we consider the hypergraph $\mathbf{G}_H$, or the hypergraph $\mathbf{G}_H\backslash\mathbf{e}_H$. The reason is simple: it works! For instance, see Lemmas 10 and 16.

Definition 9 (Connectivity) Let $H \in \mathcal{G}$. A *path of H* is a path of $\mathbf{G}_H$. It is *internal in H* if it belongs to I^+, $I^+ \times S$, $S \times I^+$ or $S \times I^+ \times S$ with $S := \mathbf{vert}_H(\mathbf{e}_H)$ and $I := (\mathbf{E}_H \cup \mathbf{V}_H) - (\{\mathbf{e}_H\} \cup \mathbf{vert}_H(\mathbf{e}_H))$. A subhypergraph K of H is *internally connected in H* if it is nonempty and if every two elements of $\mathbf{V}_K \cup \mathbf{E}_K$ are the extremities of some path of K internal in H. An *internally connected component of H* is a maximal subhypergraph of H to be internally connected in H. The e-hypergraph H is:

- *internally connected* if $\mathbf{G}_H\backslash\mathbf{e}_H$ is internally connected in H.
- *connected* if $\mathbf{G}_H\backslash\mathbf{e}_H$ is connected.
- 2-*edge-connected* if $\mathbf{G}_H$ is 2-edge-connected.

H is *empty* if $\mathbf{G}_H \backslash \mathbf{e}_H = \emptyset$. An *isolated vertex* of H is an isolated vertex of $\mathbf{G}_H \backslash \mathbf{e}_H$. H is *without-isolated-vertex* if it does not contain any isolated vertex. The set of all internally connected e-hypergraphs is denoted $\mathcal{G}_{i.c}$.

Observe that the precedent definition requires for every internally connected subhypergraph of some e-hypergraph to contain some non-source edge or some internal-vertex. Consequence of the precedent definition and of Lemma 10, we have the following characterization. The proof is left to the reader.

Lemma 10 *Every e-hypergraph H is 2-edge-connected if and only if it is connected, and if $H \backslash d$ is connected, for every $d \in \mathbf{E}_H \backslash \mathbf{e}_H$.*

The notion of an internal-connectivity e-hypergraph is less natural than the notion of a connected one. For example, two distinct internally connected component of some e-hypergraph H are not necessarily disjoint: eventually they can have in common certain source vertices of H. Notation 11 and Fact 12 permit to define the notion of an internally connected e-hypergraph thanks the notion of a connected e-hypergraph. This characterization will be useful in a large number of proofs.

Notation 11 Let $H \in \mathcal{G}$ and D be a set. If D is disjoint with $\{\mathbf{e}_H\} \cup \mathbf{vert}_H(\mathbf{e}_H)$, we denote by $H \backslash D$ the e-hypergraph $(\mathbf{e}_H, \mathbf{G}_H \backslash D)$. If D does not contain any internal vertex of H, we denote by $H \backslash\backslash D$ the e-hypergraph $(\mathbf{e}_H, \mathbf{G}_H \backslash\backslash D)$.

The proof of the next fact is easy and is omitted.

Fact 12 *For every e-hypergraph H, the following assertions are equivalent:*

- *H is internally connected.*
- *$H \backslash\backslash D$ is not defined or is internally connected, for every set D.*
- *H is connected and $H \backslash\backslash D$ is internally connected, for some set D.*
- *H and $H \backslash\backslash S$ are connected, with S the set of sources of H.*

Now, let us present the operation of substitution. This definition is more simple than in the case of concrete sourced-hypergraphs for at least two reasons: the edge to substitute in H is necessarily the source-edge of K, the substitution does not identify vertices of H.

Definition 13 Let $H, K \in \mathcal{G}$ with $\mathbf{G}_H \upharpoonright \mathbf{e}_K = \mathbf{G}_H \cap \mathbf{G}_K = \mathbf{G}_K \upharpoonright \mathbf{e}_K$ and $\mathbf{e}_H \neq \mathbf{e}_K$. We denote by $H[K]$ the e-hypergraph $(\mathbf{e}_H, (\mathbf{G}_H \cup \mathbf{G}_K) \backslash \mathbf{e}_K)$. Observing that H is the unique e-hypergraph L to verify $L[K] = H[K]$, H is said the *context* of K in $H[K]$.

Let $m \geq 1$ and $H, K_1, \ldots, K_m \in \mathcal{G}$ with $(H[K_1] \ldots)[K_m] = (H[K_{\pi(1)}] \ldots)[K_{\pi(m)}]$, for every permutation π on $[m]$. The e-hypergraph $(H[K_1] \ldots)[K_m]$ is denoted by $H[K_1, \ldots, K_m]$.

A property φ is said *substitution-closed in $\mathcal{G}$* if for every e-hypergraphs H and K that verify φ, the e-hypergraph $H[K]$ verifies φ if it is defined.

In our formalism, we don't orient the edges. The unique interest of a such choice is the simplicity in which we manipulate the object: see for example Fact 27 established in the next section. A counterpart of a such simplicity (in comparison for example with the formalism of Bauderon and Courcelle [1]) is the impossibility to extend such operations on isomorphic class of e-hypergraphs. It is easy to exhibit four e-hypergraphs H, K, H', K' with H (resp. K) isomorphic with H' (resp. K') such that $H[K]$ and $H'[K']$ are defined but are not isomorphic. To assure a such isomorphism, it would be necessarily to orient the replaced hyperedge, like it is made in [1]. However, the objects considered in this article are concrete. Hence, a such simplification can be made.

The next fact establishes a classical property of the substitution: that is context-free. The proof is easy and is omitted.

Lemma 14 *Let G, H, K be three e-hypergraphs with $(G[H])[K]$ defined. If $G[H[K]]$ (resp. $(G[K])[H]$) is defined, then it is equal to $(G[H])[K]$.*

Before to prove Lemma 16, a little fact that states that $\backslash\backslash$ commutes with the substitution. Its proof is easy, and is omitted.

Fact 15 *Let H, K two e-hypergraphs and D be a set with $(H[K])\backslash\backslash D$ defined. Then, $(H[K])\backslash\backslash D = (H\backslash\backslash D)[K\backslash\backslash D]$.*

Lemma 16 *The properties "internally connected", "connected", "2-edge-connected", "nonempty" and "without-isolated-vertex" are substitution-closed in $\mathcal{G}$.*

Proof.
Let $G, H, K \in \mathcal{G}$ such that $G = H[K]$. By definition, we have : $\mathbf{G}_G \backslash \mathbf{e}_G = (\mathbf{G}_H \cup \mathbf{G}_K) \backslash \{\mathbf{e}_H, \mathbf{e}_K\}$. Then :

1. G is connected, if H and K are connected.
   If $\mathbf{G}_H \backslash \{\mathbf{e}_H, \mathbf{e}_K\} = \emptyset$, the conclusion is obvious. Otherwise, every connected component of $\mathbf{G}_H \backslash \{\mathbf{e}_H, \mathbf{e}_K\}$ is not disjoint with $\mathbf{G}_K \upharpoonright \mathbf{e}_K$. The connectivity of $\mathbf{G}_K \backslash \mathbf{e}_K$ implies $\mathbf{G}_G \backslash \mathbf{e}_G$ and G connected.

2. G is internal-connected, if H and K are internal-connected.
   Let $S := \mathbf{vert}_G(\mathbf{e}_G)$. The e-hypergraphs H and K are connected (Fact 12), then G is connected (precedent point). The e-hypergraph $G \backslash\backslash S$, equal to $H \backslash\backslash S[K \backslash\backslash S]$ (Fact 15), is connected ($H \backslash\backslash S$ and $K \backslash\backslash S$ are connected (Fact 12)). Then, G is internal-connected (Fact 12).

3. G is 2-edge-connected, if H and K are 2-edge-connected.
   H and K are connected, then G too. For every $d \in \mathbf{E}_G \backslash \mathbf{e}_H$, the e-hypergraphs $H \backslash d$ and $K \backslash d$ are connected (Lemma 7), thus, $G \backslash d$, equal to $(N \backslash d)[P \backslash d]$, is connected. Then, G is 2-edge-connected (Lemma 7).

4,5 G is nonempty (resp. without-isolated-vertex), if H and K are nonempty (resp. without-isolated-vertex).
   Evident. □

The next fact studies the converse of the precedent result.

Fact 17 *Let $H, K \in \mathcal{G}$ with $H[K]$ defined. H is connected, if $H[K]$ is connected and if K is nonempty. H is internally connected, if $H[K]$ is internally connected and if $K \backslash\backslash \mathbf{vert}_H(\mathbf{e}_H)$ is nonempty.*

Proof.
Let $G, H, K \in \mathcal{G}$ such that $G = H[K]$. Let $S := \mathbf{vert}_H(\mathbf{e}_H)$. We have :

- H is connected, if G is connected and if K is nonempty.
  If $H \backslash \mathbf{e}_K$ is empty, the conclusion is obvious. Moreover, we suppose $H \backslash \mathbf{e}_K$ nonempty. If a connected component L of $H \backslash \mathbf{e}_K$ is disjoint with $\mathbf{G}_H \upharpoonright \mathbf{e}_K$, L is disjoint with $\mathbf{G}_K \backslash \mathbf{e}_K$, is a connected component of G disjoint with the nonempty subhypergraph $\mathbf{G}_K \backslash \mathbf{e}_K$ of G. Contradiction. Then, every connected component of $H \backslash \mathbf{e}_K$ is not disjoint with $\mathbf{G}_H \upharpoonright \mathbf{e}_K$. Hence, H is connected.

- H is internally connected, if G is internally connected and if $K \backslash\backslash S$ is nonempty.
  G is connected (Fact 12), then H is connected (precedent point). $G \backslash\backslash S$ is connected (Fact 12), is equal to $(H \backslash\backslash S)[K \backslash\backslash S]$ (Fact 15). $H \backslash\backslash S$ is connected (precedent point). H is internally connected (Fact 12). □

2 Tree-decomposition

In this section, we recall the notion of a tree-decomposition, introduced by Robertson and Seymour in [14]. The definition of a tree-decomposition we select, is the one in which every node of the tree is associates not to a set of vertices of the tree-decomposed hypergraph, but to a subhypergraph of this hypergraph. This definition can be found in [15].

That permits to define, in the same way than for hypergraphs, e-tree-decompositions from tree-decomposition: an e-tree-decomposition X is a tree-decomposition with a distinguished edge: its source-edge that is denoted by $\mathbf{e}_X$. Their set is denoted by $\mathcal{T}$. The e-tree-decompositions are very near with algebraic terms for two reasons. Firstly, every $X \in \mathcal{T}$ contains an e-hypergraph (its value denoted by $\mathbf{val}(X)$) and a arborescent description of it. Secondly, the tree can be considered as a rooted tree: it suffices to consider as its root the unique node that contains the source-edge $\mathbf{e}_X$.

This proximity permits to extend on $\mathcal{T}$ some very useful tools usually defined for terms. For example, we define the relation $\sqsubseteq$ and the operation of substitution $[\,]$. We obtain two non surprising but important properties: it is context-free and commutes with **val**.

Before to recall the notion of a tree-decomposition, let us recall the trees.

Definition 18 A *forest* is a graph with no elementary cycle and with at least one vertex. A *tree* is a connected forest. A vertex (resp. edge) of a tree is called a *node* (resp. *arc*). For every tree T, its set of nodes $\mathbf{V}_T$ is denoted by $\mathbf{N}_T$, its set of arcs $\mathbf{E}_T$ is denoted by $\mathbf{A}_T$. A tree is atomic if it contains a unique node. A *rooted tree* is a pair (T, r) consisting of a tree T and a distinguished node r called the *root*. Let s and t be two nodes of a rooted-tree (T, r). The node s is a *descendant node of* t if every path of T from s to r contains t. The node s is a *child* (resp. the *parent*) of a node t, if s and t are adjacent and if s (resp. t) is a descendant node of t (resp. s). A *leaf of* (T, r) is a node with no children.

The below definition is illustrated by Example 20.

Definition 19 (Tree-decomposition) A *tree-decomposition* is a pair (T, g) where T is a tree and where g associates with every node t of T a hypergraph $g(t)$ such that:

- $\mathbf{E}_{g(s)} \cap \mathbf{E}_{g(t)} = \emptyset$, for all distinct nodes s, t of T.
- for all nodes s, u of T, every node t of the elementary path of T with extremities s and u, verifies: $g(s) \cap g(u) \subseteq g(t)$.

The *width* of a tree-decomposition (T,g) is denoted by $\underline{wd}(T,g)$ and is the maximum of $\mathbf{card}(\mathbf{V}_{g(t)})-1$ taken over all $t \in \mathbf{N}_T$. The *tree-width* of a hypergraph G, denoted by $\underline{twd}(G)$, is the minimum width of all tree-decompositions (T,g) such that $G = \bigcup_{t\in\mathbf{N}_T} g(t)$. For every tree-decomposition (T,g) and every subset $U \subseteq \mathbf{N}_T$ (resp. subhypergraph $U \subseteq T$), we denote by $g(U)$ the hypergraph $\bigcup_{t\in U} g(t)$ (resp. $\bigcup_{t\in\mathbf{V}_U} g(t)$).

Example 20 Figure 1 represents, at the left side, a hypergraph G of treewidth 2 and, at the right side, a tree-decomposition (T,g) of G of width 3. G contains 4 vertices (represented by dark disks) and 4 edges respectively of degree $0,1,2,3$. The edge of degree 2 is represented by a simple line. The other edges are represented by a white box linked to its extremities thanks dark lines. T contains 3 nodes. Each of them is represented by a white disk. The mapping g is representing by drawing in each node of T the subhypergraph $g(t)$ of G. The dotted lines permit for every vertex of G to relate its different occurences that appear in the nodes of T.

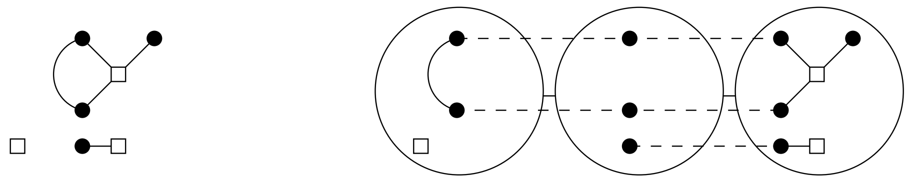

Figure 1: a tree-decomposition of a hypergraph.

Definition 21 An *e-tree-decomposition* X is a sequence $(\mathbf{e}_X, \mathbf{T}_X, \mathbf{g}_X)$, where $(\mathbf{T}_X, \mathbf{g}_X)$ is a tree-decomposition and $\mathbf{e}_X$ an edge of $\mathbf{g}_X(\mathbf{T}_X)$, the *source-edge of* X. The hypergraph $\mathbf{g}_X(\mathbf{T}_X)$, denoted by $\mathbf{G}_X$, is supposed to be disjoint with $\mathbf{T}_X$. We denote by $\mathcal{T}$ the set of all e-tree-decompositions and, for every $k \geq -1$, by $\mathcal{T}_k$ the set $\{X \in \mathcal{T} \mid \underline{wd}(X) \leq k\}$.

Let $X \in \mathcal{T}$. The e-hypergraph *denoted by* X, the *value* of X, is the couple $(\mathbf{e}_X, \mathbf{G}_X)$, denoted by $\mathbf{val}(X)$. An *edge* (resp. *vertex*, *source*, *internal vertex*) of X is an edge (resp. vertex, source, internal vertex) of $\mathbf{val}(X)$. The *root of* X, denoted by $\mathbf{r}_X$, is the unique node t of $\mathbf{T}_X$ such that $\mathbf{e}_X \in \mathbf{E}_{\mathbf{g}_X(t)}$. An *arc* (resp. *node*, *leaf*) of X is an arc (resp. node, leaf) of $(\mathbf{T}_X, \mathbf{r}_X)$. The set of all nodes, arcs, vertices, edges of X are denoted respectively by $\mathbf{N}_X$, $\mathbf{A}_X$, $\mathbf{V}_X$, $\mathbf{E}_X$.

Two e-tree-decompositions are *equivalent* if they have same value. For each $u \subseteq \mathcal{T}$, we denote by $\mathbf{val}(u)$ the set $\{\mathbf{val}(X) \mid X \in u\}$. Two subsets u, v of $\mathcal{T}$ are *equivalent* if $\mathbf{val}(u) = \mathbf{val}(v)$.

In the next definition, we extend the substitution to $\mathcal{T}$ (see Example 23).

Definition 22 (Edge substitution) Let $Y, Z \in \mathcal{T}$ with $\mathbf{T}_Y \cap (\mathbf{G}_Z \cup \mathbf{T}_Z) = \emptyset = (\mathbf{G}_Y \cup \mathbf{T}_Y) \cap \mathbf{T}_Z$ and $\mathbf{val}(Y)[\mathbf{val}(Z)]$ defined. We denote by $Y[Z]$ denote the e-tree-decomposition $(\mathbf{e}_Y, T, g)$ where:

- T is obtained from $\mathbf{T}_Y \cup \mathbf{T}_Z$ by adding the edge $\mathbf{e}_Z$ of extremities $\mathbf{r}_Z$ and the unique node s of Y that verifies: $\mathbf{e}_Z \in \mathbf{g}_Y(s)$.

- g associates with every node s of T the hypergraph $\mathbf{g}_Y(s) \backslash \mathbf{e}_Z$ if s is a node of Y and $\mathbf{g}_Z(s) \backslash \mathbf{e}_Z$, otherwise.

Let $m \geq 1$ and $Y, Z_1, \ldots, Z_m \in \mathcal{T}$ with $(Y[Z_1] \ldots)[Z_m] = (Y[Z_{\pi(1)}] \ldots)[Z_{\pi(m)}]$, for every permutation π on $[m]$. The e-tree-decomposition $(Y[Z_1] \ldots)[Z_m]$ is denoted by $Y[Z_1, \ldots, Z_m]$. For all subsets u, v of $\mathcal{T}$, we denote b $u[v]$ the union $u \cup v \cup \{H[K_1, \ldots, K_m] \mid H \in u, K_1, \ldots, K_m \in v, m \geq 1\}$.

Example 23 Figure 2 represents three e-tree-decompositions X, Y, Z that verify $X = Y[Z]$. X is drawed at the top of the figure. Y (resp. Z) is drawed at the bottom and at the left (resp. right) side. Source edge are represented with thick edges. In concordance with the definition of [ ], the arc d of X (edge of the tree $\mathbf{T}_X$) is the unique edge shared by Y and Z and is the source-edge of Y.

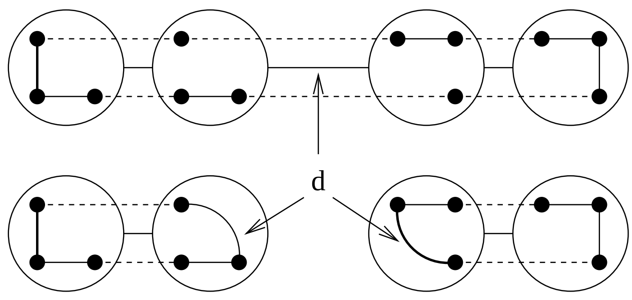

Figure 2: (hyper)edge-replacement.

The substitution defined above is context-free and commutes with the valuation mapping **val**. These both properties are the object of the both next facts. Lemma 24 is the direct consequence of Lemma 14. Its proof is omitted. The proof of Lemma 25 is obvious and is omitted.

Lemma 24 *Let* $X, Y, Z \in \mathcal{T}$ *with* $(X[Y])[Z]$ *defined. If* $X[Y[Z]]$ *(resp.* $(X[Z])[Y]$*) is defined, then it is equal to* $(X[Y])[Z]$.

Lemma 25 *Let be an e-tree-decomposition of the form* $Y[Z_1, \ldots, Z_m]$ *for some* $m \geq 1$. *Then,* $\mathbf{val}(Y[Z_1, \ldots, Z_m]) = \mathbf{val}(Y)[\mathbf{val}(Z_1), \ldots, \mathbf{val}(Z_m)]$.

As it can be observed in Example 23, every subtree of any e-tree-decomposition determines a new e-tree-decompositions, *contained* in the first. These notions are formalized below:

Definition 26 Let $X \in \mathcal{T}$ and T a subtree of $\mathbf{T}_X$. The e-tree-decomposition *generated by* X *and* T is the sequence (e, T, g), denoted by $X|T$, where:

- e is $\mathbf{e}_X$ if $\mathbf{r}_X \in \mathbf{N}_T$ and, otherwise, the unique edge incident in $\mathbf{T}_X$ with some node of T and some node of the connected component of $\mathbf{T}_X \backslash d$ that contains $\mathbf{r}_X$.
- g associates with every node t of T the union $\mathbf{g}_X(t) \cup \bigcup_{d \in D_t} G_d$ with D_t the set of all arcs of $\mathbf{A}_X - \mathbf{A}_T$ incident in $\mathbf{T}_X$ with t and where G_d designs for every arc d of X the unique connected hypergraph having for unique edge d and for vertices $\mathbf{V}_{\mathbf{g}_X(u)} \cap \mathbf{V}_{\mathbf{g}_X(v)}$ with u and v the two extremities of d in $\mathbf{T}_X$.

These definitions are extended in the obvious way to every connected set of nodes of X and to every node of X. Let d be an arc of X. We denote by $X \uparrow d$ (resp. $X \downarrow d$) the e-tree-decomposition generated by X and the maximal subtree of $\mathbf{T}_X \backslash d$ that contains (resp. does not contain) $\mathbf{r}_X$.

An e-tree-decomposition Y is *contained in* X (resp. *strictly*), denoted by $Y \sqsubseteq X$, (resp. $\sqsubset$) if $Y = X|T$ for some subtree (resp. proper subtree) T of $\mathbf{T}_X$. $X \in \mathcal{T}$ is *atomic* if $\mathbf{T}_X$ is atomic. For every $u \subseteq \mathcal{T}$, we denote by $\mathbf{atom}(u)$ the set $\{X|t \mid X \in u, t \in \mathbf{N}_X\}$.

Let $X \in \mathcal{T}$. The *degree* of some edge (resp. arc) d in X is the degree of d in $\mathbf{G}_X$ (resp. $\mathbf{G}_{X \uparrow d}$). The *type* of X is the degree of $\mathbf{e}_X$ in X. The *rank* of X is the maximal degree of all of its arcs and edges. For every integer l, the set of all e-tree-decompositions of type (resp. rank) at most l is denoted by $Type_l$ (resp. $Rank_l$). $Type_{>0}$ denotes the set of all e-tree-decompositions of non null type.

An interesting property of the substitution $[\,]$ in $\mathcal{T}$ is the fact that the e-tree-decomposition result $Y[Z]$ keeps every information contained in Y or in Z. More precisely, the pair $(Y[Z], \mathbf{e}_Z)$ determines a unique pair $\{Y, Z\}$.

This property due to the fact that edges are non-oriented is expressed by Fact 27. Its proof is easy and is omitted.

Fact 27 *For every $X, Y, Z \in \mathcal{T}$, the following assertions are equivalent:*

- $X = Y[Z]$.
- *X contains an arc d such that: $(X \uparrow d, X \downarrow d) = (Y, Z)$.*

A very useful operation over e-tree-decompositions is the contraction of an arc. Clearly, a such operation preserves the value.

Definition 28 The e-tree-decomposition *obtained from some $X \in \mathcal{T}$ by contracting some arc d of X* is the sequence $(\mathbf{e}_X, T, g)$ where:

- T is obtained from $\mathbf{T}_X \backslash d$ by identifying s with t, renamed t.
- g associates with every node u of T the hypergraph $\mathbf{g}_X(u)$ if $u \neq t$ and $\mathbf{g}_X(s) \cup \mathbf{g}_X(t)$, otherwise.

  where s and t are the extremities of d in $\mathbf{T}_X$ with t the parent of s.

The e-tree-decomposition *obtained from X by contracting some set D of arcs of X* is the one obtained from X by contracting all arcs of D, one by one.

In a natural way, we extend the operation $\backslash\backslash$ on $\mathcal{T}$.

Notation 29 Let X be an e-tree-decomposition and D be a set that does not contain any internal vertex of X. We denote by $X \backslash\backslash D$ the e-tree-decomposition $(\mathbf{e}_X, \mathbf{T}_X, h)$ where h associates with every node t of X the hypergraph $\mathbf{g}_X(t) \backslash\backslash D$.

The operation $\backslash\backslash$ verifies tow nice properties. It commutes with **val** and with [], under certain condition. These both properties are the object of the both following facts. Their proof are easy and are omitted.

Fact 30 *Each e-tree-decomposition of the form $X \backslash\backslash D$ verifies: $\mathbf{val}(X \backslash\backslash D) = \mathbf{val}(X) \backslash\backslash D$.*

Fact 31 *Let $Y, Z \in \mathcal{T}$ and let D be a set. If $(Y[Z]) \backslash\backslash D$ is defined, or if $Y \backslash\backslash D$ and $Z \backslash\backslash D$ are defined, then $(Y[Z]) \backslash\backslash D = (Y \backslash\backslash D)[Z \backslash\backslash D]$.*

To conclude this section, we extend the different notions of connectivity defined above over e-hypergraphs on e-tree-decompositions. A such extension is made in a very simple way: an e-tree-decomposition is "connected" (generic term) if every e-hypergraph it contains is "connected".

Definition 32 (Connectivity) An e-tree-decomposition Y is *connected* (resp. *internally connected*, *2-edge-connected*, *nonempty*, *without-isolated-vertex*) if for every $X \sqsubseteq Y$, the e-hypergraph $\mathbf{val}(X)$ is connected (resp. internally connected, 2-edge-connected, nonempty, without-isolated-vertex). The set of all connected (resp. internally connected, 2-edge-connected) e-tree-decompositions is denoted by $\mathcal{T}_{1.c}$ (resp. $\mathcal{T}_{i.c}$, $\mathcal{T}_{2.c}$).

Interesting properties of these three notions of connectivity, they are substitution-closed and hereditary in $\mathcal{T}$. This fact is the object of Lemma 35.

Definition 33 A property φ is *hereditary in $\mathcal{T}$* if for every e-tree-decomposition X that verifies φ, every e-tree-decomposition contained in X verifies φ. A property φ is *substitution-closed in $\mathcal{T}$* if for all e-tree-decompositions Y and Z that verify φ, the e-tree-decomposition $Y[Z]$ is not defined or verifies φ.

Fact 34 *Let φ be a property such that for every $X \in \mathcal{T}$ and every arc of X, $X \uparrow d$ and $X \downarrow d$ verify φ if X verifies φ. Then, φ is hereditary.*

Proof.
Direct consequence of Fact 27 and the fact that for all e-tree-decompositions $W \sqsubset X$, X is of the form $Y[Z_1, \dots, Z_m]$ with $W \in \{Y, Z_1, \dots, Z_m\}$ or of the form $Y[W[Z_1, \dots, Z_m]]$ for some $Y, Z_1, \dots, Z_m \in \mathcal{T}$ and some $m \geq 1$. □

Lemma 35 *Every property defined in Definition 32 is hereditary and substitution-closed in $\mathcal{T}$.*

Proof.
Let φ be a property defined in Definition 32. $\sqsubseteq$ is transitive in $\mathcal{T}$, then φ is hereditary. Let d be an arc of some $X \in \mathcal{T}$ such that $X \uparrow d$ and $X \downarrow d$ verify φ. The fact that φ is hereditary, the equality $\mathbf{atom}(\{X\}) = \mathbf{atom}(\{X \uparrow d, X \downarrow d\})$, the fact that φ is substitution-closed in $\mathcal{G}$ (Lemma 16) imply that X verifies φ. Then, φ is substitution-closed. □

Definition 32 is not practical, when we have to establish that some e-tree-decomposition is connected. The next lemma gives three definitions of these notions of connectivity, that are equivalent and more simple.

Lemma 36 *An e-tree-decomposition X is connected (resp. internally connected, 2-edge-connected) if and only if* $\mathbf{val}(X)$ *is connected (resp. internally connected, 2-edge-connected) and if every arc d of X is such that, respectively:*

- $\mathbf{val}(X\!\downarrow\! d)$ *is connected.*

- $\mathbf{val}(X\!\downarrow\! d)$ *is internally connected.*

- $\mathbf{G}_{X\uparrow d}\backslash d$ *and* $\mathbf{G}_{X\downarrow d}\backslash d$ *are connected.*

Proof.
An e-tree-decomposition X *verifies* φ_1 (resp. $\varphi_{i.c}$, φ_2) if $\mathbf{val}(X)$ is connected (resp. internally connected, 2-edge-connected) and if for every arc d of X, $\mathbf{val}(X\downarrow d)$ is connected (resp. $\mathbf{val}(X\downarrow d)$ is internally connected, $\mathbf{G}_{X\uparrow d}\backslash d$ and $\mathbf{G}_{X\downarrow d}\backslash d$ are connected). As a consequence of Lemma 35 and of the fact that every atomic e-tree-decomposition is connected (resp. internally connected, 2-edge-connected) if and only if it verifies φ_1 (resp. $\varphi_{i.c}$, φ_2), to conclude it suffices to prove that φ_1, $\varphi_{i.c}$ and φ_2 are hereditary and substitution-closed. Observe that for all arcs c, d of some $X \in \mathcal{T}$, we have: $X\!\uparrow\! c = (X\!\uparrow\! d)\!\uparrow\! c$ and $X\!\downarrow\! c = ((X\!\uparrow\! d)\!\downarrow\! c)[X\!\downarrow\! d]$, if c is an arc of $X\!\uparrow\! d$, we have: $X\!\uparrow\! c = X\!\uparrow\! d[(X\!\downarrow\! d)\!\uparrow\! c]$ and $X\!\downarrow\! c = (X\!\downarrow\! d)\!\downarrow\! c$, if c is an arc of $X\!\downarrow\! d$.

1 φ_1 is substitution-closed.
Let $X \in \mathcal{T}$ and $d \in \mathbf{A}_X$ such that $X\uparrow d$ and $X\downarrow d$ verifying φ_1. Lemma 16 involves $\mathbf{val}(X)$ connected. Let $c \in \mathbf{A}_X$. If $c = d$, $\mathbf{val}(X\!\downarrow\! c)$ is by hypothesis connected. If c is an arc of $X\uparrow d$ (resp. $X\downarrow d$), $\mathbf{val}(X\!\downarrow\! c)$ is equal to $\mathbf{val}((X\!\uparrow\! d)\!\downarrow\! c)[\mathbf{val}(X\!\downarrow\! d)]$ (resp. $\mathbf{val}((X\!\downarrow\! d)\!\downarrow\! c)$) and is connected (by Lemma 16). Then, φ_1 is substitution closed.

2 φ_1 is hereditary.
Let $X \in \mathcal{T}$ that verifies φ_1 and let $d \in \mathbf{A}_X$. The e-hypergraph $\mathbf{val}(X\uparrow d)$ (resp. $X\!\downarrow\! d$) is, by Fact 17, (resp. by hypothesis) connected. Let c be an edge of $X\uparrow d$. The e-hypergraph $H = \mathbf{val}((X\uparrow d)\downarrow c)$ verifies $\mathbf{val}(X\downarrow c) = H[\mathbf{val}(X\downarrow d)]$ and, then, is connected (Fact 17). Then, $X\uparrow d$ verifies φ_1. Let c be an edge of $X\downarrow d$. The e-hypergraph $\mathbf{val}((X\downarrow d)\downarrow c)$ is equal to $X\!\downarrow\! c$ and then is connected. Then, $X\!\downarrow\! d$ verifies φ_1. Thus, φ_1 is hereditary.

3,4 $\varphi_{i.c}$ is substitution-closed and hereditary.
The proof is obtained from the ones of Point 1 and 2, by replacing "connected" by "internally connected".

5 φ_2 is substitution-closed.
Let $X \in \mathcal{T}$ and let $d \in \mathbf{A}_X$ such that $Y := X\!\uparrow\! d$ and $Z := X\!\downarrow\! d$ verify φ_2. $\mathbf{val}(X)$ is 2-edge-connected (Lemma 16). Let $c \in \mathbf{A}_X$. If $c = d$, $\mathbf{G}_Y\backslash d$ and $\mathbf{G}_Z\backslash d$ are by hypothesis connected. Otherwise, $(c, \mathbf{G}_{X\uparrow c})$,

equal to $(c, \mathbf{G}_{Y\Uparrow c})$ (resp. $(c, \mathbf{G}_{Z\Uparrow c})[(d, G_Y))$ if c is an arc of Y (resp. Z), is connected, the e-hypergraph $(c, \mathbf{G}_{X\Downarrow c})$, equal to $\mathbf{val}(Y \downarrow c)[\mathbf{val}(Z)]$ (resp. $\mathbf{G}_{Y\Downarrow c}$) if c is an arc of Y (resp. Z), is connected. Then, φ_2 is substitution-closed.

6 φ_2 is hereditary.
Let $X \in \mathcal{T}$ that verifies φ_2 and let $d \in \mathbf{A}_X$. Let $Y := X \uparrow d$ and $Z := X \downarrow d$. Let $c \in \mathbf{E}_Y$. If $c = d$, $(c, \mathbf{G}_Y)$ is, by hypothesis, connected. Otherwise, $(c, \mathbf{G}_Y)$ verifies $(c, \mathbf{G}_X) = (c, \mathbf{G}_Y)[\mathbf{val}(Z)]$ with $(c, \mathbf{G}_X)$ connected and is connected (Fact 17). Then, $\mathbf{val}(Y)$ is 2-edge-connected. Let $c \in \mathbf{A}_Y$. Let $L := (c, \mathbf{G}_{Y\Uparrow c})$ and $M := (c, \mathbf{G}_{Y\Downarrow c})$. The e-hypergraph L verifies $(c, \mathbf{G}_{X\Uparrow c}) = L[\mathbf{val}(Z)]$ (resp. is equal to $(c, \mathbf{G}_{X\Uparrow c})$) if d is an arc of $X \uparrow c$ (resp. $X \downarrow c$) and, by Fact 17 (resp. by hypothesis), is connected. M is equal to $(c, \mathbf{G}_{X\Downarrow c})$ (resp. verifies $(c, \mathbf{G}_{X\Downarrow c}) = M[\mathbf{val}(Z)]$) if d is an arc of $X \uparrow c$ (resp. $X \downarrow c$) and, by Fact 17 (resp. by hypothesis), is connected. Then, Y verifies φ_2. By a symmetrical proof than above, we prove that Z verifies φ_2. Thus, φ_2 is hereditary. □

In a similar way than in Fact 12, we compare the notions of a connected e-tree-decomposition and of an internally connected e-tree-decomposition.

Fact 37 *For every $X \in \mathcal{T}$, the following assertions are equivalent:*

- *X is internally connected.*
- *$X \backslash\backslash D$ is not defined or internally connected, for every set D.*
- *X is connected and $X \backslash\backslash D$ is internally connected, for some set D.*
- *X and $X \backslash\backslash S$ are connected, with S the set of sources of X.*

Proof.
Direct consequence of Definition 32 and Fact 12. □

3 Linearity and quasi-linearity

In this section, we define quasi-linear subsets of $\mathcal{T}$. This definition induces a new measure of complexity over e-tree-decompositions, more precisely of their rooted trees. Hence, each e-tree-decomposition has two complexities: this new one and the width. That permits to study these two ones under "connectivity" constraints. Example 39 illustrates the below definition.

Definition 38 (Quasi-linearity) An e-tree-decomposition is *linear* if it has a unique leaf. Their set is denoted by $\mathcal{L}$. We define $\mathcal{L}_0 := \emptyset$ and, for each $p \geq 0$, we define $\mathcal{L}_{p+1} := \mathcal{L}[\mathcal{L}_p]$. Every subset of $\mathcal{L}_p$ for some $p \geq 0$ is said *quasi-linear.*

Example 39 Figure 3 represents three e-tree-decompositions of $\mathcal{L}_2 = \mathcal{L}[\mathcal{L}]$. The value of each of them is an empty e-hypergraph (with no vertex and having as unique edge the source edge). The source edge is represented by a box. The first e-tree-decomposition in the left side is linear ($\in \mathcal{L}$). The other ones are not linear and, then, belong to $\mathcal{L}_2 - \mathcal{L}$.

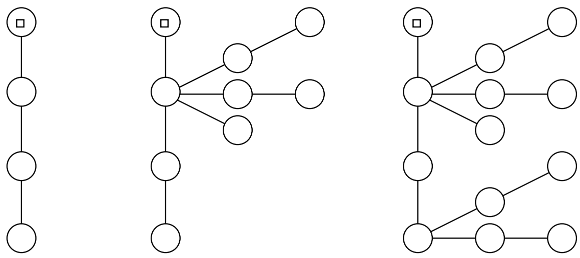

Figure 3: a quasi-linear set.

Note that above definition induces a new notion of complexity on $\mathcal{T}$ that associates with every $X \in \mathcal{T}$ the unique integer $p(X) \geq 1$ such that $X \in \mathcal{L}_{p(X)} - \mathcal{L}_{p(X)-1}$. This complexity is near with the path-width of the tree $\mathbf{T}_X$, usually denoted $\underline{pwd}$ (the path-width of a hypergraph is the smallest width of all its "path-decompositions", indeed tree-decomposition of the form (T, g) with T a path). It is not difficult to prove that for every $X \in \mathcal{T}$, the integer $p(X)$ is, almost 1, the smallest width of path-decompositions of $(\mathbf{T}_X, \mathbf{r}_X)$ that eliminates every child before every parent and, verifies: $\underline{pwd}(\mathbf{T}_X) \leq p(X) \leq 2 \cdot (1 + \underline{pwd}(\mathbf{T}_X))$.

In the next lemma, we present how to compute $p(X)$ for each $X \in \mathcal{T}$:

Lemma 40 *For every $X \in \mathcal{T}$, the integer p such that $X \in \mathcal{L}_p - \mathcal{L}_{p-1}$ is $p_X(\mathbf{r}_X)$ where p_X associates with every node t of X the integer defined recursively in the following way:*

- $p_X(t) = 1$, *if t is a leaf of X.*
- $p_X(t) = h(p_X(t_1), \ldots, p_X(t_m))$, *with $t_1, \ldots, t_m$ the childrens of t, otherwise.*

where h associates with every sequence $(n_1, \dots, n_m) \in \mathbb{N}^m$ for some $m \geq 1$ the integer $\max\{n_i \mid i \in [m]\}$ if there is a unique integer $j \in [m]$ such that $n_j = \max\{n_i \mid i \in [m]\}$ and $1 + \max\{n_i \mid i \in [m]\}$, otherwise.

Proof.
For every $X \in \mathcal{T}$, we denote by $p(X)$ the unique integer p such that $X \in \mathcal{L}_p - \mathcal{L}_{p-1}$ and by p_X (resp. h) the mapping defined in Lemma 40. Let us prove $p(X) = p_X(\mathbf{r}_X)$ for every $X \in \mathcal{T}$.

Clearly, every $X \in \mathcal{T}$ with $p_X(\mathbf{r}_X) = 1$ is linear. Suppose there is $l \geq 1$ such that every $X \in \mathcal{T}$ with $p_X(\mathbf{r}_X) \leq l$ verifies $p(X) \leq p_X(\mathbf{r}_X)$. Let $X \in \mathcal{T}$ such that $p_X(\mathbf{r}_X) = l + 1$. Denote by P the minimal subtree of $\mathbf{T}_X$ that contains every node t of X such that: $p_X(t) = l + 1$. As a consequence of the definition of h, $(P, \mathbf{r}_X)$ contains a unique leaf. Hence, $X|P \in \mathcal{L}$. Denote by v the set of all e-tree-decompositions generated by X and by some tree of $\mathbf{T}_X \backslash \mathbf{N}_P$. As a consequence of the definition of h, $p_Z(t) = p_X(t) \leq l$, for every $Z \in v$ and every node t of Z. By induction, $p(Z) \leq l$ for every $i \in [m]$. The inclusion $X \in \{X|P\}[v]$ implies $p(X) \leq l+1 \leq p_X(\mathbf{r}_X)$. Then, $p(X) \leq p_X(\mathbf{r}_X)$ for every $X \in \mathcal{T}$.

Clearly, every $X \in \mathcal{L}$ verifies $p_X(\mathbf{r}_X) = 1$. Suppose there is $l \geq 1$ such that every $X \in \mathcal{L}_l$ verifies: $p_X(\mathbf{r}_X) \leq p(X)$. Let $X \in \mathcal{L}_{l+1} - \mathcal{L}_l$. Thus, X is equal to $Y[Z_1, \dots, Z_m]$ for some $m \leq 1$, $Y \in \mathcal{L}$ and some $Z_1, \dots, Z_m \in \mathcal{T}_l$. By induction, every $i \in [m]$ verifies $p_{Z_i}(\mathbf{r}_{Z_i}) \leq p(Z_i) \leq l$. Clearly, $p_X(t) = p_{Z_i}(t)$, for every $i \in [m]$ and every node t of Z_i. Every leaf t of Y verifies $p_X(t) \leq 1 + l$. We have: $h(a_1 + b_1, \dots, a_n + b_n) \leq h(a_1, \dots, a_n) + l$, for every sequences $(a_1, \dots, a_n) \in \mathbb{N}_+^n$ and $(b_1, \dots, b_n) \in [0, l]^n$ for some $n \geq 1$. Then, every node t of X verifies $p_X(t) \leq p_Y(t) + l \leq p(X)$. Thus, $p_X(\mathbf{r}_X) \leq p(X)$ for every $X \in \mathcal{T}$. □

The precedent lemma has two following corollaries.

Corollary 41 *Every e-tree-decomposition of the form $Y[Z]$ verifies:*

- $p(Y[Z]) = p(Z)$ *if $p(Y) < p(Z)$.*
- $p(Y[Z]) \leq p(Y) + 1$ *if $p(Y) \geq p(Z)$.*

where for each $X \in \mathcal{T}$, $p(X)$ denotes the unique p such that $X \in \mathcal{L}_p - \mathcal{L}_{p-1}$.

Proof.
For every $X \in \mathcal{T}$, we denote by $p(X)$ the unique integer p such that $X \in \mathcal{L}_p - \mathcal{L}_{p-1}$, by p_X (resp. h) the mapping defined in Lemma 40. Let $X, Y, Z \in \mathcal{T}$

with $X = Y[Z]$ and q the integer $\max\{p(Y), p(Z)\}$. Denote by U the e-tree-decomposition generated by Y and the minimal subtree of $\mathbf{T}_Y$ that contains $\mathbf{r}_X$, the node of Y adjacent in $\mathbf{T}_X$ with $\mathbf{r}_Z$ and every node t that verifies: $p_Y(t) = q$. Then, $Y = U[U_1, \ldots, U_m]$ for some $U_1, \ldots, U_m \in \mathcal{T}$ and some $m \geq 1$. As a consequence of Lemma 40, $p(U_i) < q$ for every $i \in [m]$. Denote by V the e-tree-decomposition generated by Z and the minimal subtree of $\mathbf{T}_Z$ that contains every node t of Z such that: $p_Z(t) = p(Z)$. Then, $Z = V[V_1, \ldots, V_m]$ for some $V_1, \ldots, V_n \in \mathcal{T}$ and some $n \geq 1$. Clearly, $p(V_i) < q$ for every $i \in [n]$. The equality $X = (U[V])[U_1, \ldots, U_m, V_1, \ldots, V_n]$ implies $p(X) \leq p(U[V]) + q - 1$. Two cases appear:

- $p(X) = p(Z)$ if $p(Y) < p(Z)$.
  $(\mathbf{T}_U, \mathbf{r}_X)$ is a rooted path. It comes: $p(U[V]) = 1$ and $p(X) = p(Z)$.

- $p(X) \leq 1 + p(Y)$ if $p(Y) \geq p(Z)$.
  The rooted trees $(\mathbf{T}_U, \mathbf{r}_X)$, $(\mathbf{T}_{U[V]}, \mathbf{r}_X)$ are the "union" of two rooted path. It comes: $p(U[V]) \leq 2$ and $p(X) \leq 1 + p(Y)$. $\square$

The proof of the next corollary uses the same technical than the precedent one. It is left to the reader. Note that the complexity of the complete binary rooted-tree with 2^n leaves is $1 + n$.

Corollary 42 *Each e-tree-decomposition having n leaves belongs to $\mathcal{L}_{\lceil log(1+n) \rceil}$.*

To conclude this section, we study the five notions defined in Definitions 9 and 32 that concern e-hypergraphs and e-tree-decompositions and the two notions of complexities defined on $\mathcal{T}$. Let us consider some $H \in \mathcal{G}$, some property φ (in Definition 32) verified by H and $k := \underline{twd}(H)$. Let us define $\mathcal{T}_H := \{X \in \mathcal{T} \mid H = \mathbf{val}(X)\}$ and $\mathcal{T}_\varphi := \{X \in \mathcal{T} \mid X \models \varphi\}$. A first natural question comes: is $\mathcal{T}_\varphi \cap \mathcal{T}_k \cap \mathcal{T}_H$ nonempty? The answer is yes. The proof presents no difficulty and is made by using Lemma 36.

Let us interest to the notion of quasi-linearity. Denote by p the integer $\min\{l \mid \mathcal{L}_l \cap \mathcal{T}_k \cap \mathcal{T}_H \neq \emptyset\}$. It follows a second natural question: is $\mathcal{L}_p \cap \mathcal{T}_\varphi \cap \mathcal{T}_k \cap \mathcal{T}_H$ nonempty? If φ is the property "nonempty" or "without-isolated-vertex", the answer is yes. This result is the object of Lemma 43. Otherwise, the answer depends on the graph. Nevertheless, if φ is the property "internally connected", $\mathcal{L}_{p\cdot(k+1)} \cap \mathcal{T}_\varphi \cap \mathcal{T}_k \cap \mathcal{T}_H \neq \emptyset$. This result is the object of Lemma 44. Note that, by using similar technical, we can extend this result to the properties "connected" and "2-edge-connected".

Lemma 43 *Let k, l be two integers. Every $X \in \mathcal{L}_l \cap \mathcal{T}_k$ admits an equivalent e-tree-decomposition $Y \in \mathcal{L}_l \cap \mathcal{T}_k$ such that:*

- *Y is nonempty, if* $\mathbf{val}(X)$ *is nonempty.*
- *Y is without-isolated-vertex, if* $\mathbf{val}(X)$ *is without-isolated-vertex.*
- *Y verifies every property φ defined in Definition 32 and verified by X.*

Proof.
Let k, l be two integers. For every $X \in \mathcal{T}$, we denote by $\|X\|$ the sum $\mathbf{card}(\mathbf{N}_X) + \sum_{t \in \mathbf{N}_X} \mathbf{card}(\mathbf{V}_{\mathbf{g}_X(t)})$. An *isolated-pair of* X is pair (t, x) where t is a node of $\mathbf{N}_X \backslash \mathbf{r}_X$ and x is an isolated vertex and a source of $\mathbf{val}(X|t)$. Clearly, every not without-isolated-vertex $X \in \mathcal{T}$ admits a pair of the form (t, x) with $t \in \mathbf{N}_X$ and x an isolated vertex of $\mathbf{val}(X|t)$. If $\mathbf{val}(X)$ is without-isolated-vertex, x is incident in $\mathbf{G}_X$ with at least one edge of $\mathbf{G}_X \backslash \mathbf{e}_X$ and then is a source of $X|t$. Then, every without-isolated-vertex e-tree-decomposition that denotes a without-isolated-vertex e-hypergraph admits an isolated-pair.

Suppose there is $n \geq 0$ such that every $X \in \mathcal{L}_l \cap \mathcal{T}_k$ with $\|X\| \leq n$ admits an equivalent $Y \in \mathcal{L}_l \cap \mathcal{T}_k$ that verifies the condition of Lemma 43. Let $X \in \mathcal{L}_l \cap \mathcal{T}_k$ be an e-tree-decomposition with $\|X\| = n$. Denote by Y the e-tree-decomposition X if X is nonempty and the e-tree-decomposition obtained from X by contracting an arc of X incident with a leaf l of X such that: $\mathbf{g}_X(l) = \emptyset$. Denote by Z the e-tree-decomposition Y if Y is without-isolated-vertex and, otherwise, the e-tree-decomposition $(\mathbf{e}_Y, \mathbf{T}_Y, g)$ where g associates with every node s of Y the hypergraph $\mathbf{g}_Y(s)$ if $s \neq t$ and $\mathbf{g}_Y(s) \backslash x$ otherwise, for some isolated-pair (t, x) of Y. Clearly, Y and Z belong to $\mathcal{L}_l \cap \mathcal{T}_k$ and are equivalent with X. Every property defined in Definition 32 and verified by X, is verified by Y and by Z. If X is nonempty and without-isolated-vertex, the conclusion is immediate. Otherwise, Z verifies $\|Z\| < n$. The induction hypothesis permits to conclude. □

Now, we establish that every quasi-linear subset of $\mathcal{T}_k$ having for value a set of internally connected e-hypergraphs can be rewritten into an equivalent subset of $\mathcal{T}_{i.c} \cap \mathcal{T}_k$ that is quasi-linear too.

Lemma 44 *For all $k, l \geq 0$, the set $\{X \in \mathcal{L}_l \cap \mathcal{T}_k \mid \mathbf{val}(X) \in \mathcal{G}_{i.c}\}$ is equivalent with a subset of $\mathcal{L}_{l \cdot (1+k)} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$.*

Proof.
This proof comport two parts. A first one we treat the linear case. A second one we treat the general case. For every $X \in \mathcal{T}$, we denote by $p(X)$ the unique integer p such that $X \in \mathcal{L}_p - \mathcal{L}_{p-1}$. We denote by $\mathcal{I}$ the set of

e-tee-decompositions that denote internally connected e-hypergraphs.

Part 1
For every $X \in \mathcal{T}$, we denote by $\|X\|$ the number of its nodes and by $f(X)$ the integer $\max\{1, 1 + \underline{wd}(X)\}$.

Suppose there is $n \geq 0$ such that every $X \in \mathcal{I} \cap \mathcal{L}$ of size $\|X\| \leq n$ admits an equivalent e-tree-decomposition in $\mathcal{L}_{f(X)} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{\underline{wd}(X)}$. Let $X \in \mathcal{I} \cap \mathcal{L}$ of size $\|X\| = n + 1$. If X is atomic, if X contains no vertex or if $\mathbf{val}(X)$ contains an isolated vertex (necessary, this vertex is unique), X admits an equivalent atomic e-tree-decomposition in $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{\underline{wd}(X)}$. Moreover, we suppose X not atomic, with at least one vertex and $\mathbf{val}(X)$ without-isolated-vertex. We can suppose X nonempty and without-isolated-vertex (Lemma 43). Denote by e the unique arc of X incident with $\mathbf{r}_X$ and by V_e the set of vertices $\mathbf{V}_{\mathbf{g}_X(s)} \cap \mathbf{V}_{\mathbf{g}_X(t)}$ with s and t the two extremities of e in $\mathbf{T}_X$. If $V_e = \emptyset$, we have $\mathbf{g}_X(\mathbf{r}_X) \backslash \mathbf{e}_X = \emptyset$, the e-tree-decomposition X' obtained from X by contracting e is equivalent with X, of width at most $\underline{wd}(X)$ and verifies $\|X'\| < \|X\|$. The induction suffices to conclude. Moreover, we suppose $V_e \neq \emptyset$. Let $Z = X|(\mathbf{T}_X \backslash \mathbf{r}_X)$. The set of sources of Z is the set V_e and, then, is nonempty. Z is nonempty and without-isolated-vertex (Lemma 35), then there is an edge d in $\mathbf{g}_Z(l) \backslash \mathbf{e}_Z$, with l the unique leaf of Z. The e-hypergraph $\mathbf{val}(X)$ is connected, then $\mathbf{G}_Z$ is connected (eventually $\mathbf{G}_Z \backslash \mathbf{e}_Z$ is not connected). In consequence, there is an internal-path p of $\mathbf{val}(Z)$ from d to a source of Z, noted s.

Denote by $G_1, \ldots, G_m$ the internally connected components of $\mathbf{val}(Z)$. Without pert of generality, we can suppose $d \in \mathbf{E}_{G_1}$. For every $i \in [m]$, denote by K_i the hypergraph obtained from G_i by adding a new edge noted d_i of extremities the set of vertices of $G_i \cap \mathbf{g}_X(s) \cap \mathbf{g}_X(t)$ with s and t the extremities of $\mathbf{e}_Z$ in $\mathbf{T}_X$. By construction, (d_i, K_i) is internally connected. For every $i \in [m]$, denote by Z_i the sequence $(d_i, \mathbf{T}_Z, g)$ where g associates with every node t of Z the intersection $\mathbf{g}_Z(t) \cap K_i$ if $t \neq \mathbf{r}_Z$ and $(\mathbf{g}_Z(t) \cap K_i) \cup (K_i \restriction d_i)$ if $t = \mathbf{r}_Z$. Clearly, for every $i \in [m]$, Z_i denotes (d_i, K_i) and belongs to $\mathcal{I} \cap \mathcal{L} \cap \mathcal{T}_{\underline{wd}(X)}$. By induction, Z_1 admits an equivalent e-tree-decomposition $V_1 \in \mathcal{L}_{f(X)} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{\underline{wd}(X)}$.

The path p is an internal-path of (d_1, K_1), contains no vertex of $(d_i, K_i)\backslash\backslash\{s\}$ for every $i \in [2, m]$. Then, for every $i \in [2, m]$, we have: $\underline{wd}(Z_i\backslash\backslash\{s\}) < \underline{wd}(X)$ and, then, $f(Z_i\backslash\backslash s) < f(X)$. Then, for every $i \in [2, m]$, $Z_i\backslash\backslash\{s\}$ belongs to $\mathcal{I} \cap \mathcal{L} \cap \mathcal{T}_{\underline{wd}(X)-1}$ and admits, by induction, an equivalent $W_i \in \mathcal{L}_{f(X)-1} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{\underline{wd}(X)}$. For every $i \in [2, m]$, denote by V_i the sequence $(\mathbf{e}_{W_i}, \mathbf{T}_{W_i}, g)$ where g associates with every node t of W_i the unique hypergraph:

- $\mathbf{g}_{V_i}(t)$, if $t \notin N_i$.
- that contains s and verifies $\mathbf{g}_{V_i}(t) \backslash\backslash \{s\} = \mathbf{g}_{W_i}(t)$, if $t \notin N_i$.

  where N_i is the minimal subtree of $\mathbf{T}_{V_i}$ that contains every node u of V_i such that an edge of $\mathbf{g}_{V_i}(u)$ is incident in $\mathbf{G}_{Z_i}$ with S.

Clearly, for every $i \in [2, m]$, V_i belongs to $\mathcal{L}_{f(X)-1} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{\underline{wd}(X)-1+1}$ and denotes $\mathbf{val}(Z_i)$. Let U be an atomic e-tree-decomposition that denotes $(\mathbf{e}_X, \mathbf{g}_X(\mathbf{r}_X) \cup \bigcup_{i \in [m]} K_i \restriction d_i)$. Rather to take isomorphic and equivalent copies, we can suppose $U[V_1, \ldots, V_m]$ defined. The equality $\mathbf{val}(X) = \mathbf{val}(U[V_1, \ldots, V_m])$, the internal-connectivity of $\mathbf{val}(X)$, Lemma 25 and Fact 17, imply $\mathbf{val}(U)$ and U internally connected (U is atomic). Hence, $U[V_1, \ldots, V_m]$ is equivalent with X and belongs to $\mathcal{L}_{f(X)} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{\underline{wd}(X)}$.

Part 2

Let k be an integer. Suppose, there is some n such that $\mathcal{I} \cap \mathcal{L}_n \cap \mathcal{T}_k$ is equivalent with some subset $\mathcal{L}_{n \cdot (1+k)} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$. Let $X \in \mathcal{I} \cap \mathcal{L}_{n+1} \cap \mathcal{T}_k$. If X contains no vertex or if $\mathbf{val}(X)$ contains an isolated vertex (necessary this vertex is unique), X admits an equivalent e-tree-decomposition $Y \in \mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$. If $n = 1$, Part **1** suffices to conclude. Moreover, we suppose $X \notin \mathcal{L} \cap \mathcal{T}_k$, with at least one vertex and $\mathbf{val}(X)$ without-isolated-vertex. By Lemma 43, we can suppose X nonempty and without-isolated-vertex.

Denote by P the subpath of $\mathbf{T}_X$ that contains every node t of X associated to the value $p_X(t) = p_X(\mathbf{r}_X) = p(X)$ (see Lemma 40). For every node t of P, denote by u_t the set of all e-tree-decompositions generated by X and some maximal subtree of $\mathbf{T}_X \backslash \mathbf{N}_P$ that contains a node adjacent with t. We denote by u the union $\bigcup_{t \in \mathbf{N}_P} u_t$. It comes, $u \subseteq \mathcal{L}_{n-1} \cap \mathcal{T}_k$. In the same way than in Part **1**, we can transform every $Z \in u$ into a set $v_Z \subseteq \mathcal{I} \cap \mathcal{L}_{n-1} \cap \mathcal{T}_k$ such that $\bigcup_{L \in v_Z} \mathbf{G}_L \backslash \mathbf{e}_L$ is the union of the distinct internally connected component of $\mathbf{val}(Z)$ and such that for every $L \in z_Z$, we have: $\mathbf{vert}_Z(\mathbf{e}_Z) \cap \mathbf{V}_L = \mathbf{vert}_L(\mathbf{e}_L)$. For every $t \in \mathbf{N}_P$, we denote by v_t the set $\bigcup_{Z \in u_t} v_Z$. By induction, every v_t is equivalent with a subset of $w_t \mathcal{L}_{(n-1) \cdot (1+k)} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$.

Denote by H the hypergraph $\mathbf{g}_X(P) \cup \bigcup_{W \in w_t, t \in \mathbf{N}_P} \mathbf{G}_W \upharpoonright \mathbf{e}_W$. Denote by Y the sequence $(\mathbf{e}_X, P, g)$ where g associates with every node t of P the hypergraph $\mathbf{g}_X(t) \cup \bigcup_{W \in w_t} \mathbf{G}_W \upharpoonright \mathbf{e}_W$. Then, Y belongs to $\mathcal{L} \cap \mathcal{T}_k$ and verifies : $\mathbf{val}(X) = \mathbf{val}(Y)[\mathbf{val}(W_1), \ldots, \mathbf{val}(W_m)]$ with $\{W_1, \ldots, W_m\} = \bigcup_{t \in \mathbf{N}_P} w_t$. $\mathbf{val}(X)$ is internally connected, then $\mathbf{val}(Y)$ is internally connected (Fact 17). By induction, Y admits an equivalent $U \in \mathcal{L}_{1+k} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$. Without pert of generality, we can suppose that $U[W_1, \ldots, W_m]$ is defined. Then, X is equivalent with $U[W_1, \ldots, W_m]$ that belongs to $\mathcal{L}_{(1+k)\cdot n} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$. □

4 Criticality

This section comports three important results about criticality (Theorems 50, 60 and 61).

Firstly, we define the notions of a nowhere-critical e-tree-decomposition and of a (everywhere-)critical e-tree-decomposition. Their sets are respectively denoted by $\mathcal{T}_{i.c}^{\mathbf{nc}}$ and $\mathcal{T}_{i.c}^{\mathbf{c}}$. Remarkable property, they suffice to product $\mathcal{T}_{i.c}$ thanks the higher-order substitution $\otimes$, we define in this section. Theorem 50 establishes: $\mathcal{T}_{i.c} = \mathcal{T}_{i.c}^{\mathbf{nc}} \otimes \mathcal{T}_{i.c}^{\mathbf{c}}$.

The second result "decomposes" the critical and linear case thanks $+$. More precisely, Theorem 60 states that every linear and critical e-tree-decomposition can be produced thanks the mapping $+$ by using linear and internally connected e-tree-decompositions of smaller width.

The third result "decomposes" the critical case thanks []. More precisely, Theorem 61 compares criticality and quasi-linearity and states that for every k, the set $\mathcal{T}_{i.c}^{\mathbf{c}} \cap \mathcal{T}_k$ is equivalent to a quasi-linear subset of $\mathcal{T}_k$. This result is the most difficult result of this paper.

The proofs of Theorems 60 and 61 have the same structure. In particular, they use two other notions of "criticalities" related to the notions of a connected e-tree-decomposition and of a 2-edge-connected e-tree-decomposition. These notions of a 1-critical e-tree-decomposition and of a 2-critical one are presented here.

An edge is said critical in an e-hypergraph if it is needed to internally connect its sources. This notion permits to define two kinds of internally connected e-tree-decompositions: the nowhere-critical ones and the (everywhere)-critical ones. These notions are formalized below.

Definition 45 (Criticality) An edge d is *critical* in some e-hypergraph H if $d \in \mathbf{E}_H \backslash \mathbf{e}_H$ and if every internally connected subhypergraph of $H \backslash d$ does not contain every source of H.

An e-tree-decomposition X is *critical* if it is internally connected and if every arc d of X is critical in $\mathbf{val}(X \uparrow d)$. Their set is denoted by $\mathcal{T}^{\mathbf{c}}_{i.c}$. An e-tree-decomposition X is *nowhere-critical* if for every arc d of X and every $Y \sqsubseteq X$, d is not critical in $\mathbf{val}(Y)$. Their set is denoted by $\mathcal{T}^{\mathbf{nc}}_{i.c}$.

The both notions presented above are hereditary. This nice property is expressed by Lemma 47. Previously, a little fact.

Fact 46 *Let $G, H, K \in \mathcal{G}_{i.c}$ with $G = H[K]$. Then:*

- *$\mathbf{e}_K$ is critical in H if K contains at least one critical edge of G.*
- *for each $d \in \mathbf{E}_H \backslash \mathbf{e}_K$, d is critical in H iff d is critical in G.*
- *every edge of $\mathbf{E}_K$ critical in G is critical in K.*

Proof.
The sentence "internally connected" is abbreviated in "i.c". Let G be an e-hypergraph of the form $H[K]$ for some i.c e-hypergraphs H and K. Denote by S the set of sources of G and by D the set of critical edges of G. Let d be an edge.

- $\mathbf{e}_K$ is critical in H if $D \cap \mathbf{E}_K \neq \emptyset$.
  Every i.c subhypergraph of $H \backslash \mathbf{e}_K$ is i.c in $G \backslash d$, for some $d \in \mathbf{E}_K \cap D$. It does not contain S.

- d is critical in H if $d \in D \cap \mathbf{E}_H$.
  Let L be an i.c subhypergraph of $H \backslash d$. The hypergraph M, equal to L if $\mathbf{e}_K \notin \mathbf{E}_L$ and to $(L \cup \mathbf{G}_G) \backslash \mathbf{e}_K$, otherwise, is i.c. in G (Lemma 16), in $G \backslash d$ and does not contain S. L does not contain S, d is critical.

- $d \in D$ if d is critical in H and if $d \neq \mathbf{e}_K$.
  Let M be an i.c subhypergraph of G. The hypergraph $\mathbf{G}_K$ is i.c. in K, in G and in $G \backslash d$. If $M \cap \mathbf{G}_K = \emptyset$, M is i.c in H and does not contain S. Otherwise, M contains $\mathbf{G}_K$, is of the form $(L \cup \mathbf{G}_K) \backslash \mathbf{e}_K$ with $L \restriction \mathbf{e}_K = L \cap \mathbf{G}_K = \mathbf{G}_K \restriction \mathbf{e}_K$. The hypergraph L is i.c in H (Lemma 16) and does not contain S. In the both cases, M does not contain S. Then, $d \in D$.

- d is critical in K if $d \in D \cap \mathbf{E}_K$.
  Let L be an i.c subhypergraph of $K \backslash d$. Suppose $\mathbf{vert}_K(\mathbf{e}_K) \subseteq \mathbf{V}_L$ and denote by K' the i.c e-hypergraph $(\mathbf{e}_K, (K \restriction \mathbf{e}_K) \cup L)$. The e-hypergraph $H[K']$ is i.c (Lemma 16). The hypergraph $\mathbf{G}_{H[K']}$ is a

subhypergraph of $G\backslash d$ i.c in G that contains S. Contradiction. Then, $\mathbf{vert}_K(\mathbf{e}_K) \not\subseteq \mathbf{V}_L$. d is critical in K. $\square$

Lemma 47 *The memberships $X \in \mathcal{T}^{\mathbf{nc}}_{i.c}$ and $X \in \mathcal{T}^{\mathbf{c}}_{i.c}$ are hereditary.*

Proof.
By Fact 34, to conclude it suffices to prove that for every $X \in \mathcal{T}^{\mathbf{c}}_{i.c}$ (resp. $\in \mathcal{T}^{\mathbf{nc}}_{i.c}$) and every arc d of X, the inclusion $\{X \uparrow d, X \downarrow d\} \subset \mathcal{T}^{\mathbf{c}}_{i.c}$ (resp. $\subset \mathcal{T}^{\mathbf{nc}}_{i.c}$). Let d be an arc of some $X \in \mathcal{T}_{i.c}$. By Lemma 35, we have $\{X \uparrow d, X \downarrow d\} \subset \mathcal{T}_{i.c}$. If $X \in \mathcal{T}^{\mathbf{c}}_{i.c}$, Fact 46 implies $\{X \uparrow d, X \downarrow d\} \subset \mathcal{T}^{\mathbf{c}}_{i.c}$. If $X \in \mathcal{T}^{\mathbf{nc}}_{i.c}$, $\mathbf{val}(\mathbf{atom}(\{X\})) = \mathbf{val}(\mathbf{atom}(\{X \uparrow d\})) \cup \mathbf{val}(\{\mathbf{atom}(X \downarrow d)\})$ implies $\{X \uparrow d, X \downarrow d\} \subset \mathcal{T}^{\mathbf{nc}}_{i.c}$. $\square$

Now, let us define the main operation of this articles. This higher-order substitution transforms e-tree-decompositions of some set u by replacing simultaneously each of its nodes by some e-tree-decomposition of some set $v \subseteq \mathcal{T}$. The below definition is illustrated by Example 49.

Definition 48 (Higher-order substitution) Let u and v be two subsets of $\mathcal{T}$. We denote by $u \otimes v$ the set of all e-tree-decompositions X that contains a set of arcs D such that:

- u contains the e-tree-decomposition obtained from X by contracting $\mathbf{A}_X \backslash D$.
- v contains every e-tree-decomposition generated by X and some maximal subtree of the forest $\mathbf{T}_X \backslash D$.

Example 49 Figure 4 represents three subsets $u = \{X\}, v = \{Y\}, w = \{Z_1, Z_2, Z_3\}$ of $\mathcal{T}$ that verify $u = v \otimes w$ and a set $D \subseteq \mathbf{A}_X$. X is the e-tree-decomposition with 6 nodes. D is the set of arcs of X that are thick drawing. Y, the e-tree-decomposition with 3 nodes, is obtained from X by contracting $\mathbf{A}_X - D$. w contains the three e-tree-decompositions with two nodes. Each of them is generated by X and some maximal subtree of $\mathbf{T}_X \backslash D$.

Now a easy but important result that decomposes $\mathcal{T}_{i.c}$ into $\mathcal{T}^{\mathbf{nc}}_{i.c}$ and $\mathcal{T}^{\mathbf{c}}_{i.c}$.

Theorem 50 $\mathcal{T}_{i.c} = \mathcal{T}^{\mathbf{nc}}_{i.c} \otimes \mathcal{T}^{\mathbf{c}}_{i.c}$.

Proof.
As a consequence of the inclusion $\mathbf{atom}(\mathcal{T}^{\mathbf{nc}}_{i.c} \otimes \mathcal{T}^{\mathbf{c}}_{i.c}) \subseteq \mathbf{atom}(\mathcal{T}^{\mathbf{c}}_{i.c}) \subseteq \mathcal{T}_{i.c}$ and of Lemma 35, we have: $\mathcal{T}_{i.c} \supseteq \mathcal{T}^{\mathbf{nc}}_{i.c} \otimes \mathcal{T}^{\mathbf{c}}_{i.c}$.

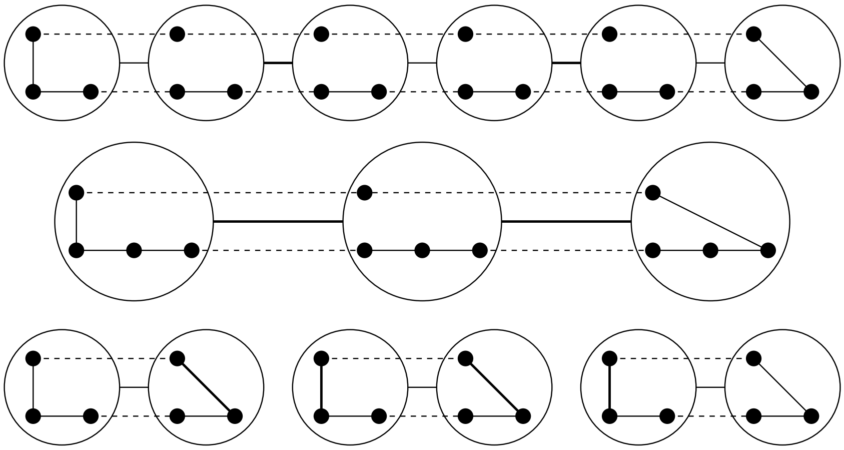

Figure 4: the higher-order substitution $\otimes$.

For every $X \in \mathcal{T}$, we denote by $||X||$ the number of its nodes. Trivially, we have: $\mathbf{atom}(\mathcal{T}_{i.c}) \subseteq \mathcal{T}_{i.c}^{\mathbf{nc}} \otimes \mathcal{T}_{i.c}^{\mathbf{c}}$. Suppose there is $n \geq 1$ such that every $X \in \mathcal{T}_{i.c}$ with $||X|| \leq n$ belongs to $\mathcal{T}_{i.c}^{\mathbf{nc}} \otimes \mathcal{T}_{i.c}^{\mathbf{c}}$. Let $X \in \mathcal{T}_{i.c}$ with $||X|| = n+1$. Denote by $\mathcal{D}$ the set of all e-tree-decompositions $Y \sqsubseteq X$ such that $\mathbf{r}_Y = \mathbf{r}_X$ and such that every arc d of Y is critical in $\mathbf{val}(Y \uparrow d)$. Trivially, $\mathcal{D}$ contains the e-tree-decomposition generated by X and its root and is contained in $\mathcal{T}_{i.c}^{\mathbf{c}}$.
Let Y be a maximal element of $(\mathcal{D}, \sqsubseteq)$. Denote by A the atomic e-tree-decomposition obtained from Y by contracting all of its arcs. It comes $Y \in \{A\} \otimes \{Y\}$ with $A \in \mathcal{T}_{i.c}^{\mathbf{nc}}$. If $X = Y$, that suffices to conclude. Otherwise, X is of the form $Y[Z_1, \ldots, Z_m]$ for some $Z_1, \ldots, Z_m \in \mathcal{T}_{i.c}$ (Lemma 35). By induction, for every $i \in [m]$ there is an e-tree-decomposition $P_i \in \mathcal{T}_{i.c}^{\mathbf{nc}}$ and a subset $Q_i \subseteq \mathcal{T}_{i.c}^{\mathbf{c}}$ such that $Z_i \in \{P_i\} \otimes Q_i$. It follows that X belongs to $\{A[P_1, \ldots, P_m]\} \otimes (\{Y\} \cup Q_1 \cup \ldots \cup Q_m\}$ and then to $\{A[P_1, \ldots, P_m]\} \otimes \mathcal{T}_{i.c}^{\mathbf{c}}$. To conclude, it suffices to prove: $A[P_1, \ldots, P_m] \in \mathcal{T}_{i.c}^{\mathbf{nc}}$.

Let d be an arc of $A[P_1, \ldots, P_m]$ and $H \in \mathbf{val}(\mathbf{atom}(\{A[P_1, \ldots, P_m]\}))$. If d is an arc of some P_i with $i \in [m]$, if $H \in \mathbf{val}(\mathbf{atom}(\{P_1, \ldots, P_m\}))$, by hypothesis d is not critical in H, if $H = \mathbf{val}(A)$, d is not an edge of $\mathbf{val}(A)$ and then is not critical in $\mathbf{val}(A)$. Thus, every arc of some P_i with $i \in [m]$, is not critical in any e-hypergraph of $\mathbf{val}(\mathbf{atom}(\{A[P_1, \ldots, P_m]\}))$. Let d be an arc of the form $\mathbf{e}_{Z_i}$ for some $i \in [m]$. If $H \in \mathbf{val}(\mathbf{atom}(\{P_1, \ldots, P_m\}))$, d is trivially not critical in H. Suppose $H = \mathbf{val}(A)$. Let W be the e-tree-decomposition generated by X and by the union of the set of nodes of Y and the extremities of d in $\mathbf{T}_X$. By hypothesis, $W \notin \mathcal{D}$. Then, there is an arc d' of W not critical in $W \uparrow d'$. If we suppose $d \neq d'$, d' is an arc of X that is not critical in $Y \uparrow d'$ (Fact 46). Contradiction. Thus, $d = d'$ and d is not critical in H. It comes $A[P_1, \ldots, P_m] \in \mathcal{T}_{i.c}^{\mathbf{nc}}$. □

In the almost same way than for internal-connectivity, we can associate with the notion of a connected e-tree-decomposition the notion of a 1-critical e-tree-decomposition. This extension is made with a little difference: we impose that every leaf of some 1-critical e-tree-decomposition contains some 1-critical edge. It follows that every arc is 1-critical.

Definition 51 An edge d is 1-*critical* in some e-hypergraph H if $d \in \mathbf{E}_H \backslash \mathbf{e}_H$ and if every connected subhypergraph of $H \backslash d$ does not contain every source of H. An e-tree-decomposition X is 1-*critical* if $X \in \mathcal{T}_{1.c}$ and if for every leaf l of X, the hypergraph $\mathbf{g}_X(l)$ contains at least one 1-critical edge of $\mathbf{val}(X)$. Their set is denoted by $\mathcal{T}_{1.c}^{\mathbf{c}}$.

The similarity of the notions of a 1-critical edge and of a critical edge has for consequence the next fact, that can be compared with Fact 46.

Fact 52 *Let $G, H, K \in \mathcal{G}$ with $G = H[K]$ and H, K connected. We have:*

- $\mathbf{e}_K$ *is* 1-*critical in* H *if* K *contains at least one* 1-*critical edge of* G.
- *for each* $d \in \mathbf{E}_H \backslash \mathbf{e}_K$, d *is* 1-*critical in* H *iff* d *is* 1-*critical in* G.
- *every edge of* $\mathbf{E}_K$ 1-*critical in* G *is* 1-*critical in* K.

Proof.
The proof is obtained from the proof of Fact 46, by replacing the sentence "internally connected" by "connected". □

To associate with the notion of a 2-edge-connected e-tree-decomposition the notion of 2-critical one, we define a *circular-decomposition.* Roughly speaking, a circular-decomposition of an e-tree-decomposition X decomposes $\mathbf{val}(X)$ into a circuit, all whose nodes are subgraphs of $\mathbf{val}(X)$ and all whose edges are edges of $\mathbf{val}(X)$ such that every leaf of X contains at least one of these edges. These notions are formalized below:

Definition 53 A *circular-decomposition* of some hypergraph G is a pair (R, S), with R a sequence $(d_0, \ldots, d_{l-1})$ of l distinct edges of G for some $l \geq 2$, with S a sequence $(G_0, \ldots, G_{l-1})$ of l disjoint hypergraphs with $G_0 \cup \ldots \cup G_{l-1} = G \backslash \{d_0, \ldots, d_{l-1}\}$ such that for every $i, j \in [0, l-1]$ the following assertions are equivalent:

- $G \restriction d_i$ is not disjoint with G_j.

- $j \in \{i, (i+1) mod(l)\}$.

A *circular-decomposition* of some e-hypergraph H is a circular-decomposition (R, S) of $\mathbf{G}_H$ with $\mathbf{e}_H$ the first edge of R. For every sequence $S = (G_1, \ldots, G_n)$ and every hypergraph G (resp. e-hypergraph $H = (e, G)$) with $G_i \cap G$ defined for every $i \in [n]$, we denote by $S \sqcap G$ (resp. $S \sqcap H$) the empty sequence if S is empty, and, otherwise, the sequence obtained from $(G_1 \cap G, \ldots, G_n \cap G)$ by deleting every empty hypergraph.

A *circular-decomposition* of some $X \in \mathcal{T}$ is a circular-decomposition (R, S) of $\mathbf{val}(X)$ such that R contains at least an edge of every hypergraph of the form $\mathbf{g}_X(l)$ with l a leaf of X. An e-tree-decomposition X is 2-*critical* if $X \in \mathcal{T}_{2.c}$ and if X admits a circular-decomposition. Their set is denoted by $\mathcal{T}^{\mathbf{c}}_{2.c}$.

The notions of 1-critical e-tree-decomposition and of a 2-critical e-tree-decomposition are hereditary. This result is the object of Lemma 57. Previously, let us establish the three following technical facts.

The next fact establishes the fact that every 1-critical edge disconnect the sources, under an additional condition. This condition is required by the degenerate case: the e-hypergraph $H = (e, \emptyset, \{d, e\}, \mathbf{vert})$ with no vertex and with exactly two edges is connected, has no source, admits d as 1-critical edge but admits no circular-decomposition. The reason is the fact that the e-hypergraph $H \backslash d$ is empty and, then, has no connected component!

Fact 54 *For every edge d of some connected e-hypergraph H that contains at least one vertex, the following assertions are equivalent:*

- *d is a 1-critical edge of H.*

- *H admits circular-decomposition of the form $((\mathbf{e}_H, d), S)$ for some S.*

Fact 55 *Let $G, H, K \in \mathcal{G}$ with $G = H[K]$ and H, K connected. Let be a circular-decomposition of G of the form $((\mathbf{e}_G, d), S)$ for some edge d and some sequence S. Then:*

- *$((\mathbf{e}_H, \mathbf{e}_K), S \sqcap H)$ is a circular-decomposition of H, if $d \in \mathbf{E}_K$.*

- *$((\mathbf{e}_K, d), S \sqcap K)$ is a circular-decomposition of K, if $d \in \mathbf{E}_K$.*

- *$((\mathbf{e}_H, d), S \sqcap H)$ is a circular-decomposition of H, if $d \in \mathbf{E}_H$.*

Proof.
The proof is obtained by using Fact 54 and similar arguments that those in proof of Fact 52. □

Fact 56 *Let $G, H, K \in \mathcal{G}$ with $G = H[K]$ and H, K 2-edge-connected. Let (R, S) be a circular-decomposition of G. If R contains some edge of K, then:*

- *$(R_1, S \sqcap H)$ is a circular-decomposition of H.*
- *$(R_2, S \sqcap K)$ is a circular-decomposition of K, where:*
  - *R_1 is obtained from S by replacing every maximal subsequence of S of edges of $\mathbf{E}_K$ by $\mathbf{e}_K$.*
  - *R_2 is the concatenation of $(\mathbf{e}_K)$ with the restriction of S on $\mathbf{E}_K$.*

Proof.
For every sequence of edges $R = (d_0, \ldots, d_l)$ for some $l \geq 0$, we denote by U_R the set $\{d_0, \ldots, d_l\}$. For every sequence of disjoint hypergraphs $S = (G_0, \ldots, G_l)$ for some $l \geq 0$, we denote by U_S the hypergraph $G_0 \cup \ldots \cup G_l$. Let (e, G) be an e-hypergraph of the form $(e, H)[(d, K)]$ for some 2-edge-connected e-hypergraphs (e, H) and (d, K). Let (R, S) be a circular-decomposition of (e, G) with $R = (d_0, \ldots, d_{l-1})$, $S = (G_0, \ldots, G_{l-1})$ such that R contains at least one edge of K. Denote by R_1 the sequence obtained from R by replacing every maximal subsequence of R contained in $\mathbf{E}_K^+$ by d. Denote by R_2 the concatenation of (d) with the restriction of R on $\mathbf{E}_H$.

By hypothesis, R contains at least one edge of K, the sequence R_1 contains d and for first edge e, the sequence R_2 contains, by construction, for first edge d and contains at least one edge of $\mathbf{E}_K \backslash d$. The intersection $U_S \cap H$ and $U_S \cap K$ are defined. Then, $U_{S \sqcap H} = H \backslash U_{R_1}$ and $U_{S \sqcap K} = K \backslash U_{R_2}$.

By construction, the edges of R_2 are distinct. Denote by i_0 (resp. j_0) the minimal (resp. maximal) integer of $[0, l]$ such that $d_{i_0+1} \in \mathbf{E}_K$ (resp. $d_{j_0} \in \mathbf{E}_K$). Note I the set $[0, i_0] \cup [j_0 + 1, l - 1]$ and J the set $[i_0 + 1, j_0]$. Let d_j be an edge of $\mathbf{E}_H$ with $j \in [0, l - 1]$. The hypergraph $H \backslash d$ is connected and is a subhypergraph of $G \backslash \{d_{i_0+1}, d_{j_0}\}$. Then, d_0 and d_j belong to some common connected component of $G \backslash \{d_{i_0+1}, d_{j_0}\}$. It follows: $j \in I$. With a symmetrical proof, we prove that for every edge $d_j \in \mathbf{E}_K$ with $j \in [0, l - 1]$, we have: $j \in J$. Then, $U_R \cap \mathbf{E}_H = \{d_j \mid j \in I\}$ and $U_R \cap \mathbf{E}_K = \{d_j \mid j \in J\}$.

The edges of R_1 are distinct.

Let $j \in J$ be an integer. Let us prove $G_j \cap H = \emptyset$ for every $j \in]i_0, j_0[$. The result is trivial in the case $i_0 + 1 = j_0$. Suppose $i_0 + 1 < j_0$. Let $j \in [i_0 + 1, j_0 - 1]$ be an integer. The hypergraph $G_{i_0+1} \cup \ldots \cup G_{j_0-1} \cup (G \upharpoonright \{d_{i_0+2}, \ldots, d_{j_0-1}\})$ is a connected subhypergraph of $G \backslash \{d_{i_0+1}, d_{j_0}\}$ that contains d_{i_0+1} and, then, that does not contain d_0. The hypergraph $H \backslash d$ is connected. Then, it is a connected subhypergraph of $G \backslash \{d_{i_0+1}, d_{j_0}\}$ that contains d_0 and is disjoint with $G_{i_0+1} \cup \ldots \cup G_{j_0-1}$. With symmetrical argument, we prove $G_j \cap K = \emptyset$ for every $j \in [0, i_0[\cup]j_0, l-1]$. The hypergraph G_{i_0} (resp. G_{j_0}) is by hypothesis not disjoint with $G \upharpoonright d_{i_0}$ and $G \upharpoonright d_{i_0+1}$ (resp. $G \upharpoonright d_{j_0}$ and $G \upharpoonright d_{(j_0+1) mod\, l}$) and then is not disjoint with H and K. It follows: $S \sqcap K = (G_{i_0} \cap K, G_{j_0} \cap K)$ if $i_0 + 1 = j_0$, $S \sqcap K = (G_{i_0} \cap K, G_{i_0+1}, \ldots, G_{j_0-1}, G_{j_0} \cap K)$ if $i_0 + 1 < j_0$. Denote by $(f_0, \ldots, f_m)$ the sequence $(d, d_{i_0+1}, \ldots, d_{j_0})$ and by $(K_0, \ldots, K_m)$ the sequence $S \sqcap K$. From precedent remarks, it follows that for every hypergraph of the form K_j with $j \in [i_0, j_0]$ not disjoint with some hypergraph of the form $K \upharpoonright f_i$ with $i \in [i_0, j_0]$, we have: $j \in \{i, (i+1) mod\, m\}$. The 2-edge-connectivity of K implies that every hypergraph of the form K_j with $j \in [0, m-1]$ is not disjoint with $K \upharpoonright f_j$ and $K \upharpoonright f_{j'}$ with $j' = (j+1) mod m$. Then, $(R_1, S \sqcap K)$ is a circular-decomposition of (d, K). A symmetrical proof, permits to establish $(R_2, S \sqcap H)$ circular-decomposition of (e, H). $\square$

Lemma 57 *The memberships $X \in \mathcal{T}^{\mathbf{c}}_{1.c}$, $X \in \mathcal{T}^{\mathbf{c}}_{2.c}$ are hereditary.*

Proof.
Direct consequence of Lemma 35 and Facts 52 and 56. $\square$

Now, let us define how to add one vertex to every e-tree-decomposition of some set $u \subseteq \mathcal{T}$. This definition is illustrated by Example 59.

Definition 58 (+) For every $u \subseteq \mathcal{T}$, we denote by $+(u)$ the union of u and the set of all e-tree-decompositions X that contains a vertex x such that u contains $(\mathbf{e}_X, \mathbf{T}_X, g)$ with $g(t) = \mathbf{g}_X(t) \backslash\backslash \{x\}$ for every node t of X.

Example 59 Figure 5 represents two subsets $u = \{X\}$ and v of $\mathcal{T}$ that verifies $v = +(u)$. X is represented at the left of the figure. v contains X and all the others e-tree-decompositions represented in Figure 5. Except X, every e-tree-decomposition of v is obtained from X by adding a new vertex (drawed with a white disk) making incident to some edges of X.

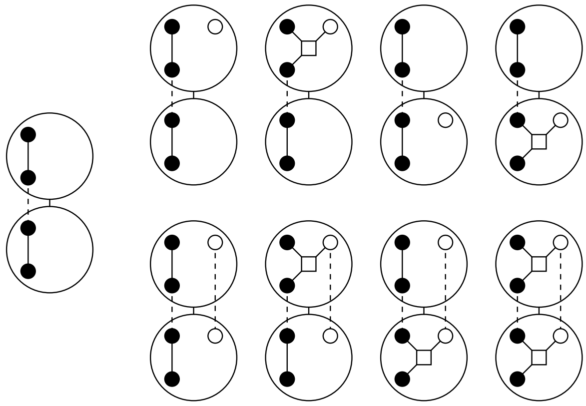

Figure 5: the unary operation $+$.

The next theorem states that $+$ produces an equivalent set of $\mathcal{L}\cap\mathcal{T}_{i.c}^{\mathbf{c}}\cap\mathcal{T}_k$ from a subset of $\mathcal{L}\cap\mathcal{T}_{i.c}^{\mathbf{c}}\cap\mathcal{T}_{k-1}$. The whole Appendix a is devoted to its proof.

Theorem 60 *For each $k \geq 0$, the set $\mathcal{L}\cap\mathcal{T}_{i.c}^{\mathbf{c}}\cap\mathcal{T}_k$ is equivalent with a subset of $\mathcal{L}\cap\mathcal{T}_{i.c}\cap\mathcal{T}_k\cap+^{2+6\cdot k}(\mathcal{L}\cap\mathcal{T}_{i.c}\cap\mathcal{T}_{k-1})$.*

The next theorem states that $[\,]$ produces an equivalent set of $\mathcal{T}_{i.c}^{\mathbf{c}}\cap\mathcal{T}_k$ from $\mathcal{L}\cap\mathcal{T}_k$. The whole Appendix b is devoted to its proof.

Theorem 61 *For each $k \geq 0$, $\mathcal{T}_{i.c}^{\mathbf{c}}\cap\mathcal{T}_k$ is equivalent with a subset of $\mathcal{L}_{2\cdot(1+k)^2}\cap\mathcal{T}_k$.*

5 An algebra of sets of e-tree-decompositions

In this section, we define Π and present a fundamental property of Π, that states that Π produces an equivalent subset of $\mathcal{T}_k$, for each k. One of the operations of Π intersects every subset of $\mathcal{T}$ with some given MSO-definable set. For this purpose, we recall briefly monadic second-order logic and relational structures defined on a ranked alphabet R, (R-structure, for short).

Definition 62 (MSO logic) Let R be a ranked alphabet such that each element r in R has a rank $\rho(r)$ in $\mathbb{N}_+$. A symbol $r \in R$ is considered as a $\rho(r)$-ary relation symbol. A *R-structure* is a tuple $S = (\mathbf{D}_S, (r_s)_{r\in R})$

where $\mathbf{D}_S$ is a finite (possibly empty) set, called the *domain of* S, and r_S is a subset of $\mathbf{D}_S^{\rho(r)}$ for each r in R. We denote by $\mathcal{S}(R)$ the set of R-structures.

Let S be an R-structure for some alphabet R. The formulas of *monadic second-order logic* (called *MSO-formulas* for short) are written with variables of two types, namely lower case letters $x, y, \ldots$ called *object variables*, denoting elements of $\mathbf{D}_S$, and upper case letters $X, Y, \ldots$ called *set variables*, denoting subsets of $\mathbf{D}_S$. The atomic formulas are of the form $x = y$, $x \in X$, $r(x_1, \ldots, x_n)$ (where $r \in R$ and $n = \rho(r)$), and formulas are formed with propositional connectives and quantifications over the variables. For every finite set W of object and set variables, we denote by $L(R, W)$ the set of all formulas that are written with relational symbol from R and have their free variables in W. We also denote by $L(R)$ the set of closed formulas $L(R, \emptyset)$.

Let $\varphi \in L(R, W)$ and let γ be a W-assignment in S (i.e., $\gamma(X)$ is a subset of $\mathbf{D}_S$ for every set variable X in W, and $\gamma(x) \in \mathbf{D}_S$ for every object variable x in W). We write $(S, \gamma) \models \varphi$ if and only if φ holds for S for γ. We write $S \models \varphi$ in the case where φ has no free variable. A set of R structures L is *MSO-definable* if there is a formula φ in $L(R)$ such that L is the set of all R-structures S such that $S \models \varphi$.

Any e-hypergraph H is represented by the R-structure $|H| = (\mathbf{V}_H \cup \mathbf{E}_H, (r_{|H|})_{r \in R})$ with $R = \{\mathbf{vr}, \mathbf{ed}, \mathbf{sr}, \mathbf{ic}\}$, where $\mathbf{vr}_{|H|}$, $\mathbf{ed}_{|H|}$ and $\mathbf{sr}_{|H|}$ are the unary predicates that define the vertex-set, the edge-set and the source-edge, respectively, and where $\mathbf{ic}_{|H|}(d, x) :\Leftrightarrow x \in \mathbf{vert}_H(d)$ is the binary incidence relation. Clearly, for all $G, H \in \mathcal{G}$, $|G| = |H|$ if and only if $G = H$.

Any e-tree-decomposition X is represented by the R'-structure $|X| = (\mathbf{V}_X \cup \mathbf{E}_X \cup \mathbf{N}_X \cup \mathbf{A}_X, (r_{|X|})_{r \in R'})$ with $R' = \{\mathbf{vr}, \mathbf{ed}, \mathbf{sr}, \mathbf{nd}, \mathbf{ar}, \mathbf{ic}, \mathbf{mp}\}$, where $\mathbf{vr}_{|X|}$, $\mathbf{ed}_{|X|}$, $\mathbf{nd}_{|X|}$, $\mathbf{ar}_{|X|}$ and $\mathbf{sr}_{|X|}$ are the unary predicates that define the vertex-set, the edge-set, the node-set, the arc-set and the source-edge, respectively, where $\mathbf{ic}_{|X|}(d, x) :\Leftrightarrow x \in \mathbf{vert}_{\mathbf{G}_X}(d) \vee x \in \mathbf{vert}_{\mathbf{T}_X}(d)$ is the binary incidence relation, and where $\mathbf{mp}_{|X|}(t, x) :\Leftrightarrow x \in \mathbf{V}_{\mathbf{g}_X(t)} \cup \mathbf{E}_{\mathbf{g}_X(t)}$ is the binary mapping relation. Clearly, for all $X, Y \in \mathcal{T}$, $|X| = |Y|$ if and only if $X = Y$. Hence, $|X|$ "contains" $|H|$, the value of X.

Notation 63 (The algebra Π) We denote by Π the algebra $(\mathcal{P}(\mathcal{T}), \mathcal{F})$ with $\mathcal{F} := \{+, \mathbf{1}\} \cup \{\mathbf{m}_\varphi \mid \varphi \in L(R')\} \cup \{\mathbf{n}_k, \mathbf{o}_k, \mathbf{p}_k \mid k \geq 0\}$ where:

- $\mathbf{1}$ is the nullary operation $\to \mathbf{atom}(\mathcal{T})$.
- $\mathbf{m}_\varphi$ is the unary operation $u \to \{X \in u : |X| \models \varphi\}$, for each $\varphi \in L(R')$.

and where for each $k \geq 0$:

- $\mathbf{n}_k$ is the binary operation $(u, v) \to u \otimes (v \cap Type_k)$.
- $\mathbf{o}_k$ is the binary operation $(u, v) \to u[\{X \in v \cap Type_k \mid \mathbf{val}(X) \in \mathcal{G}_{i.c}\}]$.
- $\mathbf{p}_k$ is the nullary operation $\to \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k$.

A subset of $\mathcal{T}$ is *produced by* Π if it is denoted by some finite and well-formed term built with symbols of $\mathcal{F}$.

The next theorem resumes all precedent results of this paper. This property is the first important one of Π. Prealably, a litle and obvious fact.

Lemma 64 *The sets $\mathcal{L}$, $\mathbf{atom}(\mathcal{T})$, $\mathcal{T}_{i.c}$, $\mathcal{T}_k$ for some k are MSO-definable.*

Theorem 65 *For every k, Π produces an equivalent subset of $\mathcal{T}_k$.*

Proof.
Let $k \geq 0$. We have:

1. $\mathcal{T}_k$ contains and is equivalent with $\mathbf{o}_{k+1}(\mathbf{1} \cap \mathcal{T}_k, u)$ with $u = \mathcal{T}_{i.c} \cap \mathcal{T}_k$.

   Let $G \in \mathcal{G}$ with $\underline{twd}(G) \leq k$. Denote by $L_1, \ldots, L_m$ the internally connected components of G. For every $i \in [m]$, let K_i be the e-hypergraph obtained from L_i by adding a new edge, its source-edge, of extremities the set of vertices of $\mathbf{G}_G \cap L_i$. Clearly, K_i is an internally connected e-hypergraph of tree-width at most k and is equal to $\mathbf{val}(Y_i)$ for some $Y_i \in \mathcal{T}_k \cap \mathcal{T}_{i.c}$ (Lemma 44). Let $H = (\mathbf{e}_G, (\mathbf{G}_G \upharpoonright \mathbf{e}_G) \cup (K_1 \upharpoonright \mathbf{e}_{K_1}) \cup \ldots \cup (K_m \upharpoonright \mathbf{e}_{K_m}))$. Every vertex of H is a source of K, then K has at most $k+1$ vertices, belongs to $\mathbf{atom}(\mathcal{T}_k)$ and verifies $G = H[K_1, \ldots, K_m]$. Then, $\mathcal{T}_k$ contains and is equivalent with $\mathbf{atom}(\mathcal{T}_k)[\mathcal{T}_k \cap \mathcal{T}_{i.c}] = \mathbf{o}_{k+1}(\mathbf{1} \cap \mathcal{T}_k, u)$ with $u = \mathcal{T}_{i.c} \cap \mathcal{T}_k$.

2. $\mathcal{T}_{i.c} \cap \mathcal{T}_k = \mathbf{n}_{k+1}(\mathbf{p}_{k+1}, u)$ with $u = \mathcal{T}^{\mathbf{c}}_{i.c} \cap \mathcal{T}_k$.

   Direct consequence of $\mathcal{T}_{i.c} = \mathcal{T}^{\mathbf{nc}}_{i.c} \otimes \mathcal{T}^{\mathbf{c}}_{i.c}$ (Theorem 50) and of the obvious equalities $(\mathcal{T}^{\mathbf{nc}}_{i.c} \otimes \mathcal{T}^{\mathbf{c}}_{i.c}) \cap \mathcal{T}_k = (\mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_{k+1}) \otimes (\mathcal{T}^{\mathbf{c}}_{i.c} \cap \mathcal{T}_k)$ and $\mathcal{T}_k \cap Type_{k+1} = \mathcal{T}_k$.

3. $\mathcal{T}^{\mathbf{c}}_{i.c} \cap \mathcal{T}_k$ is equivalent with a subset of $\mathbf{o}_{k+1}[\ldots \mathbf{o}_{k+1}[u, u] \ldots, u]$ $(2 \cdot (1+k)^3$ times) with $u = \mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$.

   Direct consequence of Theorem 61 and 44.

4. $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k = \mathbf{n}_{k+1}(\mathbf{p}_{k+1} \cap \mathcal{L}, u) \cap \mathcal{L}$ with $u = \mathcal{L} \cap \mathcal{T}^{\mathbf{c}}_{i.c} \cap \mathcal{T}_k$.

Direct consequence of Point 2 and of the obvious equality $(u \otimes v) \cap \mathcal{L} = ((u \cap \mathcal{L}) \otimes (v \cap \mathcal{L})) \cap \mathcal{L}$ for every $u, v \subseteq \mathcal{T}$.

5. $\mathcal{L} \cap \mathcal{T}_{i.c}^{\mathbf{c}} \cap \mathcal{T}_k$ is equivalent with a subset of $+^{2+6 \cdot k}(u) \cap \mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$ with $u = \mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k-1}$.

   By Theorem 60.

Consequence of the obvious fact that $\mathcal{T}_{i.c} \cap \mathcal{T}_{-1}$ is equivalent with $\mathbf{1} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{-1}$, of the fact that the relation "contains and is equivalent with" is transitive, of the fact that $\mathbf{o}_{k+1}[\ldots \mathbf{o}_{k+1}[u, u] \ldots, u]$ $(2 \cdot (1 + k)^3$ times) with $u = \mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$ is contained in $\mathcal{T}_{i.c} \cap \mathcal{T}_k$, of Lemma 64 and of Points 1,..,5, we can construct for every k a term of Π that denotes $\mathcal{T}_k$. $\square$

Note that the precedent proof permits to extend Theorem 65 to every set of the form $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$ or $\mathcal{T}_{i.c} \cap \mathcal{T}_k$ for some k.

6 Each operation of Π preserves MSO-parsability

In this last section, we establish our main result. For this purpose, we will review, briefly, the notion of MSO-transduction of relational structures. We show a fundamental property of Π that states the MSO-parsability of every set produced by Π.

The notion of a MSO-transduction of relational structures is already used in [4] and [5] and surveyed in [6]. A MSO-transduction transforms a structure S into a structure S' by defining S' "inside" S by means of MSO-formulas. More precisely, S' is defined inside an intermediate structure made of k disjoint copies of S, for some fixed k. This makes it possible to construct S' with a domain larger than that of S (larger within the factor k). The MSO-formulas that define S' from S are collected in a tuple, called a *definition scheme.* The definition scheme is thus the syntactic description of the transduction.

Definition 66 (MSO-transduction) Let R and R' be two ranked alphabets of relation symbols. Let W be a finite set of set variables, the set of *parameters.* An (R', R)-*definition scheme* is a tuple of formulas of the form $\Delta = (\varphi, \psi_1, \ldots, \psi_k, (\theta_{r,j})_{r \in R', j \in [k]^{\rho(r)}})$ where:

- $k > 0$.
- $\varphi \in L(R, W)$.

- $\psi_i \in L(R, W \cup \{x_1\})$ for $i \in [k]$.
- $\theta_{r,j} \in L(R, W \cup \{x_1, \ldots, x_{\rho(r)}\})$ for $r \in R'$, $j \in [k]^{\rho(r)}$.

Let $S \in \mathcal{S}(R)$, and let γ be a W-assignment in S. An R'-structure S' is *defined by* Δ *in* (S, γ), denoted by $S' = \mathbf{def}_\Delta(S, \gamma)$, if:

- $(S, \gamma) \models \varphi$.
- $\mathbf{D}_{S'} = \{(d, i) \mid d \in \mathbf{D}_S, i \in [k], (S, \gamma, d) \models \psi_i\}$.
- for each $r \in R'$: $r_{S'} = \{((d_1, i_1), \ldots, (d_t, i_t)) \mid (S, \gamma, d_1, \ldots, d_t) \models \theta_{r,j}\}$, where $j = (i_1, \ldots, i_t)$ and $t = \rho(r)$.

The *transduction defined by* Δ is the relation denoted by $\mathbf{def}_\Delta$ that contains every pair of the form $(S, S') \in \mathcal{S}(R) \times S(R')$ with $S' = \mathbf{def}_\Delta(S, \gamma)$ for some assignment γ in S. A transduction f is *MSO-definable*, a *MSO-transduction* for short, if there is a definition scheme Δ such that $f = \mathbf{def}_\Delta$ or such that for every $(a, b) \in f$ there is $(a, c) \in \mathbf{def}_\Delta$ with b and c isomorphic (with the usual notion of isomorphism).

Definition 67 (MSO-parsability) A subset u of $\mathcal{T}$ is *MSO-parsable* if $\{(|\mathbf{val}(X)|, |X|) \mid X \in u\}$ is a MSO-transduction.

Our proof requires a few properties of MSO-transductions. The both next results are due to Courcelle in [4, 6].

Proposition 68 *The composition of two MSO-definable transductions is MSO-definable. The inverse image of a MSO-definable set of structures under a MSO-definable transduction is MSO-definable.*

Proposition 69 *The domain of a MSO-definable transduction is MSO-definable.*

Now, let us consider the constant sets of Π. Obviously, the set of all atomic e-tree-decompositions is MSO-parsable (We recall that the domain of every e-hypergraph contains the source-edge and, thus, is not empty). The MSO-parsability of the second constant-set of Π is the object of the next theorem. Appendix c is devoted to its proof.

Theorem 70 *For every k, the set $\mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_{k+1}$ is MSO-parsable.*

The previous theorem permits to extend a well-known result due to Courcelle: it enables us to define a class of context-free MSO-parsable hypergraph-grammars that contains strictly the "regular" one defined by Courcelle in [4]. It suffices to consider rule of productions (u, G) such that every non terminal edge of G is not critical in G.

Remarkable property of $\otimes$: it preserves MSO-parsability, under an additional condition.

Theorem 71 *For each k, the operation $u, v \to u \otimes (v \cap Type_k)$ preserves MSO-parsability.*

Proof.
Let $k \geq 0$. This proof comports 4 parts. A first one, we translate the problem into transduction of $\mathcal{T}^2$. A second one, we present e-tree-decomposition with an "assignment", their set is denoted by $\mathcal{T}_{assi}$. A third one, we transform such e-tree-decompositions into disjoint union of e-tree-decompositions, their set is denoted by $\mathcal{T}_{disj}$. A fourth one, we conclude.

Part 1
For every $X \in \mathcal{T}$ and every set D of arcs of X, we denote by:

- $\underline{contr}(X, D)$ the e-tree-decomposition obtained from X by contracting $\mathbf{A}_X \backslash D$.
- $\underline{part}(X, D)$ the set of all e-tree-decompositions generated by X and some maximal subtree of $\mathbf{T}_X \backslash D$.

Then, for every $X \in \mathcal{T}$, $\underline{contr}(X, \emptyset)$ is atomic. The domain of every e-hypergraph is nonempty (it contains the source edge). Then, for every set $u \subseteq \mathbf{atom}(\mathcal{T})$, the following assertions are equivalent:

- $\mathbf{val}(u)$ is MSO-definable.
- u is MSO-parsable.
- u is MSO-definable.

Then, every set $u \subseteq \mathcal{T}$ is MSO-parsable if and only if $\{(\underline{contr}(X, \emptyset), X) \mid X \in u\}$ is MSO-definable.

Part 2
Let $X \in \mathcal{T}$ and $D \subseteq \mathbf{A}_X$. A *vertex-partition of* (X, D) is a sequence of length k of the form $(V_1, \ldots, V_l, \emptyset, \ldots, \emptyset)$ for some $l \in [k]$ such that $(V_1, \ldots, V_l)$ is a

partition of $\bigcup_{d\in D,\{t,u\}=\mathbf{vert}_{\mathbf{T}_X}(d)} \mathbf{V}_{\mathbf{g}_X(t)}\cap\mathbf{V}_{\mathbf{g}_X(u)}$ that verify $\mathbf{V}_{\mathbf{g}_X(t)}\cap\mathbf{V}_{\mathbf{g}_X(u)}\cap V_i \cap V_j = \emptyset$ for every $d \in D$ and all $1 \le i < j \le k$ with $\{t,u\} = \mathbf{vert}_{\mathbf{T}_X}(d)$.

$\mathcal{T}_{assi}$ denotes the set of all sequences of the form $(X, D, V_1, \dots, V_k)$ with $X \in \mathcal{T}$, D a set of arcs of X and $(V_1, \dots, V_m)$ a vertex-partition of (X, D). For every $U \in \mathcal{T}_{assi}$, we denote by $\underline{contr}(U)$ the sequence $(\underline{contr}(X, D), D, V_1, \dots, V_k)$ with $U = (X, D, V_1, \dots, V_k)$. Clearly, $\mathcal{T}_{assi}$ is MSO-definable (the notion of "vertex-partition" is MSO-definable) and contains $\underline{contr}(\mathcal{T}_{assi})$.

Part 3

An *e-forest-decomposition* X is a sequence of the form (D, F, h) where D, F and h are respectively of the form $\{\mathbf{e}_{X_i} \mid i \in [m]\}$, $\mathbf{T}_{X_1} \cup \ldots \cup \mathbf{T}_{X_m}$ and $\mathbf{g}_{X_1} \cup \ldots \cup \mathbf{g}_{X_m}$ for some $X_1, \dots, X_m \in \mathcal{T}$ of disjoint domains. Clearly, a such decomposition into e-tree-decompositions is unique. It is denoted by $\underline{part}(X)$. We denote by $\mathcal{T}_{disj}$ the set of all sequences U of the form $(e, D, F, h, \gamma, \theta_1, \dots, \theta_k)$ where:

- $e \notin D$.
- $(\{e\} \cup D, F, h)$ is an e-forest-decomposition. The set $\underline{part}(\{e\} \cup D, F, h)$ is denoted by $\underline{part}(U)$.
- γ is a function $\mathbf{E}_{h(F)} \to \mathbf{E}_{h(F)}$ such that the transitive closure of γ is a partial order with e as the unique maximal element and such that every e-hypergraph $H \in \mathbf{val}(\underline{part}(U))$ verifies: $\{\mathbf{e}_H\} = \gamma(\mathbf{E}_H \backslash \mathbf{e}_H)$.
- for every $i \in [k]$, θ_i is a mapping $\mathbf{V}_{h(F)} \to \mathbf{V}_{h(F)}$ such that $\mathbf{Dom}(\theta_1), \dots, \mathbf{Dom}(\theta_k)$ is a partition of $\mathbf{V}_{h(F)}$, two distinct sources of some $Y \in \underline{part}(U)$ do not belong to same domain $\mathbf{Dom}(\theta_i)$ with $i \in [k]$ and such that for every $i \in [k]$ and every $Y \in \underline{part}(U)$ with $\mathbf{e}_Y \neq e$, we have: $\theta_i(\mathbf{vert}_{h(F)}(\mathbf{e}_Y)) = \theta_i(\mathbf{vert}_{h(F)}(\gamma(\mathbf{e}_Y)))$.

Clearly, $\mathcal{T}_{disj}$ is MSO-definable. For every $U = (e, D, F, h, \gamma, \theta_1, \dots, \theta_k) \in \mathcal{T}_{disj}$, we denote by $\underline{fus}(U)$ the sequence $(e, T, g, D, \mathbf{Im}(\theta_1), \dots, \mathbf{Im}(\theta_k))$, where T is obtained from F by adding every edge $d \in D$ of extremities the node t of F verifying $d \in \mathbf{E}_{h(t)}$ and the node s of F satisfying $\gamma^{-1}(d) \in \mathbf{E}_{h(s)}$ and where g associates with every node s of T the hypergraph obtained from $h(t) \backslash D$ by identifying every node $x \in \mathbf{Dom}(\theta_i)$ for some $i \in [k]$ with $\theta_i(x)$. Then, we have:

- $\underline{fus}(\mathcal{T}_{disj}) \subseteq \mathcal{T}_{assi}$.
  Consequence of the definitions of $\mathcal{T}_{assi}$ and $\mathcal{T}_{disj}$.

- $\{(U, f\underline{us}(U)) \mid U \in \mathcal{T}_{disj}\}$ is MSO-definable.
  Direct consequence of the definition of $f\underline{us}$.

- $f\underline{us}(\mathcal{T}_{disj}) = \mathcal{T}_{assi}$.
  Let $U = (X, D, V_1, \ldots, V_k)$ be a sequence of $\mathcal{T}_{assi}$. Denote by F the forest $\mathbf{T}_X \backslash D$. For every node t of X, we denote by X_t the e-tree-decomposition generated by X and by the maximal subtree of F that contains t. Denote by h the mapping that associates with every node t of F the hypergraph obtained from $\mathbf{g}_{Y_t}(t)$ by renaming $(\mathbf{e}_{Y_t}, i)$ every vertex of V_i and $(\mathbf{e}_{Y_t}, 0)$ the source-edge of Y_t. Denote by γ the mapping that associates with every edge d of X the edge $(\mathbf{e}_{Y_t}, 0)$ where t is some node of X verifying $d \in \mathbf{E}_{\mathbf{g}_X(X_t)}$ and to every edge (of $h(F)$), of the form $(d, 0)$ with d an arc of X, the edge d. For every $i \in [k]$, denote by θ_i the mapping that associates with every node of the form (d, i) for some arc d of X, the original vertex of $\mathbf{V}_{g(\mathbf{T}_X)}$. The sequence $W = (e, D, F, h, \gamma, \theta_1, \ldots, \theta_k)$ belongs to $\mathcal{T}_{disj}$ and verifies $U = f\underline{us}(W)$. Then $f\underline{us}(\mathcal{T}_{disj}) = \mathcal{T}_{assi}$.

- $\{(f\underline{us}(U), U) \mid U \in \mathcal{T}_{disj}\}$ is MSO-definable.
  Clearly, the mapping described in the precedent point is MSO-definable. Then, $\{(f\underline{us}(U), U) \mid U \in \mathcal{T}_{disj}\}$ is MSO-definable.

For every $U = (e, D, F, h, \gamma, \theta_1, \ldots, \theta_k) \in \mathcal{T}_{disj}$, we denote by $\underline{contr}(U)$ the sequence $(e, D, F', h', \gamma, \theta_1, \ldots, \theta_k)$ where F' is obtained from F by contracting every maximal subtree of F into an isolated vertex and where h' associates with every node s of F' the hypergraph $h(T_t)$ with T_t the maximal subtree of F that contains t. Clearly, $\underline{contr}(\mathcal{T}_{disj}) \subseteq \mathcal{T}_{disj}$ and $f\underline{us}(\underline{contr}(U)) = \underline{contr}(f\underline{us}(U))$, for every $U \in \mathcal{T}_{disj}$.

Part 4

Let $u, v \subseteq \mathcal{T}$ be two MSO-parsable sets. The set $Type_k$ is MSO-definable, then $v \cap Type_k$ is MSO-parsable. In order to simplify the proof, we can suppose without pert of generality that $v \cap Type_k = v$.

Denote by v_{disj} the set $\{X \in \mathcal{T}_{disj} \mid \underline{part}(X) \subseteq v\}$ and by v_{assi} the set $\{X \in \mathcal{T}_{assi} \mid \underline{part}(X) \subseteq v\}$. Clearly, $\{(\underline{contr}(X), X) \mid X \in v_{disj}\}$ is MSO-definable. Then, $\{(f\underline{us}(\underline{contr}(X)), X) \mid X \in v_{disj}\}$, $\{(f\underline{us}(\underline{contr}(X)), f\underline{us}(X)) \mid X \in v_{disj}\}$, $\{(\underline{contr}(U), U) \mid U \in v_{assi}\}$ and $\{(\underline{contr}(X, D), X) \mid (X, D, V_1, \ldots, V_k) \in v_{assi}\}$ are MSO-definable.

Every $X \in u \otimes v$ admits $D \subseteq \mathbf{A}_X$ such that $\underline{contr}(X, D) \in u$, $\underline{part}(X, D) \subseteq v$ and such that (X, D) admits a vertex-partition (the proof is obvious and is omitted). Then, $\{(\mathbf{val}(X), X) \mid X \in u \otimes v\}$ is the composition of $\{(\mathbf{val}(X), X) \mid X \in u\}$ and $\{(\underline{contr}(X, D), X) \mid (X, D, V_1, \ldots, V_k) \in v_{assi}\}$. By Proposition 68, it is a MSO-transduction. Hence, $u \otimes (v \cap Type_k)$ is MSO-parsable. $\square$

The previous result can be extended to every operation of Π. The fact that $u \to \{X \in u \mid X \models \varphi\}$ for some MSO-formula and $+$ preserves MSO-parsability are the obvious consequence of their MSO-definability (as transduction) and of Proposition 68. The fact that the fourth operation of Π preserves MSO-parsability is actually a consequence of Theorem 71. All these results are contained in the second important property of Π:

Theorem 72 *Every operation of* Π *preserves MSO-parsability.*

Proof.
We denote by $\mathcal{I}$ the set $\{X \in \mathcal{T} \mid \mathbf{val}(X) \in \mathcal{G}_{i.c}\}$ and, for every $k \geq 0$, by $\mathbf{star}_k$ the mapping that associates with all subsets u, v of $\mathcal{T}$ the set $\{Y[Z_1, \ldots, Z_m] \mid Y \in u, Z_1, \ldots, Z_m \in v \cap \mathcal{I} \cap Type_k, m \geq 1\}$. As a consequence of Theorem 71, of Proposition 68, of the fact that the transductions $+ : u \to +(u)$ and $u \to u \cap \{X \in \mathcal{T} \mid X \models \varphi\}$ for every $\varphi \in L(R')$, are MSO-definable and then preserve MSO-parsability, of the fact that the operation union $((u, v) \to u \cup v)$ preserves MSO-parsability and of the equality $u[v \cap \mathcal{I} \cap Type_k] = u \cup v \cup \mathbf{star}_k(u, v)$ for every $k \geq 0$ and all $u, v \subseteq \mathcal{T}$, to conclude it suffices to prove that for every k the binary operation $\mathbf{star}_k$ preserves MSO-parsability.

Let $k \geq 0$. Let us denote by $Star_{i.c}$ the set of all not atomic e-tree-decompositions X such that every leaves of X is adjacent with its root and such that the value of any e-tree-decomposition generated by X and some of its leaf is internally connected. For every $X \in Star_{i.c}$, we denote by $f(X)$ the set of all e-tree-decompositions generated by X and some of its leaf, we denote by $g(X)$ the e-tree-decomposition generated by X and its root. For every $H \in \mathcal{G}$, every subset $D \subseteq \mathbf{E}_H \backslash \mathbf{e}_H$ and every subset $U \supseteq \mathbf{vert}_H(\mathbf{e}_H) \cup (\mathbf{V}_H - \mathbf{vert}_H(D))$, we denote by $h(H, D, U)$ the e-hypergraph $(\mathbf{e}_H, \mathbf{V}_H, \mathbf{e}_H \cup D, j)$ with j the mapping that associates with every $d \in \mathbf{e}_H \cup D$ the set $\mathbf{vert}_H(d)$ if $d \in D$ and the set U if $d = \mathbf{e}_H$. For every $H \in \mathcal{G}$, we denote by $h(H)$ the set of all e-hypergraphs of the form $h(H, D, U)$ for some sets D and U. Clearly, $\{(H, h(H)) \mid H \in \mathcal{G}\}$ is MSO-definable. For every $X \in Star_{i.c}$, we denote by $h(X)$ the e-hypergraph $h(\mathbf{val}(X), D, U)$ with D the set of edges of X that does not belong to $\mathbf{g}_X(\mathbf{r}_X)$ and with U the set of vertices of X that belong to $\mathbf{g}_X(\mathbf{r}_X)$. It follows that the transduction

$\{(\mathbf{val}(X), h(\mathbf{val}(X)) \mid X \in Star_{i.c}\}$ is equal to $\{(H, h(H)) \mid H \in \mathcal{G}\}$ and, then, is MSO-definable.

For every $X \in Star_{i.c}$, the set $\{\mathbf{G}_Y \backslash \mathbf{e}_Y \mid Y \in f(X)\}$ is the set of all internally connected component of $h(X)$. The notion of an internally connected component of some e-hypergraph is MSO-definable. Then, $\{(\mathbf{val}(X), \{\mathbf{G}_Y \backslash \mathbf{e}_Y \mid Y \in f(X)\}) \mid X \in Star_{i.c}\}$ is MSO-definable. For all distinct leaves s, t of X, the hypergraphs $\mathbf{G}_Y \backslash \mathbf{e}_Y$ and $\mathbf{G}_Z \backslash \mathbf{e}_Z$ are nonempty and distinct, where Y (resp. Z) designs the e-tree-decomposition generated by X and s (resp. t). Thus, $\{(\mathbf{val}(X), f(X)) \mid X \in Star_{i.c}\}$ is MSO-definable.

To define $g(X)$ in terms of $\mathbf{val}(X)$, it suffices to define the subhypergraph $\mathbf{g}_X(\mathbf{r}_X)$, obvious, augmented with each source-edge of some e-tree-decomposition of $f(X)$. Then, $\{(\mathbf{val}(X), (f(X), g(X))) \mid X \in Star_{i.c}\}$ is MSO-definable. Thus, $\{(\mathbf{val}(X), X) \mid X \in Star_{i.c}\}$ is MSO-definable. The set $Star_{i.c}$ is MSO-parsable.

For every subsets $u, v, w \subseteq \mathcal{T}$, let us denote by $u \otimes' (v, w)$ the set that contains every e-tree-decomposition of $u \otimes (v \cup w)$ obtained from some e-tree-decomposition of u by replacing its root by an element of v and other nodes by elements of w. By a proof similar with the one of Theorem 71, we prove that $(u, v, w) \to (Star_{i.c} \otimes' (v, w \cap Type_k))$ preserves MSO-parsability. Let u, v be two MSO-parsable subsets of $\mathcal{T}$. Clearly $\mathbf{star}_{i.c}(u, v)$ is equal to $Star_{i.c} \otimes (u, v \cap Type_k)$. Hence, $\mathbf{star}_{i.c}$ preserves MSO-parsability. $\square$

Thus, thanks to Theorem 65 and 72, we obtain:

Theorem 73 *For every k, $\mathcal{T}_k$ contains an equivalent MSO-parsable set.*

Proof.
Direct consequence of Theorems 65 and 72. $\square$

Note that the result of Theorem 65 can be extended to every set of the form $\mathcal{T}_{i.c} \cap \mathcal{T}_k$, $\mathcal{L}_l \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$ and $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$ for some k, l. Thus, by Theorem 72, all these sets are MSO-parsable. In particular: $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$. This result, that concerns "linear, internally connected k-trees" is similar with the result of Kabanets [11] that concerns "k-paths".

As a consequence of Theorem 73, we obtain by Theorem 74 our main important result. The proof of Theorem 74 requires a few notations and definitions, we will not present. We invite the reader to read [4]. We can say simply that this result is equivalent to prove the existence of a transduction that associates with every hypergraph of bounded tree-width an "reduced term". A "term" is represented by a ordered tree, written with function symbols of a fixed arity, constants, and variables. If an operation symbol like $+$ is associative and commutative, then a term like $+(x, +(y, z))$ can be written equally well $+(x, y, z)$ or $+(y, x, z)$. The order of arguments is irrelevant (in other words, they form a set and not a sequence). Then, the successors of a node labelled $+$ form a set (as opposed to a sequence), the cardinality of which is not fixed. This idea has been introduced by Franchi-Zannettacci in the context of attribute grammars [9]. A "reduced term" is a term a term built with associative and commutative operation symbols and operation symbol, denoting operations having no special property.

Theorem 74 *For every k, the set of graphs of tree-width at most k is strongly context-free.*

Proof.
Let k be an integer. Denote by $\mathcal{R}$ contains every atomic e-tree-decomposition that designs an e-hypergraph H that verifies at least one of the following assertions:

- H has at most one non-source edge.
- H has no isolated vertex and at most three edges.
- H has no isolated vertex and every pair of edges of $\mathbf{E}_H \backslash \mathbf{e}_H$ have same extremities.

We suppose in this proof, that if $\mathcal{T}_k$ and $\mathcal{T}_k \otimes \mathcal{R}$ are equivalent and if $\mathcal{T}_k \otimes \mathcal{R}$ contains an equivalent MSO-parsable set, then $\mathbf{val}(\mathcal{T}_k)$ is strongly context-free.

Denote by $\mathcal{S}$ the set of all $X \in \mathcal{T}$ such that:

- X contains at most $k+1$ vertices.
- $\mathbf{atom}(\{X\}) \subseteq \mathcal{R}$.
- two edges of H distinct with $\mathbf{e}_X$ having same extremities in $\mathbf{G}_X$ belong to some common hypergraph of the form $\mathbf{g}_X(t)$ with t a leaf of X.
- for every node t of X, $\mathbf{g}_X(t)$ contains at least one edge, has at least two childrens in $(\mathbf{T}_X, \mathbf{r}_X)$ or one vertex that does not belong to $\mathbf{g}_X(r)$ with r its parent in $(\mathbf{T}_X, \mathbf{r}_X)$ if $t \neq \mathbf{r}_X$.

$mic(\mathcal{S}) \subseteq \mathcal{R}$ implies: $\mathcal{T}_k \otimes \mathcal{S} \subseteq \mathcal{T}_k \otimes \mathcal{R} \subseteq \mathcal{T}_k$. Clearly, every e-hypergraph having at most $k+1$ vertices belongs to $\mathbf{val}(\mathcal{S})$. Then, $\mathcal{T}_k$, $\mathcal{T}_k \otimes \mathcal{R}$ and $\mathcal{T}_k \otimes \mathcal{S}$ are equivalent.

For every $G \in \mathbf{val}(\mathcal{S})$, the cardinality of $\mathbf{V}_G$ is at most $k+1$ and the cardinality of $\{\{c \mid c \in \mathbf{E}_G \backslash \mathbf{e}_G, \mathbf{vert}_G(c) = \mathbf{vert}_G(d)\} \mid d \in \mathbf{E}_G \backslash \mathbf{e}_G\}$ is at most 2^{k+1}. It follows that every e-tree-decomposition of $\mathcal{S}$ has at most $k+1+2^{k+1}$ leaves and at most $2 \cdot (k+1+2^{k+1})$ nodes. Then, $\mathcal{S}$ is MSO-parsable.

By Theorem 73, $\mathcal{T}_k$ contains an equivalent MSO-parsable set $\mathcal{L}$. By Theorem 71, $\mathcal{L} \otimes \mathcal{S}$ is MSO-parsable (we have: $\mathcal{S} \subseteq Type_{k+1}$). The sets $\mathbf{atom}(\mathcal{T}_k)$ and $\mathcal{S}$ are equivalent, then, $\mathcal{L} \otimes \mathcal{S}$ is equivalent with $\mathcal{L}$. Thus, $\mathcal{L} \otimes \mathcal{S}$ is an equivalent and MSO-parsable subset of $\mathcal{T}_k$. $\square$

Clearly, the above result is extended to every set of oriented (or not) hypergraphs having a bounded rank (and of tree-width at most some k).

Let us recall a fundamental result of Courcelle [3].

Theorem 75 *Every CMSO-definable set of graphs is recognizable.*

Using above theorem and the result of Courcelle [4] mentioned in the introduction, we conclude and establish our main result that states the equivalence of the notion of a recognizable set of graphs and of a CMSO-definable set of graphs.

Theorem 76 *Every set of graphs of bounded tree-width is CMSO-definable if and only if it is recognizable.*

Clearly, the above result is extended to every set of oriented (or not) hypergraphs having a bounded rank (and of tree-width bounded).

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Appendix a

This appendix is devoted to the proof of Theorem 60. The decomposition of the linear and critical case is made in two steps. Firstly, we treat thanks Lemma 79, the linear and critical case. That permits in an easy way to prove Theorem 60.

A simple way to manipulate linear e-tree-decompositions is to consider not a unique distinguished edge but a couple of distinguished edges. For this reason, we define a *2e-path-decomposition* that is an linear e-tree-decomposition X with the distinguished edge $\mathbf{e}_X$ at the root and a second one, denoted $\mathbf{d}_X$, in the leaf. The operation of substitution $[\,]$ induces naturally on such structures an operation that "concatenates" $2e$-path-decompositions like words. For this reasons, we use the symbol $\mathcal{W}$ to denote their set and the symbol $\odot$ to denote the operation of substitution induced by $[\,]$.

Definition 77 An *$2e$-hypergraph* H is a sequence $(\mathbf{e}_H, \mathbf{V}_H, \mathbf{E}_H, \mathbf{vert}_H, \mathbf{d}_H)$ where $(\mathbf{V}_H, \mathbf{E}_H, \mathbf{vert}_H)$ is a hypergraph, denoted by $\mathbf{G}_H$, and where $\mathbf{e}_H$ and $\mathbf{d}_H$ are two distinct edges of $\mathbf{G}_H$. In order to simplify, every $2e$-hypergraph ma be identified with $(\mathbf{e}_H, \mathbf{G}_H, \mathbf{d}_H)$.

Let H and K be two $2e$-hypergraphs with $\mathbf{e}_H \neq \mathbf{d}_K$ and $\mathbf{G}_H \restriction \mathbf{d}_H = \mathbf{G}_H \cap \mathbf{G}_K = \mathbf{G}_K \restriction \mathbf{e}_K$. We denote by $H \odot K$ the $2e$-hypergraph $(\mathbf{e}_H, (\mathbf{G}_H \cup \mathbf{G}_K) \backslash \mathbf{e}_K, \mathbf{d}_K)$.

An *$2e$-path-decomposition* X is a sequence $(\mathbf{e}_X, \mathbf{T}_X, \mathbf{g}_X, \mathbf{d}_X)$, where $(\mathbf{e}_X, \mathbf{T}_X, \mathbf{g}_X)$ is a linear e-tree-decomposition such that its unique leaf l verifies $\mathbf{d}_X \in \mathbf{E}_{\mathbf{g}_X(l)}$. Their set is denoted by $\mathcal{W}$. For every $k \geq -1$, $\mathcal{W}_k$ denotes the subset $\{X \in \mathcal{W} \mid \underline{wd}(X) \leq k\}$.

In order to simplify, X can be identified with the pair $(Y, \mathbf{d}_X)$. The $2e$-hypergraph *denoted by* X is the sequence $(\mathbf{e}_X, \mathbf{g}_X(\mathbf{T}_X), \mathbf{d}_X)$, denoted by $\mathbf{val}(X)$. For every $2e$-path-decomposition $Y' = (Y, c), Z' = (Z, d)$ such that $c = \mathbf{e}_Z$ and $Y \odot Z$ defined, we denote by $Y' \odot Z'$ the $2e$-path-decomposition $(Y \odot Z, d)$. For every set $u, v \subseteq \mathcal{W}$, we denote by $u \odot v$ the union $u \cup v \cup \{Y \odot Z \mid Y \in u, Z \in v\}$. In a natural way, we extend $\sqsubseteq$, $+$ to $\mathcal{W}$.

In a very natural way, we extend to $\mathcal{W}$ a few notions defined over $\mathcal{T}$.

Definition 78 A $2e$-path-decomposition X is *internally connected*, *connected*, *critical* if respectively:

- $(\mathbf{e}_X, \mathbf{T}_X, \mathbf{g}_X)$ is internally connected.

- $(\mathbf{e}_X, \mathbf{T}_X, \mathbf{g}_X)$ is connected.
- X is connected and $\mathbf{d}_X$ is critical in $(\mathbf{e}_X, \mathbf{G}_X)$.

Their sets are respectively denoted by $\mathcal{W}_{i.c}$, $\mathcal{W}_{1.c}$, $\mathcal{W}^{\mathbf{c}}_{1.c}$.

The next lemma translates critical linear case into linear one with smaller width.

Lemma 79 *For every k, the set $\mathcal{W}^{\mathbf{c}}_{1.c} \cap \mathcal{W}_{i.c} \cap \mathcal{W}_{k+1}$ is equivalent with some subset of $\mathcal{W}_{i.c} \cap \mathcal{W}_{k+1} \cap +^{6 \cdot k}(\mathcal{W}_{i.c} \cap \mathcal{W}_k)$.*

Proof.
Let k be an integer. For every $2e$-path-decomposition X and every arc (resp. edge) d of X, we denote by $\overline{\mathbf{g}}_X(d)$ the hypergraph with no edge $\mathbf{g}_X(s) \cap \mathbf{g}_X(t)$ with s and t the extremities of d in $\mathbf{T}_X$ (resp. $(\mathbf{G}_X \restriction d) \backslash d$). In this proof, Ω is a given $2e$-path-decomposition, $\mathcal{M}$ a couple of hypergraphs (L, M) such that $((\mathbf{e}_\Omega, \mathbf{d}_\Omega), \mathcal{M})$ is a circular-decomposition of $(\mathbf{e}_\Omega, \mathbf{T}_\Omega, \mathbf{g}_\Omega)$. The union $\{\mathbf{e}_\Omega, \mathbf{d}_\Omega\} \cup \mathbf{A}_\Omega$ is denoted by $\mathcal{A}$. We denote by $\prec$ the total order on $\mathcal{A}$ induced by the path $\mathbf{T}_\Omega$ by considering $\mathbf{d}_\Omega$ (resp. $\mathbf{e}_\Omega$) the minimal (resp. maximal) element of $(\mathcal{A}, \prec)$. For every hypergraph G, we denote by $\|G\|$ the number of its vertices. For every $2e$-path-decomposition X, we denote by:

- $\|X\|$ the minimum of $\|\mathbf{g}_X(t)\|$ taken over all node t of X (that is not $\underline{wd}(X)$!),
- $\underline{l}(X)$ the minimal integer l such that $+^l(\mathcal{W}_{i.c} \cap \mathcal{W}_k)$ contains X.

Clearly, we have:

(1) $\underline{l}(X \odot Y) \leq \underline{l}(X) + \underline{l}(Y)$, for all $Y, Z \in \mathcal{W}$ with $Y \odot Z$ defined.

For every hypergraph G, we denote by $G!L$ the hypergraph $(G \backslash\backslash \mathbf{V}_M) \backslash \mathbf{E}_M$. For every $2e$-hypergraph H with $\{\mathbf{e}_H, \mathbf{d}_H\}$ disjoint with $\mathbf{E}_{L \cup M}$, we denote by $H!L$ the $2e$-hypergraph $(\mathbf{e}_H, \mathbf{G}_H!L, \mathbf{d}_H)$. For every $X \in \mathcal{W}$ with $\{\mathbf{e}_X, \mathbf{d}_X\}$ disjoint with $\mathbf{E}_{L \cup M}$, we denote by $X!L$ the $2e$-path-decomposition $(\mathbf{e}_X, \mathbf{T}_X, g, \mathbf{d}_X)$ where g associates with every node t of X the hypergraph $\mathbf{g}_X(t)!L$. Clearly, every $X \in \mathcal{W}$ verifies:

(2) $\mathbf{val}(X!L) = \mathbf{val}(X)!L$, if $X!L$ defined.

(3) $(X!L) \uparrow d = (X \uparrow d)!L$ for every arc d of X.

(4) $(X!L) \downarrow d = (X \downarrow d)!L$ for every arc d of X.

In a symmetrical way, we define $G!M$, $H!M$ and $X!M$ and obtain similar results. For every $X \sqsubseteq \Omega$, we denote by:

- $\mathbf{first}(X)$ the edge $\min_{\prec}\{c \in \{\mathbf{d}_X, \mathbf{e}_X\} \cup \mathbf{A}_X \mid \|\overline{\mathbf{g}}_X(c)!L\| \leq \|X!L\|\}$.
- $\mathbf{last}(X)$ the edge $\max_{\prec}\{c \in \{\mathbf{d}_X, \mathbf{e}_X\} \cup \mathbf{A}_X \mid \|\overline{\mathbf{g}}_X(c)!L\| \leq \|X!L\|\}$.

We have:

(5) $\mathbf{first}(X) \prec \mathbf{last}(X)$, for every $X \sqsubseteq \Omega$.
Let $X \sqsubseteq \Omega$ and t some node of X such that: $\|\mathbf{g}_X(t)!L\| = \|X!L\|$. The edges $d = \mathbf{d}_{X|t}$ and $e = \mathbf{e}_{X|t}$ belong to $\mathcal{A}$, verify $d \prec e$ and $\overline{\mathbf{g}}_X(e) \cup \overline{\mathbf{g}}_X(d) \subseteq \mathbf{g}_X(t)$. It follows $\mathbf{first}(X) \prec d \prec e \prec \mathbf{last}(X)$.

(6) $1 \leq \|X!L\|, \|X!M\| \leq 1 + k$, for every $X \sqsubseteq \Omega$.
Let $X \sqsubseteq \Omega$. For every node t of X, $\mathbf{V}_{\mathbf{g}_X(t)}$ is equal to $\mathbf{V}_{\mathbf{g}_\Omega(t)}$, and, then, is the disjoint union of $\mathbf{V}_{\mathbf{g}_\Omega(t)} \cap \mathbf{V}_L$ and of $\mathbf{V}_{\mathbf{g}_\Omega(t)} \cap \mathbf{V}_M$. $\underline{wd}(\Omega) \leq k+1$ implies $\|X!L\| + \|X!M\| \leq \|X\| \leq 2 + k$. By hypothesis, L is not disjoint with $\mathbf{G}_\Omega \restriction \mathbf{e}_\Omega$ and not disjoint with $\mathbf{G}_\Omega \restriction \mathbf{d}_\Omega$. The fact that $\mathbf{d}_X$ (resp. $\mathbf{e}_X$) is associated to the leaf (resp. root) of Ω, implies that $\mathbf{V}_{\mathbf{g}_X(t)} \cap \mathbf{V}_L$ is nonempty, for every node t of X. Same argument for M. It follows: $1 \leq \|X!L\|, \|X!M\| \leq 1 + k$.

An 2e-hypergraph H is *$\mathcal{M}$-circular* if :

- H, $H!L$ and $H!M$ are internally connected.
- $((\mathbf{e}_X, \mathbf{d}_X), \mathcal{M} \sqcap \mathbf{G}_X)$ is a circular-decomposition of $\mathbf{G}_X$.

An 2e-path-decomposition X is *$\mathcal{M}$-circular* if $X \in \mathcal{W}_{k+1}$ and if for every $Y \sqsubseteq X$, $\mathbf{val}(Y)$ is $\mathcal{M}$-circular. Their set is denoted by $\mathcal{W}_{\mathcal{M}}$. We have:

(7) the property "$\mathcal{M}$-circular" is hereditary and substitution-closed.
Clearly, the property $\mathcal{M}$-circular is hereditary. Let X be an 2e-path-decomposition of the form $Y \odot Z$ with $Y, Z \in \mathcal{W}_{\mathcal{M}}$. By Point (2),$\mathbf{val}(X)$, $\mathbf{val}(X!L)$ and $\mathbf{val}(X!M)$, are equals respectively to $\mathbf{val}(Y) \odot \mathbf{val}(Z)$, $(\mathbf{val}(Y)!L) \odot (\mathbf{val}(Z)!L)$, $(\mathbf{val}(Y)!M) \odot (\mathbf{val}(Z)!M)$, and by Lemma 16, are internally connected. The sequence $((\mathbf{e}_X, \mathbf{d}_X), \mathcal{M} \sqcap \mathbf{G}_X)$ is equal to $((\mathbf{e}_Y, \mathbf{d}_Z), (L \cap (\mathbf{G}_Y \cup \mathbf{G}_Z), M \cap (\mathbf{G}_Y \cup \mathbf{G}_Z)))$, and, clearly, is a circular-decomposition of $\mathbf{G}_X$.

(8) every $X \sqsubseteq \Omega$ is $\mathcal{M}$-circular.
Let $Y \sqsubseteq \Omega$. By Fact 55, $((\mathbf{e}_Y, \mathbf{d}_Y), \mathcal{M} \sqcap \mathbf{G}_Y)$ is a circular-decomposition of $\mathbf{G}_Y$. The membership $Y \in \mathcal{W}_{i.c}$ implies $\mathbf{val}(Y)$ internally connected. Every internally connected component of $\mathbf{val}(Y) \backslash \mathbf{d}_Y$

is connected and, then is contained, either in L, or in M. The fact that $\mathbf{val}(Y)$ is internally connected implies that every internally connected component contained in L (resp. M) contains at least one extremity of $\mathbf{d}_Y$ that is an internal-vertex of $\mathbf{val}(Y)$ and a vertex of L (resp. M). Then, $\mathbf{val}(Y!L)$ and $\mathbf{val}(Y!M)$ are internally connected. Ω is $\mathcal{M}$-circular. By Point (7), every $Y \sqsubseteq \Omega$ is $\mathcal{M}$-circular.

Every $X \sqsubseteq \Omega$ admits an equivalent $Y \in \mathcal{W}_{\mathcal{M}}$ with $\underline{l}(Y)$ at most:

(9) 1, if X is atomic or if $\underline{wd}(X) \leq k$.
The equality $\underline{l}(X) = 1$ and Point (8) suffice to conclude.

Moreover, we suppose X not atomic and of width $1 + k$.

(10) 3, if $\mathbf{first}(X) = \mathbf{d}_X$ and $\mathbf{last}(X) = \mathbf{e}_X$.
Let f be an arc of X such that $\|\overline{\mathbf{g}}_X(f)!M\| \leq \|X!M\|$. Let f_1 and f_2 twos symbols that do not belong to X. Denote by I_1 (resp. I_2) the connected hypergraph having for set of edges $\{f_1\}$ (resp. $\{f_2\}$) and for set of vertices $(\mathbf{V}_L \cap \mathbf{vert}_{\mathbf{G}_X}(\mathbf{e}_X)) \cup (\mathbf{V}_M \cap \mathbf{vert}_{\mathbf{G}_X}(f))$. (resp. $(\mathbf{V}_L \cap \mathbf{vert}_{\mathbf{G}_X}(\mathbf{e}_X)) \cup (\mathbf{V}_M \cap \mathbf{vert}_{\mathbf{G}_X}(f)))$.

Let Y be the sequence $(\mathbf{e}_X, \mathbf{T}_{X\Uparrow f}, g, f_1)$ where g associates with every node t of $X \uparrow f$ the hypergraph $\mathbf{g}_X(t) \cap ((\mathbf{G}_X \restriction \mathbf{e}_X) \cup M \cup I_1 \backslash f_1)$ augmented with I_1 if t is the leaf of $\mathbf{T}_{X\Uparrow d}$. Clearly, Y belongs to $\mathcal{W}$. For every node t of $X \uparrow d$, $\|\mathbf{g}_Y(t)\|$ is successively equal to $\|\mathbf{g}_Y(t) \cap L\| + \|\mathbf{g}_Y(t) \cap M\|$, $\|\overline{\mathbf{g}}_X(f)!M\| + \|\mathbf{g}_X(t) \cap M\|$ and, then is at most $\|\mathbf{g}_X(t)\| \leq \underline{wd}(X) \leq k+1$. Then, $Y \in \mathcal{W}_{k+1}$. By Point (8), $X \uparrow f \in \mathcal{T}_{\mathcal{M}}$, Then, $Y!M$ is isomorphic with $X \uparrow d$. Then, $Y!M$ is internally connected. Every vertex of $\mathbf{V}_Y \cap \mathbf{V}_M$ is uniquely incident with $\mathbf{e}_X$ and f_1, it follows that Y and $Y!M$ are internally connected. Clearly, $((\mathbf{e}_X, f_1), \mathcal{M} \sqcap \mathbf{val}(Y))$ is equal to $((I_1 \backslash f_1) \cap L, \mathbf{G}_{X\Uparrow d} \cap M))$ and is a circular-decomposition of Y. By Fact 12, Y is internally connected. Then, $Y \in \mathcal{W}_{\mathcal{M}}$. $(\mathbf{G}_Y \restriction \mathbf{e}_Y) \cap (\mathbf{G}_Y \restriction \mathbf{d}_Y)$ contains $\overline{\mathbf{g}}_X(\mathbf{e}_X) \cap \overline{\mathbf{g}}_X(f) \cap I_2 \cap L$ that is nonempty $(((\mathbf{e}_X, f), \mathcal{M} \sqcap \mathbf{val}(X \uparrow f))$ is a circular-decomposition of $\mathbf{val}(X \uparrow d))$. Let s be a vertex of $(\mathbf{G}_Y \restriction \mathbf{e}_Y) \cap (\mathbf{G}_Y \restriction \mathbf{d}_Y)$. Clearly, $Y \backslash\backslash \{s\}$ belongs to $\mathcal{W}_{i.c} \cap \mathcal{W}_k$. Then, Y belongs to $+(\mathcal{W}_{i.c} \cap \mathcal{W}_k)$ and verifies $\underline{l}(Y) = 1$.

Let W be the sequence $(f_2, \mathbf{T}_{X\Downarrow f}, g, \mathbf{d}_X)$ where g associates with every node t of $X \downarrow f$ the hypergraph $\mathbf{g}_X(t) \cap (I_2 \backslash f_2 \cup M \cup (\mathbf{G}_X \restriction \mathbf{d}_X))$ augmented with I_2 if $t = \mathbf{r}_{X\Uparrow d}$. With similar arguments than for Y, we prove $W \in \mathcal{T}_{\mathcal{M}}$ and $\underline{l}(W) = 1$.

Let U be the sequence $(f_1, \mathbf{T}_X, g, f_2)$ where g associates with every node t of X the hypergraph $\mathbf{g}_X(t) \cap (I_1 \backslash f_1 \cup L \cup (I_2 \backslash f_2)$ augmented with I_1 if $t = \mathbf{r}_X$ and with I_2 if t is the leaf of X. Let V be an isomorphic and equivalent copy of U such that the domain of $\mathbf{T}_V$ is disjoint with the domain of X. With similar arguments than for Y, we prove $V \in \mathcal{T}_{\mathcal{M}}$ and $\underline{l}(W) = 1$.

From the construction of Y, V, W, it follows that $Y \odot V \odot W$ is defined and denotes $\mathbf{val}(X)$. By Point (1) and (8), $Y \odot V \odot W$ belongs to $\mathcal{W}_{\mathcal{M}}$ and verifies $\underline{l}(Y \odot V \odot W) = 3$.

(11) $3 \cdot (1 + k)$, if $\mathbf{first}(X) = \mathbf{d}_X$.
For every $X \sqsubseteq \Omega$, we denote by $f(X)$ the integer $2 + k - \|X!L\|$. By Point (6), every $X \sqsubseteq \Omega$ verifies $1 \leq f(X) \leq k + 1$. Suppose there is $n \geq 1$ such that every $X \sqsubseteq \Omega$ with $\mathbf{first}(X) = \mathbf{d}_X$ and $f(X) < n$ admits an equivalent $Y \in \mathcal{W}_{\mathcal{M}}$ with $\underline{l}(Y) \leq 3 \cdot f(X)$. Let $X \sqsubseteq \Omega$ with $f(X) = n$. Denote by $\mathbf{l}$ the element $\mathbf{last}(X)$.

If $\mathbf{l} = \mathbf{e}_X$, Point (10) and $1 \leq f(X)$ suffice to conclude. Moreover, we suppose $\mathbf{l} \prec \mathbf{e}_X$. By Point (5), $\mathbf{l} \in \mathbf{A}_X$. Clearly, we have $\mathbf{first}(X \uparrow \mathbf{l}) = \mathbf{l} = \mathbf{d}_Y$. From the maximality of $\mathbf{l}$, every node t of $X \uparrow \mathbf{l}$ verifies $\|X!L\| < \|\mathbf{g}_X(t)!L\|$, and, then $f(X \uparrow \mathbf{l}) < f(X)$. By recurrence's hypothesis, $X \uparrow \mathbf{l}$ admits an equivalent $2e$-path-decomposition $V \in \mathcal{W}_{i.c}$ such that $\underline{l}(V) \leq 3 \cdot (f(X) - 1)$.

Clearly, $\mathbf{first}(X \downarrow \mathbf{l}) = \mathbf{first}(X) = \mathbf{d}_X = \mathbf{d}_{X \downarrow \mathbf{l}}$ and $\mathbf{last}(X \downarrow \mathbf{l}) = \mathbf{l} = \mathbf{e}_{X \downarrow \mathbf{l}}$. By Point (10), $X \downarrow \mathbf{l}$ admits an equivalent $2e$-path-decomposition in $W \in \mathcal{W}_{\mathcal{M}}$ with $\underline{l}(W) = 3$. Wihtout pert of generality, we can suppose that the domains of $\mathbf{T}_W$ and V are disjoint.

Then, $V \odot W$ belongs to $\mathcal{W}_{\mathcal{W}}$ (Point (7)), verifies $\underline{l}(V \odot W) = 3 \cdot (f(X))$ (Point (1)) and is equivalent with $\mathbf{val}(V) \odot \mathbf{val}(W) = \mathbf{val}(X)$.

(12) $3 \cdot (1 + k)$, if $\mathbf{last}(X) = \mathbf{e}_X$.
Symmetrical with the precedent proof.

(13) $6 \cdot (k + 1)$.
Let f be an arc of X such that $\|\overline{\mathbf{g}}_X(f)!L\| \leq \|X!L\|$. Clearly, we have: $\mathbf{first}(X \uparrow f) = f = \mathbf{d}_{X \uparrow f}$ and $\mathbf{last}(X \downarrow f) = f = \mathbf{e}_{X \downarrow f}$. By Point (13) (resp. Point (12)), $X \uparrow f$ (resp. $X \downarrow f$) is equivalent with an internally

connected $2e$-path-decomposition V (resp. W) in $\mathcal{W}_{\mathcal{M}}$ such that: $\underline{l}(V)$ (resp. $\underline{l}(W)$) is at most $3 \cdot (1 + k)$. Rather to take isomorphic copies, we can suppose $V \odot W$ defined. $V \odot W$ belongs, by Point (7), to $\mathcal{W}_{\mathcal{M}}$, denotes $\mathbf{val}(V) \odot \mathbf{val}(W) = \mathbf{val}(X)$ and verifies, by Point (1): $\underline{l}(V \odot W) \leq 6 \cdot (k + 1)$.

(14) Every $X \in \mathcal{W}_{k+1} \cap \mathcal{W}_{i.c} \cap \mathcal{W}^{\mathbf{c}}_{1.c}$ admits an equivalent $2e$-path-decomposition in $\mathcal{W}_{i.c} \cap \mathcal{W}_{k+1} \cap +^{6 \cdot k}(\mathcal{W}_{i.c} \cap \mathcal{W}_k)$.
Let $X \in \mathcal{W}_{k+1} \cap \mathcal{W}_{i.c} \cap \mathcal{W}^{\mathbf{c}}_{1.c}$. By Fact 54, either X contains no vertex and exactly two edges and belongs to $\mathcal{W}_0 \cap \mathcal{W}_{i.c}$, or $\mathbf{G}_X$ admits a circular-decomposition of the form $((\mathbf{e}_X, \mathbf{d}_X), S)$ for some sequence S. X can be supposed equal to Ω. Points (8) and (13) suffice to conclude. $\square$

Proof of Theorem 60.
Let $k \geq 0$. As a consequence of Lemma 79, the set $\mathcal{L} \cap \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k+1}$ is equivalent with some subset of $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k+1} \cap +^{6 \cdot k}(\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k)$. The set $\mathbf{atom}(\mathcal{T}_{i.c} \cap \mathcal{T}_{k+1})$ is obviously contained in $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k+1} \cap +(\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k)$. Obviously, for all sets $u, v \subseteq \mathcal{T}$, the set $+(u)[+(v)]$ is contained in $+^2(u[v])$. Then, to conclude it suffices to prove that $\mathcal{L} \cap \mathcal{T}^{\mathbf{c}}_{i.c} \cap \mathcal{T}_{k+1}$ is equivalent with $u[v[w]]$ with $u = \mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k+1} \cap (+(\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k))$, $v = \mathcal{L} \cap \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k+1})$ and $w = \mathbf{atom}(\mathcal{T}_{i.c} \cap \mathcal{T}_{k+1})$.

Let $X \in \mathcal{L} \cap \mathcal{T}^{\mathbf{c}}_{i.c} \cap \mathcal{T}_{k+1}$. Denote by $\mathbf{l}_X$ the unique leaf of X. Denote by R the set of sources of X. Let suppose $R = \emptyset$. Two cases appear:

- X is atomic.

  Trivially, $X \in \mathbf{atom}(\mathcal{T}_{i.c} \cap \mathcal{T}_{k+1})$.

- X is not atomic.

  Let H be the e-hypergraph denoted by $X|(\mathbf{N}_X \backslash \mathbf{l}_X)$. Let d be the unique arc of X incident with $\mathbf{l}_X$. By hypothesis, there is no internally connected component of $H \backslash d$ that contains $R = \emptyset$. By hypothesis, $H \backslash d$ has no internally connected component. Then, $H \backslash d$ has no vertex and no nonsource edge. X is equivalent with an e-tree-decomposition of $\mathbf{atom}(\mathcal{T}_{i.c} \cap \mathcal{T}_{-1})$.

Moreover, we suppose $R \neq \emptyset$. Let P be the set of all nodes t of X such that $\mathbf{g}_X(t)$ contains at least a vertex of R. From Definition 21, P contains $\mathbf{r}_X$ and there is a vertex $s \in R$ that belongs to $R \cap \bigcap_{t \in P} \mathbf{g}_X(t)$.

The e-tree-decomposition $X_0 = X|P$ verifies $X_0\backslash\backslash\{s\} \in \mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k$ and, then, belongs to $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k+1} \cap +(\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_k)$. The e-tree-decomposition $X_2 = X|\mathbf{l}_X$ belongs to $\mathbf{atom}(\mathcal{T}_{i.c} \cap \mathcal{T}_{k+1})$. Then, if $\mathbf{N}_X = P \cup \{\mathbf{l}_X\}$, X is equal to X_0, X_2 or $X_0[X_2]$, the conclusion is immediate. Moreover, we suppose $\mathbf{N}_X - (P \cup \{\mathbf{l}_X\}) \neq \emptyset$. Clearly, the e-tree-decomposition $Y = X|(\mathbf{N}_X - (P \cup \{\mathbf{l}_X\}))$ belongs to $\mathcal{L} \cap \mathcal{T}_{i.c} \cap \mathcal{T}_{k+1}$ and verifies $X = X_0[Y[X_2]]$. Then, to conclude it suffices to prove: $Y \in \mathcal{T}^{\mathbf{c}}_{1.c}$.

Let d the unique arc of X incident with $\mathbf{l}_X$. Suppose there is a connected subhypergraph J of $\mathbf{val}(Y)\backslash d$ that contains every source of Y. Then, $(\mathbf{e}_Y, (\mathbf{G}_Y \upharpoonright \mathbf{e}_Y) \cup J)$ is connected. By Fact 12, $\mathbf{val}(X_0)$ and $\mathbf{val}(X_0\backslash\backslash R)$ are connected, then $(\mathbf{G}_{X_0}\backslash\mathbf{e}_{X_0}) \cup J$ and $(\mathbf{G}_{X_0}\backslash\mathbf{e}_{X_0})\backslash\backslash R \cup J = ((\mathbf{G}_{X_0}\backslash\mathbf{e}_{X_0}) \cup J)\backslash\backslash R$ are connected (by construction, $\mathbf{V}_Y \cap R = \emptyset$). By Fact 12, $(\mathbf{G}_{X_0}\backslash\mathbf{e}_{X_0}) \cup J$ is an internally connected subhypergraph of $\mathbf{val}(X \uparrow d)\backslash d$ that contains R. Contradiction. Then, every connected subhypergraph of $\mathbf{val}(Y)\backslash d$ does not contain every source of Y. Then, d is 1-critical in $\mathbf{val}(Y)$. By Fact 52, every arc d of Y is 1-critical in $\mathbf{val}(Y \uparrow d)$. Thus, $Y \in \mathcal{T}^{\mathbf{c}}_{1.c}$.

Appendix b

This appendix is devoted to the proof of Theorem 61. In this section, we use the three notions of connectivity and the three notions of criticalities defined for e-tree-decompositions. The decomposition of the critical case is made in four steps related respectively to the 2-critical case, 2-edge-connected and 1-critical case, 1-critical case and critical case. The three first steps are the objects of respectively Lemmas 80, 82 and 83. The fourth step is "made" in the proof of Theorem 61.

Lemma 80 *For each k, $\mathcal{T}^{\mathbf{c}}_{2.c} \cap \mathcal{T}_k$ is equivalent with a subset of $\mathcal{L}_{1+k} \cap \mathcal{T}_k$.*

Proof.
In this proof, k denotes a fixed integer, Ω denotes a fixed e-tree-decomposition of $\mathcal{T}^{\mathbf{c}}_{2.c} \cap \mathcal{T}_k$, $(\mathcal{R}, \mathcal{M})$ denotes a fixed circular-decomposition of Ω and $\mathcal{A}$ the union $\mathbf{A}_X \cup \{d \in \mathcal{R}\}$. We suppose that for every leaf t of Ω, the hypergraph $\mathbf{g}_\Omega(t)\backslash\mathbf{e}_\Omega$ contains at least two edges of $\mathcal{A}$. $\mathcal{T}_\Omega$ denotes the set of all e-tree-decompositions $X \sqsubseteq \Omega$ such that each of its leaves t verifies: $\mathbf{card}(\mathcal{A} \cap \mathbf{E}_{\mathbf{g}_X(t)\backslash\mathbf{e}_X}) \geq 2$. As a consequence of the equality $\mathbf{G}_\Omega\backslash\mathcal{A} = \bigcup_{L \in \mathcal{M}} L$ and of Fact 56, every $X \sqsubseteq \Omega$ admits a circular-decomposition of the form $(R, \mathcal{M} \sqcap \mathbf{val}(X))$. As a consequence of Definition 53, a such sequence R is unique. We denote it by $\mathcal{A} \sqcap X$. The

operation of substitution [ ] is extended to sequences as follow: for all nonempty and elementary sequences $u = (u_1, \ldots, u_l)$ and $v = (v_1, \ldots, v_m)$ such that $v_1 = v_i$ for some $i \in [2, l]$, we denote by $u[v]$ the sequence $(u_1, \ldots, u_{i-1}, v_2, \ldots, v_m, u_{i+1}, \ldots, u_l)$. We have:

(1) $\mathcal{A} \sqcap (Y[Z]) = (\mathcal{A} \sqcap Y)[\mathcal{A} \sqcap Z]$ for all $Y \sqsubseteq \Omega, Z \sqsubseteq \Omega$ with $Y[Z]$ defined. Let $X \sqsubseteq \Omega$ of the form $Y[Z]$ for some $Y, Z \sqsubseteq \Omega$. Let $(d_0, \ldots, d_{n-1}) = \mathcal{A} \sqcap Y$, $(e_0, \ldots, e_{p-1}) = \mathcal{A} \sqcap Z$ and $(c_0, \ldots, c_{l-1}) = (\mathcal{A} \sqcap Y)[\mathcal{A} \sqcap Z]$. Let i be the unique integer such that $d_i = e_0$. The intersections $\mathbf{G}_\Omega \cap \mathbf{G}_Y$, $\mathbf{G}_\Omega \cap \mathbf{G}_Z$, $\mathbf{G}_\Omega \cap (\mathbf{G}_Y \cup \mathbf{G}_K)$ are defined, then $\mathcal{M} \sqcap \mathbf{val}(X)$ is defined.

Let $(H_0, \ldots, H_{n-1}) = \mathcal{M} \sqcap \mathbf{val}(Y)$ and $(K_0, \ldots, K_{n-1}) = \mathcal{M} \sqcap \mathbf{val}(Z)$. By definition, $\mathbf{G}_Y \upharpoonright d_i = \mathbf{G}_Z \upharpoonright e_0$ is not disjoint with K_0, K_{n-1}, H_{i-1}, H_i and is disjoint with every hypergraph of the form H_j with $j \in [0, i-1[\cup]i, n-1]$ or of the form K_j with $j \in]0, p-1[$. From $\mathbf{G}_Y \cap \mathbf{G}_Z = \mathbf{G}_Y \upharpoonright d_i = \mathbf{G}_Z \upharpoonright e_0$, it follows that $\mathcal{M} \sqcap \mathbf{G}_X$ is the concatenation of $(H_0, \ldots, H_{i-2}, H_{i-1} \cup K_0)$, $(K_1, \ldots, K_{n-2})$ and $(K_{n-1} \cup H_i, H_{i+1}, \ldots, H_{n-1})$. Thus, $\mathbf{val}(X)$ admits as circular-decomposition $((c_0, \ldots, c_{l-1}), \mathcal{M} \sqcap \mathbf{val}(X))$. Every leaf t of X is a leaf of Z or a leaf of Z distinct with the node u of Z such that $\mathbf{e}_Z \in \mathbf{E}_{\mathbf{g}_Y(u)}$ and, then, admits an edge of $\{c_0, \ldots, c_{l-1}\}$ that belongs to $\mathbf{g}_X(t) \backslash \mathbf{e}_X$. Then, X admits as circular-decomposition $(\mathcal{A} \sqcap Y)[\mathcal{A} \sqcap Z], \mathcal{M} \sqcap \mathbf{val}(X))$.

We denote by:

- $\mathcal{B}$ the union $\bigcup_{t \in \mathbf{N}_\Omega} \{(d, t) \mid d \in \mathcal{A} \cap \mathbf{E}_{\Omega|t}\}$.
- $\mathcal{C}$ the product $\mathcal{B} \times \{0, 1\}$.
- $\underline{node}$ (resp. $\underline{arc}$) the mapping that associates to every $b \in \mathcal{B} \cup \mathcal{C}$ the unique node of X (resp. edge or arc of $\mathcal{A}$) contained in b.
- φ the mapping that associates with every
  - $c \in \mathcal{B} \times \{1\}$ the hypergraph $\mathbf{G}_{\Omega|\underline{node}(c)} \upharpoonright \underline{arc}(c)$.
  - $c \in \mathcal{B} \times \{0\}$ the first hypergraph of the sequence $\mathcal{M} \sqcap \mathbf{val}(\Omega|\underline{node}(c))$ not disjoint with $\mathbf{G}_{\Omega|\underline{node}(c)} \upharpoonright \underline{arc}(c)$ (the existence of such hypergraph is the consequence of the fact that every vertex of Ω belongs to some hypergraph of $\mathcal{M}$ and the fact that every edge of some 2-edge-connected hypergraph is incident with at least one vertex).
- $\prec$ the transitive closure of the union $\bigcup_{x \in \mathbf{A}_\Omega \cup \mathbf{N}_\Omega} \prec_x$, where for every $d \in \mathbf{A}_\Omega$ (resp. $t \in \mathbf{N}_\Omega$) the term $\prec_d$ (resp. $\prec_t$) designs respectively:

– the partial order $\{((d,u,0),(d,v,1)),((d,v,0),(d,u,1))\}$, with $\{u,v\} = \mathbf{vert}_{\mathbf{T}_\Omega}(d)$.
– the total order on $\mathcal{C} \cap \underline{node}^{-1}(t)$ defined with the sequence $((d_1,t,1),(d_2,t,0),\ldots,(d_l,t,1),(d_1,t,0))$ with $(d_1,\ldots,d_l) = \mathcal{A} \sqcap (\Omega|t)$.

For every subset $L \subseteq \mathcal{C}$, $\varphi(L)$ denotes $\bigcup_{c\in L}\varphi(c)$. We have:

(2) $\prec$ is a total order.
Direct consequence of Point (1), we prove by a simple recurrence on the number of nodes that for every $X \in \{\Omega\} \cup \{\Omega\!\downarrow\! d \mid d \in \mathbf{A}_\Omega\}$, the restriction of $\prec$ on $\mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_X)$ is a total order.

Then, we denote by σ the permutation over $\mathcal{C}$ that associates with every $c \in \mathcal{C}$ the element $\min_\prec(\mathcal{C})$ if $c = \max_\prec(\mathcal{C})$ and $\min_\prec\{d \in \mathcal{C} \mid c \prec d\}$, otherwise. For all distinct $a, b \in \mathcal{C}$, we denote by $]a,b]$ the set $\{a \prec c \preceq b\}$ if $a \prec b$ and $\mathcal{C}-]b,a]$, otherwise. We denote by $\overline{\varphi}$ the mapping that associates with every $c \in \mathcal{C}$ the intersection $\varphi(c) \cap \varphi(\sigma(c))$. We have:

(3) $(\mathbf{G}_{\Omega|t} \restriction d)\backslash d$ is the disjoint union of the nonempty hypergraphs $\overline{\varphi}(d,t,0)$ and $\overline{\varphi}(d,t,1)$, for every $(d,t) \in \mathcal{B}$.
Let $(d,t) \in \mathcal{B}$. Direct consequence of the fact that $(\mathcal{A} \sqcap (\Omega|t), \mathcal{M} \sqcap \mathbf{val}(\Omega|t), S)$ is a circular-decomposition of $\Omega|t$ and $d \in \mathcal{A}$.

(4) $\overline{\varphi}(a) = \overline{\varphi}(\sigma(a))$, for every $a \in \mathcal{C}$ of the form $(d,t,0)$ for some $t \in \mathbf{N}_\Omega$ and some $d \in \mathbf{A}_\Omega$.
Let $a = (d,t,0) \in \mathcal{C}$ for some $d \in \mathbf{A}_X$. Let u be the extremity of d in $\mathbf{T}_X$ distinct with t and b the element $(d,u,0) \in \mathcal{C}$. By definition, $\sigma(a) = (d,u,1)$ and $\sigma(b) = (d,t,1)$. Let us prove $\overline{\varphi}(a) = \overline{\varphi}(\sigma(a))$ and $\overline{\varphi}(b) = \overline{\varphi}(\sigma(b))$. Without pert of generality, we can suppose t the parent of u in Ω.

By Point (1), $\Omega|\{t,u\}$ admits as circular-decomposition the couple $(\mathcal{A} \sqcap \Omega|t[\mathcal{A} \sqcap \Omega|u], \mathcal{M} \sqcap \mathbf{val}(\Omega|\{t,u\}))$. It follows that $\varphi(a)$ (resp. $\varphi(\sigma^2(a))$) is disjoint with $\varphi(b)$ (resp. $\varphi(\sigma^2(a))$). Then, $\overline{\varphi}(a)$ (resp. $\overline{\varphi}(\sigma(a))$) is disjoint with $\overline{\varphi}(b)$ (resp. $\overline{\varphi}(\sigma(b))$). By Point (3), $\overline{\varphi}(a)$ and $\overline{\varphi}(\sigma(b))$ are disjoint, $\overline{\varphi}(b)$ and $\overline{\varphi}(\sigma(a))$ are disjoint and verifies: $\overline{\varphi}(a) \cup \overline{\varphi}(\sigma(b)) = \overline{\varphi}(b) \cup \overline{\varphi}(\sigma(a))$. It follows: $\overline{\varphi}(a) = \overline{\varphi}(\sigma(a))$ and $\overline{\varphi}(b) = \overline{\varphi}(\sigma(b))$.

For every $X \sqsubseteq \Omega$, we denote by:

- $\|X\|$ the integer $\min\{\mathbf{card}(\mathbf{V}_{\overline{\varphi}(c)}) \mid c \in \mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_X)\}$.

- $\mathbf{first}(X)$ the element $\min_{\prec}\{b \in \mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_X) \mid \mathbf{card}(\mathbf{V}_{\overline{\varphi}(c)}) = \|X\|\}$.
- $\mathbf{last}(X)$ the element $\max_{\prec}\{b \in \mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_X) \mid \mathbf{card}(\mathbf{V}_{\overline{\varphi}(c)}) = \|X\|\}$.
- $\mathbf{P}(X)$ the minimal subtree of $\mathbf{T}_\Omega$ that contains $\mathbf{r}_X$, $\underline{node}(\mathbf{first}(X))$ and $\underline{node}(\mathbf{last}(X))$.
- $\mathbf{princ}(X)$ the e-tree-decomposition generated by Ω and by the minimal subtree of $\mathbf{T}_\Omega$ that contains $\mathbf{r}_X$ and $\mathbf{N}_X \cap \underline{node}(\{\mathbf{first}(X) \preceq b \preceq \mathbf{last}(X)\})$.
- $\mathbf{Resid}(X)$ the set of all e-tree-decompositions generated by X and some subtree of $\mathbf{T}_X \backslash \mathbf{N}_{\mathbf{princ}(X)}$.

For every $X \sqsubseteq \Omega$, we have:

(5) $\|X\| < \|Z\|$ for every $Z \in \mathbf{Resid}(X)$.
Every $c \in \mathcal{C}$ with $\underline{node}(c) \in \mathbf{N}_X - \mathbf{N}_{\mathbf{princ}(X)})$ verifies: $\|X\| < \mathbf{card}(\mathbf{V}_{\overline{\varphi}(c)})$. That suffices to conclude.

For every $X \sqsubseteq \Omega$, we denote by:

- $\mathcal{C}_X$ the set $(\mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_X)) - (\mathbf{A}_X \times \mathbf{N}_X \times \{1\})$.
- σ_X the permutation that associates with every $c \in \mathcal{C}_X$ the element $\sigma^i(c)$ with $i = \min\{1 \leq j \mid \sigma^j(c) \in \mathcal{C}_X\}$.

For every $X \sqsubseteq \Omega$, we have:

(6) $\mathbf{E}_{\varphi(a)} \cap \mathbf{E}_{\varphi(b)} = \emptyset$, for all distinct $a, b \in \mathcal{C}_X$.
Let $a \neq b$ be two elements of $\mathcal{C}_X$. Let $t = \underline{node}(a)$ and $u = \underline{node}(b)$. If $t = u$, by definition of a circular-decomposition, $\mathbf{E}_{\varphi(a)} \cap \mathbf{E}_{\varphi(b)}$ is empty. Otherwise, we have: $\varphi(a) \subseteq \mathbf{g}_X(t)$, $\varphi(b) \subseteq \mathbf{g}_X(u))$ and $\mathbf{E}_{\mathbf{g}_X(t)} \cap \mathbf{E}_{\mathbf{g}_X(u)} = \emptyset$.

(7) $\mathbf{g}_X(t) = \varphi(\mathcal{C}_X \cap \underline{node}^{-1}(t))$, for every node t of X.
If X is of the form $\Omega|t$ for some node t of Ω, we have $\mathbf{G}_X = \mathbf{g}_X(t)$ and $\mathcal{C}_X = \mathcal{C} \cap \underline{node}^{-1}(t)$. $(\mathcal{A} \sqcap X, \mathcal{M} \sqcap \mathbf{val}(X))$ is a circular-decomposition of $\mathbf{val}(X)$, $\{d \in \mathcal{A} \sqcap X$ is equal to $\{d \mid (d,t,i) \in \mathcal{C}, i \in \{0,1\}\}$. Then, $\mathbf{G}_X$ is equal to $(\mathbf{G}_X \upharpoonright \mathcal{A}) \cup (\mathbf{G}_X \backslash \mathcal{A}))$ and, then, to $\varphi(\{(d,t,0) \mid d \in \mathcal{A} \cap \mathbf{E}_X\}) \cup \varphi(\{(d,t,1) \mid d \in \mathcal{A} \cap \mathbf{E}_X\})$.

Suppose there $n \geq 1$ such that every $X \sqsubseteq \Omega$ with $\mathbf{card}(\mathbf{N}_X) \leq n$ verifies the property above described. Let $X \sqsubseteq \Omega$ with $\mathbf{card}(\mathbf{N}_X) = n+1$, d be an arc of X and t be a node of X. The cases $t \in \mathbf{N}_{X \upharpoonright d}$

and $t \in \mathbf{N}_{X\Downarrow d}$ are symmetrical, and, then, we treat the second one. Suppose $t \in \mathbf{N}_{X\Downarrow d}$. Let $a = (d, \mathbf{r}_{X\Downarrow d}, 1)$. The set $\mathcal{C}_X \cap \underline{node}^{-1}(t)$ is equal to $(\mathcal{C}_{X\Downarrow d} \cap \underline{node}^{-1}(t))\backslash a$. As a consequence of Point (3) and by recurrence's hypothesis, $\varphi(\mathcal{C}_X \cap \underline{node}^{-1}(t))$ contains $\mathbf{g}_{X\Uparrow d}(t)\backslash d$ and, then contains $\mathbf{g}_X(t)\backslash d$. The unique $b \in \mathcal{C}_{X\Downarrow d} \cap \underline{node}^{-1}(t)$ such that d belongs to $\varphi(b)$ is a, it follows $\varphi(\mathcal{C}_X \cap \underline{node}^{-1}(t)) = \mathbf{g}_X(t)$.

(8) $\varphi(]b,a] \cap \mathcal{C}_X) \cap \varphi(]a,b] \cap \mathcal{C}_X) = \overline{\varphi}(a) \cup \overline{\varphi}(b)$, for all distinct $a, b \in \underline{node}^{-1}(\mathbf{N}_X)$.
The proof in the case where X is atomic is a direct consequence of the fact that $(\mathcal{A} \sqcap X, \mathcal{M} \sqcap \mathbf{val}(X))$ is a circular-decomposition of X. The extension to general case is made by recurrence by using similar arguments than in the proof of Point (7).

(9) $\mathbf{card}(\mathbf{V}_{\varphi(c)}) + \|X\| \leq k+1$, for every $c \in \mathcal{C}_X$ if $X \in \mathcal{T}_\Omega$.
Suppose $X \in \mathcal{T}_\Omega$. Let $c \in \mathcal{C}_X$ and $t = \underline{node}(c)$. As a consequence of $\mathbf{card}(\mathbf{V}_{X|t}) \leq k+1$ and $\|X\| \leq \mathbf{card}(\mathbf{V}_{\overline{\varphi}(b)})$ for every $b \in \mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_X)$, to conclude it suffices to prove there is some $b \in \mathcal{C}$ such that $\underline{node}(b) = t$ and $\overline{\varphi}(b) \cap \varphi(c) = \emptyset$. c is of the form (d, t, i) for some $d \in \mathcal{A}$ and some $i \in \{0, 1\}$. The inclusion $X|t \sqsubseteq \Omega$ implies that $\mathbf{E}_{X|t} \cap \mathcal{A}$ contains an edge $e \neq d$. Three cases appear:

- $i = 0$.
  $\varphi(c)$ and $\varphi(t, e, 0)$ are two disjoint subhypergraphs of $\mathbf{G}_{X|t}$. Then, $\varphi(c)$ and $\overline{\varphi}(t, e, 0)$ are two disjoint subhypergraphs of $\mathbf{G}_{X|t}$.
- $i = 1$ and t is a leaf of X.
  $X|t$ contains an edge $f \in \mathcal{A}\backslash\{d, e\}$. Clearly, $\varphi(t, d, 1)$ is disjoint with $\overline{\varphi}(t, e, 1)$ (and $\overline{\varphi}(t, f, 0)$) or with $\overline{\varphi}(t, f, 1)$ (and $\overline{\varphi}(t, e, 0)$).
- $i = 1$ and t is not a leaf of X.
  Let $f \in \mathbf{A}_X$ incident with t and one of its child. By definition, $(f, t, 1) \notin \mathcal{C}_X$. Then, $d \neq e$ and, $\varphi(d, t, 1)$ is disjoint with $\overline{\varphi}(f, t, 0)$ or with $\overline{\varphi}(f, t, 1)$.

(10) $1 \leq \|X\| \leq \lfloor (k+1)/3 \rfloor$, if $X \in \mathcal{T}_\Omega$.
By Point (3), every $c \in \mathcal{C}$ verifies $1 \leq \mathbf{card}(\mathbf{V}_{\overline{\varphi}(c)})$. Then, $1 \leq \|X\|$. Let t be a leaf of X and $d \in \mathcal{A}$ be an edge of $\mathbf{g}_X(t)\backslash\mathbf{e}_X$. The membership $d \in \mathbf{E}_X$ implies $(t, d, 1), (t, d, 0) \in \mathcal{C}_X$. By Point (3), $\mathbf{V}_{\varphi(t,d,1)}$ contains the two disjoints sets $\mathbf{V}_{\overline{\varphi}(t,d,1)}$ and $\mathbf{V}_{\overline{\varphi}(t,d,0)}$. That implies $2 \cdot \|X\| \leq \mathbf{card}(\mathbf{V}_{\varphi(t,d,1)})$. Point (9) implies $\mathbf{card}(\mathbf{V}_{\varphi(t,d,1)}) \leq k+1-\|X\|$. Then, $3 \cdot \|X\| \leq k+1$.

(11) $\mathbf{princ}(X)$ admits an equivalent e-tree-decomposition in $\mathcal{L}_2 \cap \mathcal{T}_k$, if $X \in \mathcal{T}_\Omega$.
In order to simplify, we consider some $W \in \mathcal{T}_\Omega$, we denote by X the e-tree-decomposition $\mathbf{princ}(W)$ and establish that X is equivalent with some e-tree-decomposition $\mathcal{L}_2 \cap \mathcal{T}_k$. Firstly, let us prove the equality $\mathbf{princ}(X) = X$. The reason for which we require for X to be of the form $\mathbf{princ}(W)$ with $W \in \mathcal{T}_\Omega$ and, not to verify the simpler condition: $X = \mathbf{princ}(X)$ is the fact that $\mathbf{princ}(W)$ does not belong necessarily to $\mathcal{T}_\Omega$ even if $W \in \mathcal{T}_\Omega$. Clearly, $\mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_X) \subseteq \mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_W)$. Thus, $\|W\| \leq \|X\|$. The element $\mathbf{first}(W)$ and $\mathbf{last}(W)$ belong to $\mathcal{C} \cap \underline{node}^{-1}(\mathbf{N}_W)$. It follows $\|W\| \geq \|X\|$. Then, $\|X\|$, $\mathbf{first}(X)$, $\mathbf{last}(X)$ and $\mathbf{P}(X)$ are respectively equal to $\|W\|$, $\mathbf{first}(W)$, $\mathbf{last}(W)$ and $\mathbf{P}(W)$. By construction, we have: $\mathbf{r}_X = \mathbf{r}_W$ and $\mathbf{N}_X \supseteq \mathbf{N}_W \cap \underline{node}(\{\mathbf{first}(W) \preceq b \preceq \mathbf{last}(W)\})$. Then, $\mathbf{princ}(X) = X = \mathbf{princ}(W)$.

If $\mathbf{T}_X = \mathbf{P}(X)$, X belongs to $\mathcal{L}_2$: the conclusion is immediate. Moreover, we suppose $\mathbf{P}(X) \neq \mathbf{T}_X$. The proof consists to define two e-hypergraphs H and K such that $\mathbf{val}(X) = H[K]$ and to construct two e-tree-decompositions V and Z such that $\mathbf{T}_V = \mathbf{P}(X)$, $V \in \mathcal{T}_k$, $H = \mathbf{val}(V)$, $Z \in \mathcal{L} \cap \mathcal{T}_k$ and $K = \mathbf{val}(Z)$.

Let $Up = \mathcal{C}_X \cap]\mathbf{last}(X), \mathbf{first}(X)]$ and $Down = \mathcal{C}_X \cap]\mathbf{first}(X), \mathbf{last}(X)]$. Let f be an element that does not belong to Ω. Let K be the e-hypergraph defined by $\mathbf{e}_K = f$ and where $\mathbf{G}_K$ is obtained from $\varphi(Down))$ by adding the new edge f having for set of extremities $\varphi(Up) \cap \varphi(Down)$. The element $(\mathbf{e}_X, \mathbf{r}_X, 1)$ is the minimal element of $(\mathcal{C}_X, \prec)$ and, then, belongs to Up. By definition, $\varphi((\mathbf{e}_X, \mathbf{r}_X, 1)) = \mathbf{G}_X \upharpoonright \mathbf{e}_X$. That permits to denote by H the e-hypergraph $(\mathbf{e}_X, \varphi(Up) \cup (\mathbf{G}_K \upharpoonright \mathbf{e}_K))$. By construction and by Points (6) and (7), it comes $\mathbf{G}_H \cap \mathbf{G}_K = \mathbf{G}_H \upharpoonright \mathbf{e}_K = \mathbf{G}_K \upharpoonright \mathbf{e}_K$, $\mathbf{G}_X = (\mathbf{G}_H \cup \mathbf{G}_K) \backslash \mathbf{e}_K$ and $\mathbf{E}_H \cap \mathbf{E}_K = \{\mathbf{e}_K\}$. Then, $H[K]$ is defined and is equal to $\mathbf{val}(X)$. By Point (8), it comes $(\mathbf{G}_H \cap \mathbf{G}_K) \backslash \mathbf{e}_K = \overline{\varphi}(\mathbf{first}(X)) \cup \overline{\varphi}(\mathbf{last}(X))$.

Denote by Z the sequence $(\mathbf{e}_K, Q, h)$ where Q is the path naturally induced by $(\{\mathbf{first}(W)\} \cup Down, \prec)$ and where h associates to every node $c \in \mathbf{N}_Q$ the hypergraph:

- $\mathbf{G}_K \upharpoonright \mathbf{e}_K$ if $c = \mathbf{first}(W)$.
- $\varphi(a) \cup \overline{\varphi}(\mathbf{last}(X))$ if $c \neq \mathbf{first}(X)$.

Without difficulty, we prove that Z is an e-tree-decomposition. Z is, by construction, linear and, obviously, denotes K. The equality $\|W\| = \|X\| = \mathbf{card}(\mathbf{V}_{\overline{\varphi}(\mathbf{last}(X))})$ and Point (9) implies $Z \in \mathcal{T}_k$. Then, Z denotes K and belongs to $\mathcal{L} \cap \mathcal{T}_k$.

Let U be the sequence $(\mathbf{e}_X, \mathbf{P}(X), g)$ where g associates with every node t of $\mathbf{P}(X)$ the hypergraph $\mathbf{g}_X(t) \cap (\mathbf{G}_H \backslash \mathbf{e}_K)$. $(\mathbf{G}_X \upharpoonright \mathbf{e}_X) \subseteq \mathbf{G}_H \backslash \mathbf{e}_K \subseteq \mathbf{G}_X$ implies $U \in \mathcal{T}_k$. In the rest of the proof, $\mathbf{f}$ denotes $\underline{node}(\mathbf{first}(X)$ and $\mathbf{l}$ denotes $\underline{node}(\mathbf{last}(X)$.

Let $b \in Up$. Let us prove that $\underline{node}(b)$ belongs to $\mathbf{P}(X)$. Let t be the node $\underline{node}(b)$ and Q the maximal subtree of $\mathbf{T}_X$ that contains t but not its eventual parent in X. Suppose $t \notin \mathbf{N}_{\mathbf{P}(X)}$. By construction, $\mathbf{f}$ (resp. $\mathbf{l}$) do not belong to Q and we have: $b \in]\mathbf{last}(X), \mathbf{first}(X)]$. Then, every $a \in \mathcal{C}$ such that $\underline{node}(a) \in \mathbf{N}_Q$ belongs to $]\mathbf{last}(X), \mathbf{first}(X)]$ and dot not belong to $]\mathbf{first}(X), \mathbf{last}(X)]$. The tree $\mathbf{T}_X \backslash \mathbf{N}_Q$ is a proper subtree of $\mathbf{T}_X$ that contains $\mathbf{r}_X$ and $\mathbf{N}_W \cap \underline{node}(\{\mathbf{first}(W) \preceq b \preceq \mathbf{last}(W)\})$ (we have: $\mathbf{first}(W) = \mathbf{first}(X)$ and $\mathbf{last}(W) = \mathbf{last}(X)$). Contradiction. Then, every node of $\underline{node}(Up)$ belongs to $\mathbf{P}(X)$. It follows $\varphi(Up) \subseteq \mathbf{g}_X(\mathbf{P}(X))$ and $H \backslash \mathbf{e}_K = \mathbf{val}(U)$.

The tree $\mathbf{P}(X)$ is strictly contained in $\mathbf{T}_X$, then there is an arc $d_0 \in \mathbf{A}_X \backslash \mathbf{A}_{\mathbf{P}(X)}$ incident with some node t_0 of $\mathbf{P}(X)$. Let R (resp. S) be the path in $\mathbf{T}_\Omega$ from $\mathbf{f}$ (resp. $\mathbf{l}$) to t_0. Let V be the sequence $(\mathbf{e}_X, \mathbf{P}(X), h)$ where h associates with every node t of $\mathbf{P}(X)$ the hypergraph:

- $\mathbf{g}_U(t) \cup (\mathbf{G}_H \upharpoonright \mathbf{e}_K)$, if $t = t_0$.
- $\mathbf{g}_U(t) \cup \overline{\varphi}(\mathbf{first}(X))$, if t belongs to R and $t \neq t_0$.
- $\mathbf{g}_U(t) \cup \overline{\varphi}(\mathbf{last}(X))$, if t belongs to S and $t \neq t_0$.
- $\mathbf{g}_U(t)$, otherwise.

The hypergraph $\overline{\varphi}(\mathbf{first}(X))$ (resp. $\overline{\varphi}(\mathbf{last}(X))$) is contained in $\mathbf{g}_U(\mathbf{f}$ (resp. $\mathbf{g}_U(\underline{node}(\mathbf{last}(X)))$, t_0 belongs to R and to S. Wihtout difficulty, we prove that V is an e-tree-decomposition. Clearly, V denotes H.

To conclude, it suffices to prove $V \in \mathcal{T}_k$ and, then, to prove that every node t of V verifies $\mathbf{card}(\mathbf{V}_{\mathbf{g}_V(t)}) \leq k+1$. Let $t \in \mathbf{N}_V$. If t

does not belong to $R \cup S$, the membership $U \in \mathcal{T}_k$ and the equality $\mathbf{g}_U(t) = \mathbf{g}_V(t)$ suffice to conclude. If $t \in R \cup S$ and if $\mathbf{f} = \mathbf{l}$, the unique node of $R \cup S$ is $t_0 = \mathbf{f} = \mathbf{l}$, the inclusions $\overline{\varphi}(\mathbf{first}(X)) \subseteq \mathbf{g}_U(\mathbf{f})$ and $\overline{\varphi}(\mathbf{last}(X)) \subseteq \mathbf{g}_U(\mathbf{l})$ imply $\mathbf{g}_V(t_0)\backslash\mathbf{e}_K = \mathbf{g}_U(t_0)$ and, then, permit to conclude. Moreover, we suppose $\mathbf{f} \neq \mathbf{l}$ and $t \in \mathbf{N}_{R\cup S}$. Different cases appear:

- $t = \mathbf{f}$ with $\mathbf{f} \neq t_0$.
  The obvious inclusion $\underline{\varphi}(\mathbf{first}(X)) \subseteq \mathbf{g}_U(t)$ implies $\mathbf{g}_U(t) = \mathbf{g}_V(t)$. That suffices to conclude.
- $t = \mathbf{l}$ with $\mathbf{l} \neq t_0$.
  Symmetrical to precedent case.
- $t \in \mathbf{N}_R - \{t_0, \mathbf{f}, \mathbf{l}\}$.
  Let d and e be the two arcs of $R \cup S$ incident with t with e on the subpath of S from t to $\mathbf{l}$. The set $](d,t,1),(e,t,0)] \cap \mathcal{C}_X$ is contained in *Down* and verifies $\varphi(](d,t,1),(e,t,0)]) \cap \mathbf{g}_X(t) \cap \varphi(Up) = \emptyset$. Then, $\varphi(e,t,0)$ is disjoint with $\mathbf{g}_U(t)$. $\|X\| = \mathbf{card}(\mathbf{V}_{\overline{\varphi}(\mathbf{last}(X))})$ and $\|X\| \leq \mathbf{card}(\mathbf{V}_{\overline{\varphi}(e,t,0)})$ implies $\mathbf{card}(\mathbf{g}_V(t)) \leq k+1$.
- $t \in \mathbf{N}_S - \{t_0, \mathbf{f}, \mathbf{l}\}$.
  Symmetrical with the proof of precedent case.
- $t = t_0$ with $t_0 = \mathbf{f}$.
  $\mathbf{f} \neq \mathbf{t}$ implies there is an unique arc e in $R \cup S$ incident with $\mathbf{f}$. The inequality $\mathbf{first}(X) \prec (d,t,1) \prec (e,t,0) \prec \mathbf{last}(X)$ implies that $\varphi(e,t,0)$ is disjoint with $\mathbf{g}_U(t)$. $\|X\| = \mathbf{card}(\mathbf{V}_{\overline{\varphi}(\mathbf{last}(X))})$ and $\|X\| \leq \mathbf{card}(\mathbf{V}_{\overline{\varphi}(e,t,0)})$ implies $\mathbf{card}(\mathbf{g}_V(t)) \leq k+1$.
- $t = t_0$ and $t_0 = \mathbf{l}$.
  Symmetrical with the proof of precedent case.
- $t = t_0$ and $t_0 \notin \{\mathbf{f}, \mathbf{l}\}$.
  Let e (resp. f) be the unique arc of R (resp. S) incident with t. The inequalities $\mathbf{first}(X) \prec (e,t,1) \prec (d,t,0) \prec (d,t,1) \prec (f,t,0) \prec \mathbf{last}(X)$ imply $\varphi(d,t,0)$, $\varphi(f,t,0)$ and $\mathbf{g}_U(t)$ disjoint. That permits to conclude.

(12) Ω admits an equivalent e-tree-decomposition in $\mathcal{L}_{2\cdot\lfloor(k+1)/3\rfloor} \cap \mathcal{T}_k$.
The inclusion $\mathbf{Resid}(X) \subseteq \mathcal{T}_\Omega$ and Points (5), (10) and (11) permit to prove, by a simple recurrence, that Ω admits an equivalent e-tree-decomposition in $\mathcal{L}_l \cap \mathcal{T}_k$ with $l = 2 \cdot \lfloor(k+1)/3\rfloor$.

(13) $\mathcal{T}^{\mathbf{c}}_{2.c}$ is equivalent with some subset of $\mathcal{L}_{1+2\cdot\lfloor(k+1)/3\rfloor} \cap \mathcal{T}_k$.
Let $X \in \mathcal{T}^{\mathbf{c}}_{2.c}$. If X is atomic, the conclusion is immediate. Moreover,

we suppose X not atomic. Let D be the set of all arcs d of X such that $X \downarrow d \in \mathcal{L}$. Then, D contains every arc of X that is incident with a leaf of X. Let Y be the e-tree-decomposition generated by X and by the maximal subtree of $\mathbf{T}_X \backslash D$. Every leaf of Y is adjacent in $\mathbf{T}_X$ with at least two nodes. Then, Y can be supposed equal to Ω. Point (12) and the fact that every e-tree-decomposition generated by X and some maximal subtree of the forest $\mathbf{T}_X \backslash \mathbf{N}_Y$ is linear suffice to conclude. $\square$

Fact 81 *Let H be an 2-edge-connected e-hypergraph of type some k and D be the set of all 1-critical edges of H. If D is nonempty, it contains a set C of cardinality at most $2 \cdot (k-2)$ such that for every connected component L of $H \backslash C$, there is a circular-decomposition (R, S) of H such that $D \cap \mathbf{E}_L = R \backslash \mathbf{e}_H$.*

ls

Proof.

For every e-hypergraph H, we denote by $||H||$ the cardinality of $V_H \cup \mathbf{E}_H$, $\mathcal{E}_H$ the set of 1-critical edges of H, $\mathcal{D}_H$ the set of circular-decomposition of H and $\mathcal{V}_H$ the set that contains every non-empty hypergraph G such that for every circular-decomposition (R, S) of H there is a hypergraph of the sequence S that contains G. For every e-hypergraph H, every subhypergraph $G \subseteq H$ and every vertex $s \in V_H$, we denote by $G \diamond (H, s)$ the e-hypergraph obtained from $H \backslash \mathbf{E}_G$ by identifying all vertices of H with s. A *good-separator* of an e-hypergraph H is a subset $C \subseteq \mathbf{E}_H \backslash \mathbf{e}_H$ such that for every connected component L of $H \backslash C$ there is a circular-decomposition (R, S) of H such that $\mathbf{E}_L \cap \mathcal{E}_H = \{d \in R\}$. The proof of the next fact is simple and is omitted.

(1) For every 2-edge-connected e-hypergraph H, for every hypergraph $G \in \mathcal{V}_H$ and for every vertex $s \in V_G$, the e-hypergraph $K = H \diamond (G, s)$ is such that:

   - K is 2-edge-connected.
   - $\mathcal{E}_H = \mathcal{E}_K$.
   - every circular-decomposition of H is a circular-decomposition of K.
   - every circular-decomposition of K is a circular-decomposition of H.
   - every good-separator of K is a good-separator of H.

A sequence R, subsequence of some sequence R', is denoted by $R \subseteq R'$. A circular-decomposition (R, S) of H is *maximal* if every circular-decomposition (R', S') with $R \subseteq R'$ verifies: $R = R'$. Before the induction proof, a little fact.

(2) for every maximal circular-decomposition (R, S) of some 2-edge-connected e-hypergraph H, every hypergraph $G \in S$ that contains no source of H belongs to $\mathcal{V}_H$.
Let (R, S) be a maximal circular-decomposition of some 2-edge-connected e-hypergraph H. Denote by $(d_0, \ldots, d_{l-1})$ the sequence R and by $(G_0, \ldots, G_{l-1})$ the sequence S. If $l = 2$, every hypergraph of S contains at least one source of H. The conclusion is trivial. Moreover, we suppose $l \geq 3$. Let L be a hypergraph of the form G_i with $i \in [l-2]$. From the 2-edge-connectivity of $\mathbf{G}_H$, every connected-component of L is not disjoint with $\mathbf{G}_H \restriction d_i$ and not disjoint with $\mathbf{G}_H \restriction d_{i+1}$. It follows $L \cup (\mathbf{G}_H \restriction d_i)$ and $L \cup (\mathbf{G}_H \restriction d_{i+1})$ connected.

Suppose there is a 1-critical edge d of H in L. Denote by L_1 (resp. L_2) the union of the connected-components of $L \backslash d$ not disjoint with $\mathbf{G}_H \restriction d_i$ (resp. $\mathbf{G}_H \restriction d_{i+1}$). The 2-edge-connectivity of $\mathbf{G}_H$, implies $L_1 \neq \emptyset$, $L_2 \neq \emptyset$ and $L \backslash d = L_1 \cup L_2$. If there is a connected component L' of $L \backslash d$ not disjoint with $\mathbf{G}_H \restriction d_i$ and not disjoint with $\mathbf{G}_H \restriction d_{i+1}$, there is a connected component of $H \backslash d$ that contains L' and then that contains every source of H. Contradiction. Then, $L_1 \cap L_2 = \emptyset$ and (R', S') is a circular-decomposition, where $R' = (d_0, \ldots, d_i, d, d_{i+1}, \ldots, d_{l-1})$ and $S' = (G_0, \ldots, G_{i-1}, L_1, L_2, G_{i+1}, \ldots, G_{l-1})$. In contradiction with the maximality of (R, S). Then, every edge of L is not 1-critical in H.

Let (R', S') be a circular-decomposition of H. If $d_i \notin R'$ (resp. $d_{i+1} \notin R'$) the connected hypergraph $\mathbf{G}_H \restriction d_i \cup L$ (resp. $(\mathbf{G}_H \restriction d_i) \cup L$) is a subhypergraph of $\mathbf{G}_H \backslash \{d \in R'\}$ and, then, a subhypergraph of some hypergraph of S'. If $\{d_i, d_{i+1}\} \subseteq R'$, every connected component of L is a subhypergraph of $\mathbf{G}_H \backslash \{d \in R'\}$, is not disjoint with $\mathbf{G}_H \restriction d_i$ and not disjoint with $\mathbf{G}_H \restriction d_{i+1}$ and, then, is a subhypergraph of G_i (see Definition 53). Then, $L \in \mathcal{V}_H$.

Suppose there is n such that every 2-edge-connected e-hypergraph H with $\mathcal{E}_H \neq \emptyset$ and $||H|| \leq n$ admits a good-separator of size at most

$2 \cdot (|\mathbf{vert}_H(\mathbf{e}_H)| - 2)$. The property is trivial for $n = 0$. Suppose $1 \leq n$. Let H be an 2-edge-connected e-hypergraph with $\mathcal{E}_H \neq \emptyset$ and $||H|| = n+1$. Let k be the number of sources of H. Different cases appear:

- there is $G \in \mathcal{V}_H$ with $||G|| \geq 1$.
  Denote by K the e-hypergraph $H \diamond (G, s)$ for some vertex $s \in V_G$. The e-hypergraph K has at most k sources, verifies the induction hypothesis (Point (1)), admits a good-separator of size at most $2 \cdot (k - 2)$, that is a good-separator of H.

  Moreover, we suppose $||G|| = 1$ for every hypergraph $G \in \mathcal{V}_H$. Then, every hypergraph of $\mathcal{V}_H$ contains no edge and contains an unique vertex, $\mathcal{V}_H$ could be identified with V_H and $\mathcal{E}_H$ with $\mathbf{E}_H \backslash \mathbf{e}_H$. Consequence of the 2-edge-connectivity of H, every edge of $\mathbf{E}_H$ has at least 2 extremities.

- every edge $d \in \mathbf{E}_H \backslash \mathbf{e}_H$ verifies $|\mathbf{vert}_H(d)| = 2$.
  Suppose there is an elementary circuit of the form $(s_1, e_1, \ldots, e_m, s_1)$ in $H \backslash \mathbf{e}_H$, it follows $G \backslash \{\mathbf{e}_H, e_1\}$ connected, in contradiction with the 1-criticality of e_1. Then, $G \backslash \mathbf{e}_H$ is tree. Denote by r one of its leaf. Denote by I the set of vertices having at least three incident edges in G. The fact that every leaf of $(G \backslash \mathbf{e}_H, r)$ is a source of H, implies: $|I| \leq |\mathbf{vert}_H(\mathbf{e}_H) \backslash r|$. Denote by C the set of edges incident in H with a vertex of I and one of its child in $(G \backslash \mathbf{e}_H, r)$. It follows: $|C| \leq 2 \cdot |\mathbf{vert}_H(\mathbf{e}_H) \backslash r|$. Every connected component L of $G \backslash C$ is a path. Then, L can be identified to some elementary path of the form $(t_0, d_1, \ldots, d_m, t_{m+1})$. The pair (R_L, S_L) with $R_L = (\mathbf{e}_H, d_1, \ldots, d_m)$ and S_L the sequence $(G_0, G \restriction t_1, \ldots, G \restriction t_m, G_{m+1})$ with G_0 (resp. G_m) the connected component of $G \backslash \{\mathbf{e}_H, d_1, \ldots, d_m\}$ that contains t_0 (resp. t_{m+1}), is a circular decomposition of H that verifies $\mathbf{E}_L = \{d \in R \mid d \neq \mathbf{e}_H\}$.

- there is an edge $d \in \mathbf{E}_H \backslash \mathbf{e}_H$ with $|\mathbf{vert}_H(d)| \geq 3$ such that a connected component of $H \backslash d$ contains no edge and an unique vertex. Denote by $G_1, \ldots, G_m$ the connected-components of $H \backslash d$ with $G_1 \in \mathcal{V}_H$. Denote by s the unide vertex of G_1, by K' the hypergraph $G \backslash\backslash s$, and by K the e-hypergraph $(\mathbf{e}_H, K')$. Clearly, $K' \backslash d$ is connected, s is a source of H incident only with $\mathbf{e}_H$ and d in H.

  Suppose there is $f \in \mathbf{E}_K \backslash \mathbf{e}_K$ with $K' \backslash f$ not connected. Let L be a connected component of $K \backslash f$ that does not contain $\mathbf{e}_H$. Every connected-component of $K\backslash f$ that does not contain d is a subhypergraph of some connected component of $H\backslash f$ and then contains $\mathbf{e}_H$. Then, L contains d and there is a circular-decomposition (R', S') of H with $R' = (\mathbf{e}_H, d, f)$ and with S' of the form $(G_1, L\backslash d, L')$ for some hypergraph L'. Denote by (R'', S'') a maximal circular-decomposition of H that contains (R', S'). The sequence S'' is of the form $(G_1, L'_1, \ldots, L'_m)$ with $L'_1 = (G \restriction d) \cap (G_2 \cup \ldots \cup G_m)$. By Point (2), L'_1 belongs to $\mathcal{V}_H$ and contains one unique vertex. Contradiction. Then, $K'\backslash f$ is connected for every $f \in \mathbf{E}_H$. K is 2-edge-connected (Lemma 7).

Every edge adjacent with d in H and distinct with $\mathbf{e}_H$ is 1-critical in K. Thus, K verifies $|K| = n$, admits a good-separator C of size at most $2 \cdot (k-3)$. Every connected component of $H\backslash(\{d\} \cup C)$, except G_1, is a connected-component of $K\backslash(\{d\} \cup C)$. Every 1-critical edge of H is a 1-critical edge of K. Then, $\{d\} \cup C$ is a good-separator of H of size at most $2 \cdot (k-2)$.

- there is an edge $d \in \mathbf{E}_H\backslash\mathbf{e}_H$ with $|\mathbf{vert}_H(d)| \geq 3$ such that a connected component of $H\backslash d$ contains at most one source of H.
  Let L be a connected component of $H\backslash d$ that contains at most one source of H. H is 2-edge-connected, then L contains one unique source of H, noted s. Let H' be the e-hypergraph $H \diamond (L, s)$. From precedent point, $H \diamond (L, s)$ admits a good-separator C of size at most $2\cdot(k-1)$ that contains d. Without difficulty, we verify that C is a good-separator of H.

- there is an edge $d \in \mathbf{E}_H\backslash\mathbf{e}_H$ with $|\mathbf{vert}_H(d)| \geq 3$ such that every connected component of $H\backslash d$ contains at least two sources of H.
  Denote by $L_1, \ldots, L_m$ the connected components of $H\backslash d$ and for every $i \in [m]$ by s_i one source of H that belongs to L_i, by H_i the e-hypergraph $H \diamond (\bigcup_{j\in[m]\backslash i} L_i, s_i)$ and by k_i the number of sources of H_i. It comes $\bigcup_{i\in[m]} k_i = k+m$ and then $1+2\cdot\bigcup_{i\in[m]}(k_i-2) \leq 2\cdot(k-2)$. Without difficulty, for every $i \in [m]$, the e-hypergraph H_i is 2-edge-connected, verifies $||H_i|| \leq n$. By induction hypothesis, for every $i \in [m]$, H_i admits a good-separator C_i of size at most $2 \cdot (k_i - 2)$. Then, $C_i \cup \{d\}$ is a good-separator of H_i. The set $\{d\} \cup \bigcup_{i\in[m]} C_i$ is a good-separator of H of cardinality at most $2 \cdot (k-1)$. □

Lemma 82 *$\mathcal{T}_{2.c} \cap \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k \subseteq (\mathcal{L}_{\lceil log(2\cdot k-1)\rceil} \cap \mathcal{T}_k)[\mathcal{T}^{\mathbf{c}}_{2.c} \cap \mathcal{T}_k]$, for each k.*

Proof.
Let $X \in \mathcal{T}_k \cap \mathcal{T}_{2.c} \cap \mathcal{T}^{\mathbf{c}}_{1.c}$. Denote by D the set of 1-critical edges of $\mathbf{val}(X)$. If $X \in \mathcal{T}^{\mathbf{c}}_{2.c}$, Lemma 80 suffices to conclude. Moreover, we suppose $X \notin \mathcal{T}^{\mathbf{c}}_{2.c}$. By Fact 81, there is a set $C \subseteq D$ with $\mathbf{card}(C) \leq 2 \cdot (k-1)$ such that for every connected component L of $\mathbf{val}(X)\backslash D$, there is a circular-decomposition (R, S) of $\mathbf{val}(X)$ such that $D \cap \mathbf{E}_L = R\backslash \mathbf{e}_X$. X is not 2-critical, then there is no circular-decomposition (R, S) of $\mathbf{val}(X)$ such that $D \subseteq R$ (see Definition 53). Then, $C \neq \emptyset$. Denote by P the minimal subtree of $\mathbf{T}_X$ that contains $\mathbf{r}_X$ and every node s of X such that $C \cap \mathbf{E}_{\mathbf{g}_X(s)} \neq 0$. By Corollary 42, we have: $X|P \in \mathcal{L}_{\lceil log(2 \cdot k-1)\rceil}$. If $P = \mathbf{T}_X$, the conclusion is immediate. Moreover, we suppose $P \neq \mathbf{T}_X$. Note that $D - C$ is nonempty, because it contains the nonempty set $(D-C) \cap \mathbf{E}_{\mathbf{g}_X(s)}$ for some leaf s of X not in P.

Denote by $\mathcal{D}$ the set of all sets $B \subseteq D - C$ such that $\mathbf{val}(X)$ admits a circular-decomposition (R, S) with $B = R\backslash \mathbf{e}_X$. As a consequence of Definition 53, every nonempty subset of some set of $\mathcal{D}$ belongs to $\mathcal{D}$. Denote by A the set of all arcs a of X such that $D \cap \mathbf{E}_{\mathbf{val}(X\downarrow a)} \in \mathcal{D}$. Clearly, $\mathbf{A}_P \cap A = \emptyset$. Let a be an arc of $\mathbf{A}_X - \mathbf{A}_P$. The hypergraph $L = \mathbf{g}_X(\mathbf{T}_{X\downarrow a})$, contains at least one edge of D, is connected (Lemma 36), does not contain any edge of $\{\mathbf{e}_X\} \cup C$, is a subhypergraph of some connected component of $\mathbf{val}(X)\backslash C$. It follows $D \cap \mathbf{E}_L \in \mathcal{D}$ and then, $a \in A$. Then, $\mathbf{A}_X$ is the disjoint union $\mathbf{A}_P \cup A$.

Let $a \in A$. By Lemma 35: $X \downarrow a \in \mathcal{T}_k \cap \mathcal{T}_{2.c} \cap \mathcal{T}^{\mathbf{c}}_{1.c}$. By definition, $\mathbf{val}(A)$ admits a circular-decomposition (R, S) with $D \cap \mathbf{E}_{\mathbf{val}(X\downarrow a)} = R\backslash \mathbf{e}_X$. By Fact 56, $\mathbf{val}(X \downarrow a)$ admits a circular-decomposition (R', S') such that $D \cap \mathbf{E}_{\mathbf{val}(X\downarrow a)} = R'\backslash a$. Every leaf of $X \downarrow a$ is a leaf of X and contains at least one edge of $D \cap \mathbf{E}_{\mathbf{val}(X\downarrow a)}$. Then, $X \downarrow a$ belongs to $\mathcal{T}^{\mathbf{c}}_{2.c}$.

Clearly, there is a nonempty sequence of arcs $(a_1, \ldots, a_m) \in A^+$ such that $X = (X|P)[X \downarrow a_1, \ldots, X \downarrow a_m]$. The inclusions $\{X|P\} \subseteq \mathcal{L}_{\lceil log(2 \cdot k-1)\rceil} \cap \mathcal{T}_k$ and $\{X \downarrow a \mid a \in A\} \subseteq \mathcal{T}^{\mathbf{c}}_{2.c} \cap \mathcal{T}_k$ suffice to conclude. $\square$

Lemma 83 *$\mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ is equivalent with a subset of $(\mathcal{T}_{2.c} \cap \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k)[\mathcal{L}_2 \cap \mathcal{T}_k]$, for each k.*

Proof.
Let $k \geq 0$. First, note that:

(0) Every $X \in \mathcal{T}^{\mathbf{c}}_{1.c}$ with $\mathbf{G}_X$ not connected belongs to $\mathcal{L} \cap \mathcal{T}_{-1}$.
Let $X \in \mathcal{T}^{\mathbf{c}}_{1.c}$ with $\mathbf{G}_X$ not connected. By definition, $\mathbf{G}_X \backslash \mathbf{e}_X$ is connected. Then, X has no source. By definition, $\mathbf{val}(X)$ contains an edge d such that every connected component of $\mathbf{val}(X) \backslash d$ does not contain every source. Then, $\mathbf{val}(X) \backslash d$ has no connected component, $\mathbf{G}_X$ contains no edge and exactly two edges. It comes $X \in \mathcal{T}_{-1}$ and $X \in \mathcal{L}$ (from $X \in \mathcal{T}_{1.c}$).

In the rest of the proof, we suppose that every $X \in \mathcal{T}^{\mathbf{c}}_{1.c}$ contains at least one source and, then, is such that $\mathbf{G}_X$ is connected. For every $X \in \mathcal{T}$, we denote by $C(X)$ the set of all $c \in \mathbf{E}_X \backslash \mathbf{e}_X$ such that $\mathbf{G}_X \backslash c$ is not connected, by $I(X)$ the connected component of $\mathbf{G}_X \backslash C(X)$ that contains $\mathbf{e}_X$ and by $\overline{C}(X)$ the set of all $c \in C(X)$ incident with some vertex of $I(X)$. A *shred* of some $X \in \mathcal{T}$ w.r.t some edge $c \in C(X)$ is an e-hypergraph K such that:

- $\mathbf{G}_K \backslash \mathbf{e}_K$ is the union of $\mathbf{G}_X \restriction c$ with every connected component of $\mathbf{G}_X \backslash c$ that do not contain $\mathbf{e}_X$.
- the set of sources of K is the set of extremities of c in $\mathbf{G}_X$ that belong to the connected component of $\mathbf{G}_X \backslash c$ that contain $\mathbf{e}_X$.
- $\mathbf{e}_K$ does not belong to X.

For every shred K of some $X \in \mathcal{T}$ w.r.t some $c \in C(X)$, we denote by:

- $t(X, c, K)$ the unique node t of X such that $c \in \mathbf{E}_{\mathbf{g}_X(t)}$.
- $f(X, c, K)$ the sequence $(\mathbf{e}_X, \mathbf{T}_X, g)$ where g associates with every $t \in \mathbf{N}_X$:
  - $\mathbf{g}_X(t) \cap \mathbf{G}_H$, if $t \neq t(X, c, K)$.
  - $(\mathbf{g}_X(t) \cap \mathbf{G}_H) \cup (\mathbf{G}_H \restriction \mathbf{e}_K)$, if $t = t(X, c, K)$.

  with H the context of K in $\mathbf{val}(X)$.
- $g(X, c, K)$ the sequence $(\mathbf{e}_K, P, g)$ where P is the path of $\mathbf{T}_X$ from $t(X, c, K)$ to $\mathbf{r}_X$ and where g associates with every node t of P:
  - $(\mathbf{g}_X(t) \cap \mathbf{G}_K) \cup (\mathbf{G}_K \restriction \mathbf{e}_K)$ if $t = t(X, d, K)$.
  - $\mathbf{g}_X(t) \cap \mathbf{G}_K$, otherwise.

Let K be a shred of some $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ w.r.t some edge $c \in C(X)$. Let H be the context of K in $\mathbf{val}(X)$. We have:

(1) For every 1-critical edge b of $\mathbf{val}(X)$, $\mathbf{val}(f(X,c,K))$ admits as 1-critical edge b if $b \neq c$ and $\mathbf{e}_K$ if $b = c$.
Let b a 1-critical edge of $\mathbf{val}(X)$ and a the edge equal to b if $b \neq c$ and to $\mathbf{e}_K$, otherwise. $\mathbf{G}_X$ and $\mathbf{G}_X \backslash \mathbf{e}_X$ are connected. Then, there are at least two disjoint connected components of $\mathbf{G}_X \backslash \{\mathbf{e}_X, b\}$ not disjoint with $\mathbf{G}_X \upharpoonright \mathbf{e}_X$ and with $\mathbf{G}_X \upharpoonright b$. It follows that b belongs to the connected component of $\mathbf{G}_X \backslash c$ that contains $\mathbf{e}_X$, if $b \neq c$. The edge a belongs to $f(X,c,K)$.

Let J be a connected component of $H \backslash a$. To conclude, it suffices to prove that J does not contain every source of X. If J does not contain $\mathbf{e}_K$, J is a connected subhypergraph of $\mathbf{val}(X) \backslash b$ and, then, does not contain every source of X. If J contains $\mathbf{e}_K$, b is distinct with c, $(J \backslash \mathbf{e}_K) \cup (\mathbf{G}_K \upharpoonright c)$ is connected, is a connected subhypergraph of $\mathbf{val}(X) \backslash b$ and, then, does not contain every source of X.

(2) $g(X,c,K)$ belongs to $\mathcal{L} \cap \mathcal{T}_k$ and denotes K.
Let $u = t(X,c,K)$ and $Z = g(X,c,K)$. First, we prove $\mathbf{G}_K \backslash \mathbf{e}_K \subseteq \mathbf{g}_X(\mathbf{T}_Z)$. Every source of K belongs to $\mathbf{g}_X(u)$, every internal vertex of K is incident with at least one edge ($\mathbf{G}_X$ is connected) that belongs necessarily to $\mathbf{E}_K \backslash \mathbf{e}_K$. Then, to prove $\mathbf{G}_K \backslash \mathbf{e}_K \subseteq \mathbf{g}_X(\mathbf{T}_Z)$, it suffices to prove that every edge of $\mathbf{E}_K \backslash \mathbf{e}_K$ belongs to $\mathbf{g}_X(\mathbf{T}_Z)$. Let $b \in \mathbf{E}_K \backslash \mathbf{e}_K$ and t the unique node of X such that $\mathbf{g}_X(t)$ contains b. Suppose that t does not belong to $\mathbf{T}_Z$. Let Q the maximal subtree of $\mathbf{T}_X$ that contains t but not its parent. By hypothesis, X belongs to $\mathcal{T}^{\mathbf{c}}_{1.c}$. Denote by a a 1-critical edge of $\mathbf{val}(X)$ that belongs to $E_{\mathbf{g}_X(Q)}$. By hypothesis, X is connected, then there is a path q in $\mathbf{g}_X(Q)$ from b to a. By construction, q does not contain c. By Point (1), a belongs to H. Then, q contains necessarily c. Contradiction. Then, t belongs to $\mathbf{T}_Z$.

From $\mathbf{G}_K \backslash \mathbf{e}_K \subseteq \mathbf{g}_X(\mathbf{T}_Z)$ and the fact that $\mathbf{g}_X(t(X,c,K))$ contains every source of K, Z belongs to $\mathcal{T}$, to $\mathcal{T}_k$, to $\mathcal{L}$ ($\mathbf{T}_Z$ is a path) and denotes K.

(3) $f(X,c,K)$ belongs to $\mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ and denotes H.
The inclusion $\mathbf{G}_H \upharpoonright \mathbf{e}_K \subseteq \mathbf{g}_X(t(X,c,K))$ implies $f(X,c,K) \in \mathcal{T}_k$ and $\mathbf{val}(f(X,c,K)) = H$.

Let d be an arc of X. Let prove that $\mathbf{val}(f(X,c,K){\downarrow}d)$ is connected. By Lemma 36, $\mathbf{val}(X{\downarrow}d)$ is connected. If $X \downarrow d$ does not contain $t(X,c,K)$, the hypergraph $\mathbf{G}_{X\downarrow d} \backslash d$ does not contain any edge of K (by Point (2)), does not contain any internal vertex of

K (because it is connected), and then, is equal to $\mathbf{G}_{X\downarrow d}\backslash d \cap \mathbf{G}_H$ by definition equal to $\mathbf{G}_{f(X,c,K)\downarrow d}\backslash d$. In this case, $\mathbf{G}_{f(X,c,K)\downarrow d}\backslash d$ and $\mathbf{val}(f(X,c,K)\downarrow d)$ are connected. If $X \downarrow d$ contains $t(X,c,K)$, $\mathbf{e}_K$ is an edge of $\mathbf{val}(f(X,c,K)\downarrow d)$. Without difficulty, it comes: $\mathbf{val}(X \downarrow d) = \mathbf{val}(f(X,c,K) \downarrow d)[K]$. By Fact 17, $\mathbf{val}(f(X,c,K) \downarrow d)$ is connected. By Lemma 36, $X \in \mathcal{T}_{1.c}$.

To conclude, it suffices to prove $f(X,c,K) \in \mathcal{T}^{\mathbf{c}}_{1.c}$. Let t be a leaf of $f(X,c,K)$. Trivially, t is a leaf of X. Then, there is a 1-critical edge b of $\mathbf{val}(X)$ that belongs to $\mathbf{g}_X(t)$. Let a be the edge b if $b \neq c$ and $\mathbf{e}_K$ otherwise. By Point (1), a is a 1-critical edge of $\mathbf{val}(f(X,c,K))$ and belongs to $\mathbf{g}_{f(X,c,K)}(t)$. Then, $f(X,c,K) \in \mathcal{T}^{\mathbf{c}}_{1.c}$.

(4) $C(f(X,c,K)) \subset C(X)$.
Clearly, $C(X) - C(f(X,c,K))$ contains c. Let b be an edge of $C(f(X,c,K))$. $H\backslash b$ is not connected, verifies $(G\backslash b) = (H\backslash b)[K]$. Fact 17 implies $G\backslash b$ not connected. Then, $C(f(X,c,K)) \subset C(X)$.

(5) $I(f(X,c,K)) \supseteq I(X) \cup (\mathbf{G}_H \restriction \mathbf{e}_K)$ if $c \in \overline{C}(X)$.
Obvious consequence of the definitions of $I(X)$ and of $f(X,c,K)$, we have: $I(X) \cup \{\mathbf{e}_K\} \subseteq \mathbf{G}_{f(X,c,K)}$. $I(X)$ is, by definition, connected and disjoint with K. Then, $I(X)$ is a connected subhypergraph of $H\backslash C(f(X,c,K))$. It comes: $I(X) \subseteq I(f(X,c,K))$.

By construction, $H\backslash \mathbf{e}_K$ is connected. It comes $\mathbf{e}_K \notin C(f(X,c,K))$. Let x be an extremity of c in G that belongs to $I(X)$. The vertex x is incident with $\mathbf{e}_K$ in H and belongs to $I(f(X,c,K))$. Then, $\mathbf{e}_K$ belongs to $I(f(X,c,K))$. We have: $I(X) \cup (H \restriction \mathbf{e}_K) \subseteq I(f(X,c,K))$.

A *good-sequence* is a sequence $\mathcal{Y} = (Y_1, \dots, Y_m)$ for some $m \geq 1$ such that:

- $Y_1 \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$.
- $Y_i \in \mathcal{L} \cap \mathcal{T}_k$ for every $i \in [2,m]$.
- $Y_1[Y_2, \dots, Y_m]$ is defined and verifies $I(Y_1[Y_2, \dots, Y_m]) \subseteq I(Y_1)$, if $m \geq 2$.

$\mathcal{Y}$ is *2-edge-connected* if $\mathbf{val}(Y_1)$ is 2-edge-connected. Its *value*, denoted by $\mathbf{val}(\mathcal{Y})$, is the e-hypergraph $\mathbf{val}(Y_1[Y_2, \dots, Y_m])$ if $m \geq 2$ and $\mathbf{val}(Y)$, otherwise. It comes:

(6) Every $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ admits a 2-edge-connected good-sequence of same value.

$\mathbf{G}_X$ is connected. Thus, by Lemma 10, for every $X \in \mathcal{T}^{\mathbf{c}}_{1.c}$, $\mathbf{val}(X)$ is 2-edge-connected if and only if $\mathbf{card}(C(X)) = 0$.

Suppose there is $n \geq 0$ such that every $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ with $\mathbf{card}(C(X)) \leq n$ admits a 2-edge-connected good-sequence of same value. Let $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ with $\mathbf{card}(C(X)) = n + 1$. Clearly, $\overline{C}(X)$ contains at least one edge c. Let K be a shred of X w.r.t c and let Z be an isomorphic copy of $g(X, c, K)$ such that $\mathbf{T}_X \cup \mathbf{G}_X$ and $\mathbf{T}_Z$ are disjoint. As a consequence of Points (2), (3), (4) and (5), we have $\mathbf{card}(C(f(X, c, K)))) \leq n$ and $(f(X, c, K), Z)$ is a good-sequence of value $\mathbf{val}(X)$. By induction hypothesis, $f(X, c, K)$ admits a 2-edge-connected good-sequence $(Y_1, \ldots, Y_m)$ of same value. Without pert of generality, we can suppose that $\mathbf{T}_{Y_1} \cup \ldots \mathbf{T}_{Y_m}$ (resp. $\mathbf{G}_{Y_1} \cup \ldots \mathbf{G}_{Y_m}$) is disjoint with $\mathbf{T}_Z \cup \mathbf{G}_Z$ (resp. $\mathbf{T}_Z$). Let $\mathcal{Y}$ be the sequence $(Y_1, \ldots, Y_m, Y_{m+1})$ with $Y_{m+1} = Z$. Trivially, we have $Y_1 \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$, $\mathbf{val}(Y_1)$ 2-edge-connected, $Y_i \in \mathcal{L} \cap \mathcal{T}_k$ for every $i \in [2, m+1]$. The inclusions $I(X) \cup (\mathbf{G}_Z \restriction \mathbf{e}_Z) \subseteq I(f(X, c, K)) \subseteq I(Y_1)$ imply $\mathbf{e}_Z \in \mathbf{E}_{Y_1}$. Then, $Y_1[Y_2, \ldots, Y_{m+1}]$ is defined, and has for value $\mathbf{val}(X)$. Then, every $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ admits a 2-edge-connected good-sequence of same value.

For every $X \in \mathcal{T}$, we denote by $\|X\|$ the sum $\sum_{t \in \mathbf{N}_X} \mathbf{card}(\mathbf{V}_{\mathbf{g}_X(t)})$, by $D(X)$ the set of all arcs $d \in \mathbf{A}_X$ such that $\mathbf{G}_{X \Uparrow d} \backslash d$ is not connected, by $\overline{D}(X)$ the set of all arcs $d \in \mathbf{A}_X$ such that $\mathbf{T}_{X \Downarrow d}$ does not contain any arc of $D(X)$ and by $J(X)$ the hypergraph $\bigcup_{d \in \overline{D}(X)} \mathbf{g}_X(\mathbf{T}_{X \Downarrow d})$. A *shred* of some $X \in \mathcal{T}$ w.r.t some arc $d \in \mathbf{A}_X$ is an e-hypergraph K such that:

- $\mathbf{G}_K \backslash \mathbf{e}_K$ is a connected component of $\mathbf{G}_{X \Uparrow d} \backslash d$ that do not contain $\mathbf{e}_X$.
- the set of sources of K is the set of extremities of d in $\mathbf{G}_{X \Uparrow d}$ that belong to the connected component of $\mathbf{G}_{X \Uparrow d} \backslash d$ introduced above.
- $\mathbf{e}_K$ does not belong to X.

For every shred K of some $X \in \mathcal{T}$ w.r.t some arc $d \in D(X)$, we denote by:

- $t(X, d, K)$ the parent of $\mathbf{r}_{X \Downarrow d}$ in X.
- $f(X, d, K)$ the sequence $(\mathbf{e}_X, \mathbf{T}_X, g)$ where g associates with every $t \in \mathbf{N}_X$:
  - $\mathbf{g}_X(t) \backslash \mathbf{V}_K$ if $t \in \mathbf{N}_{X \Uparrow d}$.
  - $\mathbf{g}_X(t) \cup (\mathbf{G}_K \restriction \mathbf{e}_K)$ if $t = \mathbf{r}_{X \Downarrow d}$.

– $\mathbf{g}_X(t)$ otherwise.

- $g(X,d,K)$ the sequence $(\mathbf{e}_K, P, g)$ where P is the path of $\mathbf{T}_X$ from $\mathbf{r}_X$ to $t(X,c,K)$ and where g associates with every node t of P:

  – $(\mathbf{g}_X(t) \cap \mathbf{G}_K) \cup (\mathbf{G}_K \upharpoonright \mathbf{e}_K)$ if $t = t(X,d,K)$.

  – $\mathbf{g}_X(t) \cap \mathbf{G}_K$, otherwise.

Let K be a shred of some $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ w.r.t some arc $d \in D(X)$. Let H be the context of K in $\mathbf{val}(X)$. We have:

(7) every 1-critical edge of $\mathbf{val}(X)$ is 1-critical in $\mathbf{val}(f(X,d,K))$.
Let b be a 1-critical edge of $\mathbf{val}(X)$. $\mathbf{G}_X$ and $\mathbf{G}_X \backslash \mathbf{e}_X$ are connected, then $\mathbf{G}_X \backslash \{\mathbf{e}_K, b\}$ contain two disjoint connected components J, L that are not disjoint with $\mathbf{G}_X \upharpoonright \mathbf{e}_X$ and with $\mathbf{G}_X \upharpoonright b$. Suppose $b \in \mathbf{E}_K$. Then, every path in $\mathbf{G}_X$ from b to $\mathbf{e}_X$ contains at least one source of K. Thus the connected hypergraph J (resp. L) contains at least one source of K. By hypothesis, $\mathbf{G}_{X \downarrow d} \backslash d$ is connected and is a subhypergraph of $\mathbf{G}_X \backslash \{\mathbf{e}_X, b\}$. Then, $J \cup L \cup (\mathbf{G}_{X \downarrow d} \backslash d)$ is a connected subhypergraph of $\mathbf{G}_X \backslash \{\mathbf{e}_X, b\}$. Contradiction. Then, b belongs to $\mathbf{val}(f(X,d,K))$.

Let M be a connected component of $H \backslash b$. To conclude, it suffices to prove that $\mathbf{V}_M$ does not contain every source of H (or every source of X). If M does not contain $\mathbf{e}_K$, M is a connected subhypergraph of $\mathbf{val}(X) \backslash b$ and, then, does not contain every source of X. Clearly, $\mathbf{G}_H$ is connected. If M contains $\mathbf{e}_K$, $(M \cup \mathbf{G}_H) \backslash \mathbf{e}_K$ is a connected subhypergraph of $\mathbf{val}(X) \backslash b$ and, then, does not contain every source of X. That suffices to conclude.

(8) $g(X,d,K)$ belongs to $\mathcal{L} \cap \mathcal{T}_k$ and denotes K.
By construction, $\mathbf{G}_K \backslash \mathbf{e}_K$ is a connected component $\mathbf{g}_X(\mathbf{T}_{X \uparrow d})$ that does not contain $\mathbf{e}_X$ and, by Point (7), that does not contain any 1-critical edge of $\mathbf{val}(X)$. Let b be an edge of $\mathbf{G}_K \backslash \mathbf{e}_K$ and t the node of $X \uparrow d$ such that $\mathbf{g}_X(t)$ contains b. If we suppose $t \notin \mathbf{N}_{g(X,d,K)}$, the maximal subtree Q of $\mathbf{T}_X$ that contains t but not its parent is such that $\mathbf{g}_X(Q)$ is connected ($X \in \mathcal{T}_{1.c}$), contains b and some 1-critical edge of $\mathbf{val}(X)$ and is a subhypergraph of $\mathbf{g}_X(\mathbf{T}_{X \uparrow d})$. Contradiction. Then, every edge of $\mathbf{G}_K \backslash \mathbf{e}_K$ belongs to $\mathbf{g}_X(\mathbf{T}_{g(X,d,K)})$. Clearly, every vertex of $K \backslash \mathbf{e}_K$ is incident with at least one edge. Then, $\mathbf{G}_K \backslash \mathbf{e}_K \subseteq \mathbf{g}_X(\mathbf{T}_{g(X,d,K)})$.

The above inclusion and $\mathbf{G}_K \upharpoonright \mathbf{e}_K \subseteq \mathbf{g}_X(t(X,d,K))$ imply $g(X,d,K) \in \mathcal{T}_k \cap \mathcal{L}$ and $\mathbf{val}(g(X,d,K)) = K$.

(9) $f(X,d,K)$ belongs to $\mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$, denotes H and verifies $\|f(X,d,K)\| < \|X\|$.
The inclusion $\mathbf{G}_H \restriction \mathbf{e}_K \subseteq \mathbf{g}_X(t(X,c,K))$ implies $f(X,c,K) \in \mathcal{T}_k$ and $\mathbf{val}(f(X,c,K)) = H$. Clearly, we have: $\|f(X,d,K)\| < \|X\|$.

The proof of the membership $f(X,d,K) \in \mathcal{T}^{\mathbf{c}}_{1.c}$ is similar with the proof of Point (3). This similarity is due to the similarities of Points (1) and (7) and of Points (2) and (8).

(10) $\mathbf{val}(f(X,d,K))$ is 2-edge-connected if $\mathbf{val}(X)$ is 2-edge-connected.
By Point (9), $\mathbf{val}(f(X,d,K))$ is connected. By Lemma 10, to conclude it suffices to prove that $\mathbf{G}_H \backslash c$ is connected fro every edge $c \in \mathbf{E}_H \backslash \mathbf{e}_H$. Let $b \in \mathbf{E}_H \backslash \mathbf{e}_H$.

If $b = \mathbf{e}_K$, $\mathbf{G}_H \backslash c$ is the union of distinct connected components of $\mathbf{G}_{X \Uparrow d} \backslash d$ with the connected hypergraph $\mathbf{G}_{X \Downarrow d} \backslash d$. All these hypergraphs are contained in $\mathbf{G}_X$. $\mathbf{G}_X$ is connected, then every connected component of $\mathbf{G}_{X \Uparrow d} \backslash d$ is not disjoint with $\mathbf{G}_{X \Downarrow d} \backslash d$. Then, $\mathbf{G}_H \backslash c$ is connected.

If $b \neq \mathbf{e}_K$, b is an edge of $\mathbf{G}_X$ and verifies $(b, \mathbf{G}_X) = (b, \mathbf{G}_H)[K]$. By Fact 17, the connectivity of $(b, \mathbf{G}_X)$ implies $(b, \mathbf{G}_H)$ connected. Then, $\mathbf{G}_H \backslash b$ is connected.

(11) $J(f(X,d,K)) \supseteq J(X) \cup (\mathbf{G}_H \restriction \mathbf{e}_K)$ if $d \in \overline{D}(X)$.
Let $Y = f(X,d,K)$. First, prove $D(Y) \subseteq D(X)$. Let $c \in D(X)$. By definition, $\mathbf{G}_{Y \Uparrow c} \backslash c$ is the disjoint union of two hypergraphs J and L. It comes:

- $c = d$ or c belongs to $\mathbf{T}_{g(X,d,K)}$.
  $\mathbf{T}_{Y \Uparrow c} \subseteq \mathbf{T}_{X \Uparrow d}$ imply $\mathbf{G}_{Y \Uparrow c}$ disjoint with $\mathbf{G}_K$. Then, $\mathbf{G}_{X \Uparrow c} \backslash c$ is the disjoint union of J and $L \cup (\mathbf{G}_K \cap (\mathbf{G}_{X \Uparrow c} \backslash c))$. Then, $c \in D(X)$.
- $c \neq d$ and c does not belong to $\mathbf{T}_{g(X,d,K)}$.
  $t(X,d,K)$ is a node of $Y \uparrow c$. Thus, $\mathbf{G}_H \restriction \mathbf{e}_K \subseteq \mathbf{G}_{Y \Uparrow c} \backslash c$. Without pert of generality, we can suppose $\mathbf{G}_H \restriction \mathbf{e}_K \subseteq L$. J disjoint with L is disjoint with $L \cup \mathbf{G}_K$. Then, $\mathbf{G}_{X \Uparrow c} \backslash c$ is the disjoint union of J and $L \cup (\mathbf{G}_K \cap (\mathbf{G}_{X \Uparrow c} \backslash c))$. Then, $c \in D(X)$.

Suppose $d \in \overline{D}(X)$ (and $d \in D(X)$). Clearly, $D(Y) \subseteq D(X)$ implies that the forest $\bigcup_{c \in \overline{D}(Y)} \mathbf{T}_{Y \Downarrow c}$ contains $\bigcup_{c \in \overline{D}(X)} \mathbf{T}_{X \Downarrow c}$. It follows $J(Y) \supseteq J(X) \cap \mathbf{G}_H$. By construction, $\mathbf{G}_H \restriction \mathbf{e}_K$ is contained in $\mathbf{g}_Y(t(X,d,K))$ and, then, in $J(Y)$. By Point (8), every edge of $\mathbf{E}_K \backslash \mathbf{e}_K$ belongs to $\mathbf{g}_X(\mathbf{T}_{g(X,d,K)})$ and, then, does not belong to $J(X)$. Then, $J(X) = J(X) \cap \mathbf{G}_H$. Thus, $J(Y) \supseteq J(X) \cup (\mathbf{G}_H \restriction \mathbf{e}_K)$.

A *nice-sequence* is a sequence $\mathcal{Y} = (Y_1, \ldots, Y_m)$ for some $m \geq 1$ such that:

- $Y_1 \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ and $\mathbf{val}(Y_1)$ is 2-edge-connected.
- $Y_i \in \mathcal{L} \cap \mathcal{T}_k$ for every $i \in [2, m]$.
- $Y_1[Y_2, \ldots, Y_m]$ is defined and $J(Y_1) \supseteq J(X)$.

$\mathcal{Y}$ is *2-edge-connected* if Y_1 is 2-edge-connected. Its *value*, denoted by $\mathbf{val}(\mathcal{Y})$, is the e-hypergraph $\mathbf{val}(Y_1[Y_2, \ldots, Y_m])$ if $m \geq 2$ and $\mathbf{val}(Y)$, otherwise. It comes:

(12) every $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ with $\mathbf{val}(X)$ 2-edge-connected admits a 2-edge-connected nice-sequence of same value.
Suppose there is $n \geq 0$ such that every $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ with $\mathbf{val}(X)$ 2-edge-connected and $\|X\| < n$ admits a 2-edge-connected nice-sequence of same value. Let $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ with $\mathbf{val}(X)$ 2-edge-connected and $\|X\| = n$. If X is 2-edge-connected, the conclusion is immediate. Moreover, we suppose X not 2-edge-connected. From Lemma 36, $D(X)$ is nonempty. Clearly, $\overline{D}(X)$ is nonempty. Let $d \in \overline{D}(X)$ and Z be an isomorphic copy of $g(X, d, K)$ such that $f(X, d, K)[Z]$ is defined and denotes $\mathbf{val}(X)$ (see Points (8) and (9)). As a consequence of Points (8), (9) (10) and (11), the pair $(f(X, d, K), Z)$ is a nice sequence of value $\mathbf{val}(X)$ with $\|f(X, d, K)\| < n$. By induction, $f(X, d, K)$ admits a 2-edge-connected nice-sequence $(Y_1, \ldots, Y_m)$ of same value. Without pert of generality, we can suppose that $\mathbf{T}_Z \cup \mathbf{G}_Z$ (resp. $\mathbf{T}_Z$) is disjoint with $\mathbf{T}_{Y_1} \cup \ldots \cup \mathbf{T}_{Y_m}$ (resp. $\mathbf{G}_{Y_1} \cup \ldots \cup \mathbf{G}_{Y_m}$). Denote by Y_{m+1} the e-tree-decomposition Z. By Point (11) and by induction, $J(X) \subseteq J(f(X, c, K)) \subseteq J(Y_1)$. By Point (11), $\mathbf{e}_K$ is an edge of $J(f(X, c, K))$ and, then, an edge of Y_1. It follows that $Y_1[Y_2, \ldots, Y_{m+1}]$ is defined and has for value $\mathbf{val}(X)$. Trivially, $Y_1 \in \mathcal{T}_{2.c} \cap \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ and $Y_i \in \mathcal{L} \cap \mathcal{T}_k$ for each $i \in [2, m+1]$. Then, X admits a 2-edge-connected nice-sequence of same value.

(13) $\mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$ is equivalent with some subset of $(\mathcal{T}_{2.c} \cap \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k)[\mathcal{L}_2 \cap \mathcal{T}_k]$.
Let $X \in \mathcal{T}^{\mathbf{c}}_{1.c} \cap \mathcal{T}_k$. By Point (6), X admits a 2-edge-connected good-sequence of same value that is of the form $(Z_1, \ldots, Z_m)$ for some $m \geq 1$. By Point (12), Z_1 admits a 2-edge-connected nice-sequence of same value that is of the form $(Y_1, \ldots, Y_l)$ for some $l \geq 1$. If $m = 1$ or if $l = 1$, the conclusion is immediate. Moreover, we suppose $2 \leq l$ and $2 \leq m$. Without pert of generality, we can suppose that $(Y_1[Y_2, \ldots, Y_l])[Z_2, \ldots, Z_{m-1}]$ is defined.

Then, $(Y_1[Y_2,\ldots,Y_l])[Z_2,\ldots,Z_{m-1}]$ denotes $\mathbf{val}(X)$ and belongs to $(\mathcal{T}_{2.c}\cap\mathcal{T}_{1.c}^{\mathbf{c}}\cap\mathcal{T}_k)[\mathcal{L}_2\cap\mathcal{T}_k]$. $\square$

Fact 84 *Let $H\in\mathcal{G}_{i.c}$ having for type some $k\geq 1$. There is a set C of critical edges of H such that:*

- $\mathbf{card}(C)\leq 2^{k-1}$.
- *for every $d\notin C$ critical in H, there are at least two distinct internally connected component of $H\backslash d$ that contain at least one edge of C.*

Proof.
Let H be an internally connected e-hypergraph of type some $k\geq 1$. If $\mathbf{G}_H=\mathbf{G}_H\restriction\{d,\mathbf{e}_H\}$ for some edge d of H, the conclusion is immediate. Moreover, we suppose that for every non-source edge d of H, $H\backslash d$ contains at least one internally connected component. Denote by G the hypergraph $\mathbf{G}_H$, by S the set of its sources and by D the set of its critical edges. For every $d\in D$, we denote by $\mathcal{I}_d$ the set of internally connected components of $H\backslash d$, by $\mathcal{I}$ the union $\bigcup_{d\in D}\mathcal{I}_d$ and by $\mathcal{I}_0$ the set $\{L\in\bigcup_{d\in D}\mathcal{I}_d\mid\mathbf{card}(\mathbf{E}_L\cap D)=0\}$. Every edge $d\in D$ verifies:

(1) $L\cup(G\restriction d)$ is internally connected in H for every $d\in D$ and every $L\in\mathcal{I}_d$.
Let $d\in D$ be an edge and $L\in\mathcal{I}_d$ be a hypergraph. H is internally connected, then $(\mathbf{V}_L\cap\mathbf{vert}_H(\mathbf{e}_H(d)))\backslash S\neq\emptyset$. The hypergraphs L and $G\restriction d$ are internally connected in H, then $L\cup(G\restriction d)$ is internally connected in H.

(2) $\mathcal{I}_c$ and $\mathcal{I}_d$ are disjoint, for every distinct edges $c,d\in D$.
Let c and d two distinct edges of D. Suppose there is a hypergraph $L\in\mathcal{I}_c\cap\mathcal{I}_d$. The hypergraph $L\cup(G\restriction c)$ is distinct from L, internally connected in H, does not contain d and, then, is contained in some internally connected component of $H\backslash d$. Then, $L\notin\mathcal{I}_d$. Contradiction.

Consequence of Point (2), for every $L\in\mathcal{I}$, we denote by $\overline{L}$ the hypergraph $(G\restriction d)\cup\bigcup_{J\in\mathcal{I}_d\backslash L}$ with d defined by $L\in\mathcal{I}_d$. For every distinct edges $c,d\in D$, we denote by $L_{c,d}$ the unique hypergraph of $\mathcal{I}_c$ that contains d. We have:

(3) $\overline{L_{d,c}}\subseteq L_{c,d}$, for every distinct edges $c,d\in D$.
$\overline{L_{d,c}}$ does not contain c, is internally connected in $H\backslash c$ (Point (1)) and contains d. That suffices to conclude.

For every $d \in D$, we denote by $\mathcal{J}_d$ the set that contains every hypergraph $L \in \mathcal{I}_d$ such that for every $K \in \mathcal{I}_d$, we have: $(G \restriction S) \cap L \not\subset (G \restriction S) \cap K$. We denote by $\mathcal{J}$ the union $\bigcup_{d \in D} \mathcal{J}_d$. For every distinct edges $c, d \in D$, we have:

(4) $2 \leq \mathbf{card}(\mathcal{J}_d)$.
Obvious.

(5) $L_{c,d} \in \mathcal{J}$.
Let c and d two distinct edges of D. Suppose there is $L \in \mathcal{I}_c$ with $L_{c,d} \cap (G \restriction S) \subset L \cap (G \restriction S)$. We have: $L \subseteq L_{d,c}$ (Point (3)) and, then: $L_{c,d} \cap (G \restriction S) \subseteq L_{d,c} \cap (G \restriction S)$. The inclusion $\overline{L_{d,c}} \subseteq L_{c,d}$ (Point (3)) implies $\overline{L_{d,c}} \cap (G \restriction S) \subseteq L_{d,c} \cap (G \restriction S)$ and, then, $L_{d,c} \cap (G \restriction S) = G \restriction S$. The edge d is not critical in H. Contradiction. Then, $L_{c,d} \in \mathcal{J}$.

(6) $L \cap (G \restriction S) \not\subseteq M \cap (G \restriction S)$ for all $L \in \mathcal{J}_c \cap \mathcal{I}_0$, $M \in \mathcal{J}_d \cap \mathcal{I}_0$.
Let $L \in \mathcal{J}_c \cap \mathcal{I}_0$ and $M \in \mathcal{J}_d \cap \mathcal{I}_0$. Suppose $L \cap (G \restriction S) \subset M \cap (G \restriction S)$. The inclusion $M \subseteq \overline{L_{d,c}} \subseteq L_{c,d}$ implies $L \cap (G \restriction S) \subset L_{c,d} \cap (G \restriction S)$. Contradiction. Then $L \cap (G \restriction S) \not\subset M \cap (G \restriction S)$. Suppose $L \cap (G \restriction S) = M \cap (G \restriction S)$. The inclusion $L \cap (G \restriction S) \subseteq L_{c,d} \cap (G \restriction S)$ and the membership $L, L_{c,d} \in \mathcal{J}_c$ implies: $L \cap (G \restriction S) = M \cap (G \restriction S) = \overline{L_{d,c}} \cap (G \restriction S) = L_{c,d} \cap (G \restriction S)$. The inclusion $L \subseteq L_{d,c}$ implies $L \cap (G \restriction S) \subseteq L_{d,c} \cap (G \restriction S)$ and, then, $\overline{L_{d,c}} \cap (G \restriction S) \subseteq L_{d,c} \cap (G \restriction S)$. The equality $\overline{L_{d,c}} \cup L_{d,c} = G \backslash \mathbf{e}_H$ implies $L_{d,c} \cap (G \restriction S) = (G \restriction S)$ and d not critical in H. Contradiction. Then $L \cap (G \restriction S) \not\subseteq M \cap (G \restriction S)$.

Denote by C the set $\{d \in D \mid \mathbf{card}(\mathcal{J}_d - \mathcal{I}_0) \leq 1\}$. By Point (4), for every $c \in C$, the set $\mathcal{J}_c$ contains at least two hypergraphs and, then, at least one in $\mathcal{I}_0$. The maximal number of subsets $S_1, \ldots, S_n$ of S such that $S_i \not\subseteq S_j$ for all $i \neq j \in [n]$ is at most 2^{k-1}. By Point (6), it comes $\mathbf{card}(C) \leq 2^{k-1}$. To conclude, it suffices to prove that every hypergraph of $\mathcal{J} - \mathcal{I}_0$ contains at least one edge of C.

Suppose there is $n \geq 0$ such that every hypergraph of $\mathcal{J} - \mathcal{I}_0$ that contains at most n edges of D contains at least one edge of C. Property trivially true, for $n = 0$. Let $L \in \mathcal{J} - \mathcal{I}_0$ be a hypergraph with $\mathbf{card}(D \cap \mathbf{E}_L) = n + 1$. Let d be the unique edge defined by $L \in \mathcal{I}_d$. Let $c \in D \cap \mathbf{E}_L$ be an edge. It comes $L = L_{d,c}$. If $c \in C$, the conclusion is immediate. Moreover, we suppose $c \notin C$. In consequence, there is $M \in \mathcal{J}_c$ distinct from $L_{d,c}$ that does not belong to $\mathcal{I}_0$. The inclusions $M \subseteq L_{d,c}$ and $D \cap \mathbf{E}_M \subseteq (D \cap \mathbf{E}_L) \backslash c$, implies that M verifies the induction hypothesis and contains at least one edge of C. Then $C \cap \mathbf{E}_L \neq \emptyset$. $\square$

To conclude this appendix, let us prove Theorem 61 that states that for every $k \geq 0$, $\mathcal{T}_k \cap \mathcal{T}_{i.c}^{\mathbf{c}}$ is equivalent with some subset of $\mathcal{L}_{2\cdot(1+k)^2} \cap \mathcal{T}_k$.

Proof of Theorem 61.
Let k be an integer. Every e-hypergraph of $\mathbf{val}(\mathcal{T}_0 \cap \mathcal{T}_{i.c})$ contains at most one vertex. Obviously, $\mathcal{T}_0 \cap \mathcal{T}_{i.c}$ is equivalent with some subset of $\mathcal{L} \cap \mathcal{T}_0$. Moreover, we suppose: $k \geq 1$. We have:

(1) every $X \in \mathcal{T}_{i.c} \cap \mathcal{T}_k$ with $X \backslash\backslash R$ 1-critical for some set R, admits an equivalent e-tree-decomposition of $\mathcal{L}_m \cap \mathcal{T}_k$ with $m = 3 \cdot k + 2 \cdot k^2$.
Let R be a set. For every $n \geq 0$, we denote by $f(n)$ the integer $(n+1)\cdot(k+\lceil log(2\cdot k-1)\rceil+3)$ and for every $X \in \mathcal{T}$, by $\|X\|$ the integer $\mathbf{card}(R \cap \mathbf{V}_X)$. We denote by $\mathcal{I}$ the set of all e-tree-decompositions $X \in \mathcal{T}_{i.c} \cap \mathcal{T}_k$ such that $X \backslash\backslash R$ is 1-critical, and for every $n \geq 0$, by $\mathcal{I}_n$ the set $\{X \in \mathcal{I} \mid \|X\| \leq n\}$. Observe that for every $X \in \mathcal{I}$, every vertex of $R \cap \mathbf{V}_X$ is a source of X. Let $X \in \mathcal{I}$ with $\|X\| \geq k$. The e-tree-decomposition $X \backslash\backslash R$ contains at most one source, is, by hypothesis, 1-critical and, then, contains no vertex and exactly two edges. Thus, X is equivalent with some e-tree-decomposition of $\mathbf{atom}(\mathcal{T}_k)$. Then, to conclude it suffices to prove that $\mathcal{I}_{k-1}$ is equivalent with some subset of $\mathcal{L}_{f(k-1)} \cap \mathcal{T}_k$ (without difficulty, it comes: $f(k-1) \leq 3\cdot k+2\cdot k^2$).

Clearly, $\mathcal{I}_0 \subseteq \mathcal{T}_{1.c}^{\mathbf{c}} \cap \mathcal{T}_k$. As a consequence of Lemmas 80, 82 and 83, $\mathcal{I}_0$ is equivalent with a subset of $\mathcal{L}_{f(0)} \cap \mathcal{T}_k$. Suppose there is some $n \geq 1$ such that $\mathcal{I}_{n-1}$ is equivalent with a subset of $\mathcal{L}_{f(n-1)} \cap \mathcal{T}_k$.

Let $X \in \mathcal{I}_n$. Obviously, we have: $R \cap \mathbf{V}_{\mathbf{g}_X(\mathbf{r}_X)} = R \cap \mathbf{V}_X$. Let Y be the e-tree-decomposition generated by X and by the minimal subtree of $\mathbf{T}_X$ that contains every node t of X such that: $R \cap \mathbf{V}_{\mathbf{g}_X(t)} = R \cap \mathbf{V}_X$. It follows: $\underline{wd}(Y) = \underline{wd}(Y \backslash\backslash R) + \|Y\|$ and $Y \backslash\backslash R \sqsubseteq X \backslash\backslash R$. Then, $Y \backslash\backslash R$ is 1-critical (Lemma 57), is internally connected (Lemma 35) and by Lemmas 80, 82 and 83, admits an equivalent e-tree-decomposition $U \in \mathcal{L}_{f(0)} \cap \mathcal{T}_{k-\|Y\|}$. Let W be the e-tree-decomposition $(\mathbf{e}_Y, \mathbf{T}_U, g)$ where g associates with every node s of U the unique hypergraph $\mathbf{G}_Y \upharpoonright (R \cap \mathbf{V}_X) \subseteq g(s) \subseteq \mathbf{G}_Y$ such that that $g(s) \backslash\backslash R = \mathbf{g}_U(s)$. Clearly, W verifies $\mathbf{val}(W) = \mathbf{val}(Y)$ and belongs to $W \in \mathcal{L}_{f(0)} \cap \mathcal{T}_k$ (we have: $\underline{wd}(W) \leq \underline{wd}(Y \backslash\backslash R) + \|Y\| \leq k$).

If $Y = X$, the conclusion is immediate. Otherwise, X is of the form $Y[Z_1, \ldots, Z_m]$ for some $Z_1, \ldots, Z_m \in \mathcal{T}_k$. For every $i \in [m]$, $Z_i \backslash\backslash R$ is contained in $X \backslash\backslash R$, is 1-critical (Lemma 57), verifies $\|Z_i\| < \|Y\|$ and, by induction hypothesis, is equivalent with some $M_i \in \mathcal{L}_{f(n-1)} \cap \mathcal{T}_k$. Without pert of generality, we can suppose $W[M_1, \ldots, M_m]$ defined. Then, $W[M_1, \ldots, M_m]$ is equivalent with X, belongs to $\mathcal{T}_k$ and to $\mathcal{L}_{f(0)}[\mathcal{L}_{f(n-1)}] = \mathcal{L}_{f(n)}$.

(2) every e-tree-decomposition $X \in \mathcal{T}_k \cap \mathcal{T}_{i.c}$ such that for every leaf t of X, the hypergraph $\mathbf{g}_X(t)$ contains at least one critical edge of $\mathbf{val}(X)$, admits an equivalent e-tree-decomposition in $\mathcal{L}_{k,1+4\cdot k+2\cdot k^2} \cap \mathcal{T}_k$.
Let $X \in \mathcal{T}_k \cap \mathcal{T}_{i.c}$ be an e-tree-decomposition such that for every leaf t of X, $\mathbf{g}_X(t)$ contains at least one critical edge of $\mathbf{val}(X)$. Denote by R the set of sources of X and by D the set of the critical edges of $\mathbf{val}(X)$. From Fact 84, there is a set $C \subseteq D$ a set of cardinality at most 2^k such that for every $d \in D - C$, there are at least two distinct internally connected component of $\mathbf{val}(X) \backslash d$, each of them containing at least one edge of C Denote by Y the e-tree-decomposition generated by X and by the minimal subtree P of $\mathbf{T}_X$ that contains $\mathbf{r}_X$ and verifies $C \subseteq \mathbf{E}_{\mathbf{g}_X(P)}$. Then, Y contains at most 2^k leaves and, by Corollary 42, belongs to $\mathcal{L}_{1+k} \cap \mathcal{T}_k$.

If $Y = X$, the conclusion is obvious. Moreover, we suppose $Y \neq X$. Let $Z_1, \ldots, Z_m$ be the e-tree-decompositions generated by X and the subtrees of $\mathbf{T}_X \backslash \mathbf{N}_Y$.

Let $Z = Z_i$ for some $i \in [m]$. Let t be a leaf of Z. By construction t is a leaf of X, the hypergraph $\mathbf{g}_Z(t)$, equal to $\mathbf{g}_X(t)$, contains one critical edge d of $\mathbf{val}(X)$. Denote by H_d and K_d two internally connected component of $\mathbf{val}(X) \backslash d$, each of them containing at least one edge of C. $\mathbf{val}(X)$ is internally connected. Then, H_d (resp. K_d) contains at least one extremity of d, denoted by x (resp. y) and one extremity of some edge of C denoted by z (resp. w) that does not belong to R. Consequence of Fact 12, $H_d \backslash\backslash R$ and $K_d \backslash\backslash R$ are two distinct connected-component of $(\mathbf{val}(X) \backslash d) \backslash\backslash R$ and, then, are disjoint. By construction, every edge of C does not belong to Z. Then, $H_d \backslash\backslash R$ (resp. $K_d \backslash\backslash R$) contains at least one source of Z. Every connected-component L of $(\mathbf{val}(Z) \backslash d) \backslash\backslash R$ is either contained in $K_d \backslash\backslash R$, either contained in $H_d \backslash\backslash R$ or disjoint with $H_d \backslash\backslash R \cup K_d \backslash\backslash R$.

In each case, it does not contain every source of $(\mathbf{val}(Z)\backslash d)\backslash\backslash R$. Thus, d is 1-critical in $\mathbf{val}(Z)\backslash\backslash R$. In consequence, for every node t of Z, $\mathbf{g}_Z(t)$ contains one 1-critical edge of $Z\backslash\backslash R$. Thus, $Z\backslash\backslash R$ is 1-critical.

By Point (1), for each $i \in [m]$, Z_i is equivalent with some $U_i \in \mathcal{L}_{-1+3\cdot k+2\cdot k^2} \cap \mathcal{T}_k$. Without pert of generality, we can suppose $Y[U_1, \ldots, U_m]$ defined. Then, $Y[U_1, \ldots, U_m]$ is equivalent with X, belongs to $\mathcal{T}_k$ and to $\mathcal{L}_{1+k}[\mathcal{L}_{3\cdot k+2\cdot k^2}] = \mathcal{L}_{1+4\cdot k+2\cdot k^2}$.

(3) $\mathcal{T}_k \cap \mathcal{T}_{i.c}^{\mathbf{c}}$ is equivalent with some subset of $\mathcal{L}_{2\cdot(1+k)^2} \cap \mathcal{T}_k$.
Let $X \in \mathcal{T}_k \cap \mathcal{T}_{1.c}^{\mathbf{c}}$. If X is atomic, the conclusion is immediate. Moreover, we suppose X not atomic. Let Y be the e-tree-decomposition generated by X and the maximal subtree of $\mathbf{T}_X$ that contains no leaf of X. Then, $X = Y[Z_1, \ldots, Z_m]$ for some atomic e-tree-decompositions $Z_1, \ldots, Z_m \in \mathcal{T}_k$.

Y belongs to $\mathcal{T}_{i.c} \cap \mathcal{T}_k$. Every leaf t of Y is not a leaf of X, and admits at least one child in X that is a leaf of X. For every leaf t of Y, denote by s_t a leaf of X incident with t in $\mathbf{T}_X$ and by e_t the arc of X incident with t and s_t. For every leaf t of Y, the arc e_t belongs to $\mathbf{g}_Y(t)$, verifies $\mathbf{val}(Y)\backslash e_t = (\mathbf{e}_X, \mathbf{g}_X(\mathbf{T}_X\backslash s_t))$ and is critical in $\mathbf{val}(Y)$ (X is critical). By Point (2), Y admits an equivalent e-tree-decomposition $U \in \mathcal{L}_{-1+2\cdot(1+k)^2} \cap \mathcal{T}_k$. Without pert of generality, we can suppose $U[Z_1, \ldots, U_m]$ defined. Then, $U[Z_1, \ldots, U_m]$ is equivalent with X, belongs to $\mathcal{T}_k$ and to $\mathcal{L}_{1+4\cdot k+2\cdot k^2}[\mathcal{L}] = \mathcal{L}_{2\cdot(1+k)^2}$. □

Appendix c

To encode every nowhere-critical e-tree-decomposition in terms of its value, we define a new relational structure: the e-discrete-decomposition. Their set is denoted by $Discret$. This structure is an intermediate structure between the e-hypergraph and the e-tree-decomposition. This new structure is composed by an e-hypergraph and by a multiset of couples of hypergraphs (H, S) where the hypergraph with no edge S can be viewed as the "source-edge" of H (we have $S \subseteq H$). More precisely, the e-discrete-decomposition induced by some $X \in \mathcal{T}$, denoted by $\underline{discr}(X)$, is obtained by considering for each node t of X the hypergraph $\mathbf{g}_X(t)$ and the discrete subhypergraph $\mathbf{s}_X(t) \subseteq \mathbf{g}_X(t)$ having for nodes the sources of $X|t$ and by transforming the tree $\mathbf{T}_X$ into the "discrete graph" $\mathbf{N}_X$. Thanks $\underline{discr}$, the MSO-parsability of $\mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k$ is proved in two steps. That is the direct consequence of the both following results:

(1) $\{(|\mathbf{val}(X)|, |\underline{discr}(X)|) \mid X \in \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k\}$ is MSO-definable, for each k.

(2) $\{(|\underline{discr}(X)|, |X|) \mid X \in \mathcal{T}^{\mathbf{nc}}_{i.c}\}$ is MSO-definable.

The property (2), expressed by Lemma 88, is the consequence of the fact that every node of some $X \in \mathcal{T}^{\mathbf{nc}}_{i.c}$ contains a proper information that permits to define $\mathbf{T}_X$ from $\underline{discr}(X)$. This property does not hold in the general case (see Example 85).

Example 85 Let L be the set of all $X \in \mathcal{T}_{i.c}$ that contains no vertex and exactly two edges. Hence, L contains every linear e-tree-decomposition X that contains no vertex and uniquely two edges, one contained in its root, the second in its leaf. The internal nodes of every $X \in L$ contain the empty-graph, are similar and cannot be totally ordered by a MSO-formula. Then, $\underline{discr}^{-1}$ is not MSO-definable.

The reason for which, we keep for every node of X a couple of hypergraphs: its hypergraph and its sources, is the fact that the multiset of the hypergraphs denoted by some node of X does not determine X (see Example 86).

Example 86 Let $G = (a, \{1, 2, 3, 4, 5\}, \{a, b, c, d, e, f\}, v)$ where v associates with a, b, c, d, e, f respectively the set $\{1\}$, $\{1, 2\}$, $\{2, 4\}$, $\{3, 5\}$, $\{2, 5\}$, $\{3, 5\}$. Let T be the tree having three nodes A, B, C with A and C as leaves. Let $X = (a, T, g)$ and $Y = (a, T, h)$ where g (resp. h) associates respectively to A, B (resp. C) and C (resp. B) the hypergraph $G \upharpoonright \{a, b\}$, $G \upharpoonright \{c, d\}$ and $G \upharpoonright \{e, f\}$. X and Y are distinct, belong to $\mathcal{T}^{\mathbf{nc}}_{i.c}$ and verify $\{\mathbf{g}_X(t) \mid t \in \mathbf{N}_X\} = \{\mathbf{g}_Y(t) \mid t \in \mathbf{N}_Y\}$.

The property (1), expressed by Lemma 99, shows how transform by a MSO-transduction every e-hypergraph G into sets L of the form $\{\mathbf{g}_X(t) \mid t \in \mathbf{N}_X\}$ for some $X \in \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k$ such that $G = \mathbf{val}(X)$. A simple way to define such sets is to consider sets of (nonempty) connected and disjoint subgraphs of G. Unfortunately, our main operation on $\mathcal{G}$ is not the disjoint union but the edge-substitution $H[K]$ that union two hypergraphs having in the general case a common intersection ($\mathbf{V}_H \cap \mathbf{V}_K \neq \emptyset$). To address this difficulty, it suffices to refine the relational structure of an e-hypergraph by enrich its domain with a set $\mathbf{B}_H$ of *tentacles* that describe the incidences between nodes and edges (see Definition 89 and proof of

Lemma 99). To transform an hypergraph H into an hypergraph-with-tentacles, it suffices to consider as vertex each vertex and each edge of H and as edge each tentacle. This transformation is not a MSO-transduction, but its restriction on every rank-bounded set of hypergraphs is MSO-definable (see Theorem 90). If some e-hypergraph G is of the form $H[K]$, then the set A of tentacles of $\mathbf{G}_G$ is the disjoint union of B and C, respectively the set of tentacles of $\mathbf{G}_H \backslash \mathbf{e}_K$ and of $\mathbf{G}_K \backslash \mathbf{e}_K$. And if $\mathbf{G}_H \backslash \mathbf{e}_K$ is connected and if $\mathbf{G}_K \backslash \mathbf{e}_K$ is internally connected, then B and C are connected.

With the precedent remarks, it follows that every e-tree-decomposition X can be define in a MSO-way from $\mathbf{val}(X)$ if for every $t \in \mathbf{N}_X$, $\mathbf{val}(X|t)$ admits for internally connected subhypergraph $\mathbf{g}_X(t)$ $(= \mathbf{G}_{\mathbf{val}(X|t)} \backslash A_X)$. To do this, we use a *logical-set* that is a sequence of 5 sets of tentacles of $\mathbf{val}(X)$. Unfortunately, this condition on X is stronger than this one required for the membership of $\mathcal{T}_{i.c}^{\mathbf{nc}}$: we require for every $t \in \mathbf{N}_X$ and every arc d of X the fact that $\mathbf{val}(X|t) \backslash d$ contains an internally connected component containing every source. Corollary 98 shows how a sequence of 2^{k+2} logical-sets can MSO-define in terms of $\mathbf{val}(X)$ the e-discrete-decomposition $\underline{discr}(X)$ for every $X \in \mathcal{T}_{i.c}^{\mathbf{nc}} \cap Rank_k \cap Type_{>0}$.

Definition 87 An *e-discrete-decomposition* is a sequence X denoted by $(\mathbf{e}_X, \mathbf{G}_X, \mathbf{N}_X, \mathbf{g}_X, \mathbf{s}_X)$ where $(\mathbf{e}_X, \mathbf{G}_X)$ is an e-hypergraph, denoted by $\mathbf{val}(X)$, $\mathbf{N}_X$ the set of *nodes* of X, $\mathbf{g}_X$ a mapping that associates with every $t \in \mathbf{N}_X$ a subhypergraph of $\mathbf{G}_X$ such that $\mathbf{E}_{\mathbf{g}_X(u)} \cap \mathbf{E}_{\mathbf{g}_X(v)} = \emptyset$ for all $u \neq v \in \mathbf{N}_X$ and $\mathbf{s}_X$ a mapping that associates with every $t \in \mathbf{N}_X$ a subhypergraph of $\mathbf{g}_X(t)$ with no edge. We denote by *Discret* their set.

For all $X, Y \in Discret$ with $\mathbf{val}(X) = \mathbf{val}(Y)$ and $\mathbf{N}_X \cap \mathbf{N}_Y = \emptyset$, we denote by $X \cup Y$ the e-discrete-decomposition $(\mathbf{e}_X, \mathbf{G}_X, \mathbf{N}_X \cup \mathbf{N}_Y, \mathbf{g}_X \cup \mathbf{g}_Y, \mathbf{s}_X \cup \mathbf{s}_Y)$.

Let $X \in \mathcal{T}$. We denote by $\mathbf{s}_X$ the mapping that associates with every node t of X the hypergraph $\mathbf{G}_{X|t} \restriction \mathbf{e}_{X|t} \backslash \mathbf{e}_{X|t}$. We denote by $\underline{discr}(X)$ the e-discrete-decomposition $(\mathbf{e}_X, \mathbf{G}_X, \mathbf{N}_X, \mathbf{g}_X, \mathbf{s}_X)$ and, for every $P \subseteq \mathbf{N}_X$, by $\underline{discr}(X, P)$ the e-discrete-decomposition $(\mathbf{e}_X, \mathbf{G}_X, P, g, s)$ with g and s the restriction on P of respectively $\mathbf{g}_X$ and $\mathbf{s}_X$.

Every $X \in Discret$ can be represented by a relational structure denoted by $|X|$ and defined in a similar way than for e-tree-decompositions such that we have for all $X, Y \in \underline{discr}$ $X = Y$ if and only if $|X| = |Y|$.

Here, we prove the MSO-definability of the converse of the restriction of $\underline{discr}$ on $\mathcal{T}_{i.c}^{\mathbf{nc}}$.

Lemma 88 $\{(|\underline{discr}(X)|, |X|) \mid X \in \mathcal{T}^{\mathbf{nc}}_{i.c}\}$ *is MSO-definable.*

Proof.
For every $X \in \mathcal{T}$, we denote by $<_X$ the partial order on the nodes of X induced by $(\mathbf{T}_X, \mathbf{r}_X)$. Clearly, to conclude it suffices to prove that there is a MSO-formula that defines for every $X \in \mathcal{T}^{\mathbf{nc}}_{i.c}$ the partial order $<_H$ (or the relation $\leq_X$) in terms of $\underline{discr}(X)$. An *element* of some hypergraph H is an element of the domain $\mathbf{V}_H \cup \mathbf{E}_H$. Let $X \in \mathcal{T}^{\mathbf{nc}}_{i.c}$. For all nodes $s, t \in \mathbf{N}_X$, we have:

(1) $\mathbf{g}_X(s)$ contains at least one vertex not in $\mathbf{s}_X(s)$, if s is not a leaf of X.

Let $H = \mathbf{val}(X|s)$. Suppose that every vertex of H is a source. H is internally connected and, then, contains exactly one edge d distinct with $\mathbf{e}_H$. By hypothesis, s is not a leaf. Then, d is an arc of X, that is not critical in H ($X \in \mathcal{T}^{\mathbf{nc}}_{i.c}$). $\mathbf{G}_H \backslash \{\mathbf{e}_H, d\}$ contains no internally connected subhypergraph (see Definition 45). Contradiction. Then, H contains at least one non-source vertex.

(2) $\mathbf{g}_X(s)$ contains at least one edge if s is a leaf of X distinct with $\mathbf{r}_X$.

Let H be $\mathbf{val}(X|s)$. Suppose that the unique edge of H is $\mathbf{e}_H$. H is internally connected and, then, contains at least one non source vertex x. By hypothesis, no edge of H is incident with x, x is isolated in $\mathbf{val}(X)$. s is distinct with $\mathbf{r}_X$ and, then, admits a parent u. Let y be a vertex of $X|u$ (Point (1)). y is dictinct with x. Then, $\mathbf{val}(X)$ is not connected. Contradiction. Then, H contains at least one non-source edge.

(3) $s \leq_X t$ if and only if there is a path of $\mathbf{val}(X)$ from some element of $\mathbf{g}_X(s)$ to some element of $\mathbf{g}_X(t)$ that does not contain any vertex of $\mathbf{s}_X(t)$.

Let $s \leq_X t$. By Points (1) and (2), there are two elements a, b respectively of $\mathbf{g}_X(s)$ and $\mathbf{g}_X(t)$ that does not belong to $\mathbf{s}_X(s)$ and $\mathbf{s}_X(t)$. Denote by G the e-hypergraph denoted by the e-tree-decomposition generated by X and the union of $\{t\}$ and the set of all its descendant. By hypothesis, X is internally connected, then G too. By construction a and b does not belong to $\mathbf{g}_X(t)$, and then are the extremities of some internal-path of G. By definition, this path does not contain any element of $\mathbf{s}_X(t)$.

Let p a path of $\mathbf{val}(X)$ from some element of $\mathbf{g}_X(s)$ to some element of $\mathbf{g}_X(t)$ that does not contain any element of $\mathbf{s}_X(t)$. Suppose $s \not\leq_X t$

and denote by p the parent of t. Then, the path of $\mathbf{T}_X$ from s to t contains p. From Definition 19, it follows that $\mathbf{g}_X(p)$ contains some vertex of p. Then, p contains some element of $\mathbf{s}_X(t)$. Contradiction. Then, $s \leq_X t$. $\mathbf{val}(X|u)$. □

Now, let us define formally tentacles:

Definition 89 Let G be a hypergraph. A *tentacle* of G is a pair of the form $\{e, x\}$ where e is an edge of G and x one of its extremity. A tentacle $\{e, x\}$ is said *incident with* e (resp. x). Two tentacles are *vertex-adjacent* (resp. *edge-adjacent*) if they are incident to the same vertex (resp. edge). A set of tentacles A is *connected* if it is nonempty and if for all tentacles $\{d, x\}, \{e, y\}$ of A, there is at least one path $(o_1, \dots, o_m)$ of G such that $\{d, x\} = \{o_1, o_2\}$, $\{e, y\} = \{o_{m-1}, o_m\}$ and $\{o_i, o_{i+1}\} \in A$ for every $i \in [m-1]$.

In the general case, the set of tentacles is not MSO-definable. Nevertheless, in the rank bounded case, it is. It is the direct consequence of Theorem 90 due to Courcelle [7] and the fact that every tentacle $\{e, x\}$ of an oriented hypergraph of rank at most k can be represented by the pair (e, i) with $i \in [k]$ if x is the i^{th} extremity of e (An oriented-hypergraph H is a sequence $(\mathbf{V}_H, \mathbf{E}_H, \mathbf{vert}_H)$ where $\mathbf{vert}_H$ associates with every edge not a set but a sequence of vertices. The degree of an edge is the length of this sequence. The rank of an hypergraph is the maximal degree of all of its edges).

Theorem 90 *For every $k \geq 2$, the transduction $\{(\mathbf{und}(H), H) \mid H \in \mathcal{GO}_k\}$ is MSO-definable where $\mathcal{GO}_k$ contains every oriented-hypergraph of rank at most k and where* **und** *associates with every oriented-hypergraph its (unoriented-)hypergraph induced.*

Now, present the logical-set that will permit to encode $\underline{discr}(X)$ from $\mathbf{val}(X)$. Previously, some useful notations.

Notation 91 Let m be an integer. For every sequence L of length m and for every $i \in [m]$, we denote by L_i the i^{th} element of L. Let L and M two sequences of sets of length m. We denote by $L \cup M$ the sequence $(L_1 \cup M_1, \dots, L_m \cup M_m)$. L is said *contained in* M, denoted by $L \subseteq M$, if $L_l \subseteq M_l$ for every $l \in [m]$. L and M are *disjoint* if $\bigcup_{i\in[m]} L_i \cap \bigcup_{i\in[m]} L_i = \emptyset$.

Definition 92 Let $H \in \mathcal{G}$. A *logical-set of* H is a sequence L of 5 sets of tentacles of $\mathbf{G}_H$ such that $L_1 \cap L_2 = \emptyset$ and $L_1 \cup L_2 \supseteq L_3 \cup L_4 \supseteq L_5$. A logical-set A is *internally connected* if A_5 is a singleton and if either $A_2 = \emptyset$

and all tentacles of A_1 are edge-adjacent, or if A_2 is connected and if every tentacle of A_1 is edge-adjacent with at least one tentacle of A_2. An *internally connected component* of some logical-set L is a logical-set $A \subseteq L$ maximal w.r.t $\subseteq$ to be internally connected.

In the next notation, we show how associate in MSO to logical-set an e-discrete-decomposition. This definition required the next fact. Its proof is easy and will be admitted.

Fact 93 *Let L be a logical-set of some $H \in \mathcal{G}$. For each $a \in L_5$, there is an internally connected component A of L such that $a \in A_5$. Two distinct internally connected components of L are disjoint.*

Notation 94 Let L be a logical-set of some $H \in \mathcal{G}$. We denote by $\underline{discr}(H, L)$ the e-discrete-decomposition $\bigcup_{a \in L_5} X_a$ where for every $a \in L_5$, the term X_a denotes the e-discrete-decomposition $(\mathbf{e}_H, \mathbf{G}_H, \{a\}, g, s)$ where:

- $g(a)$ is the minimal subhypergraph of $\mathbf{G}_H$ that contains every vertex (resp. edge) of H incident with a tentacle of $A_1 \cup A_3 \cup A_4$ (resp. A_4).
- $s(a)$ is the minimal subhypergraph of $\mathbf{G}_H$ that contains every vertex of H incident with a tentacle of A_1.

  with A the unique internally connected component of L such that $\{a\} = A_5$.

The next fact shows how to "fuse" logical-sets by preserving the e-discrete-decompositions they define.

Fact 95 *Let $H \in \mathcal{G}$. Let M and N two logical-sets such that for every $a \in M_1 \cup M_2$ and every $d \in N_1 \cup N_2$, we have:*

- *a and d are distinct and are not edge-adjacent.*
- *a and d are not adjacent if $a \in M_2$ and $b \in N_2$.*

$M \cup N$ is a logical set such that: $\underline{discr}(H, M \cup N) = \underline{discr}(H, M) \cup \underline{discr}(H, N)$.

Proof.
Internal-connected is abbreviated in "i.c". Let M and N two logical-sets of some $H \in \mathcal{G}$ that verifies the conditions described above. Obviously, $M \cup N$ is a logical-set. Clearly, to conclude it suffices to prove that every logical-set L is an i.c component of $M \cup N$ if and only if L is an i.c component of M

or of N. The proof comports two parts. A first one, we prove that every i.c logical-set contained in $M \cup N$ is contained in M or in N. A second one, we conclude.

Part 1

Let $A \subseteq M \cup N$ be an i.c logical-set. Let us prove $A \subseteq M$ or $A \subseteq N$. Suppose A not disjoint with M (the other case is symmetrical). Four cases appear:

(1) $A_2 = \emptyset$.
All tentacles of A_1 are edge-adjacent and, then, do not belong to $N_1 \cup N_2$ and belong to M_1. A is contained in M.

(2) A_2 is connected and is not contained in M_2 and in N_2.
A_2 is connected and is partitioned $\{A_2 \cap M_2, A_2 \cap N_2\}$. Then, there are two adjacent tentacles $a \in M_2$ and $b \in N_2$. Contradiction.

(3) A_2 is connected and is contained in M_2.
Every tentacle of $A_1 \cap N_1$ is edge-adjacent with at least one tentacle of $A_2 \subseteq M_2$ and, then does not belong to N_1. Thus, $A \subseteq M$.

(4) A_2 is connected and is contained in N_2.
Symmetrical proof with the precedent one.

Part 2

Let A be an i.c component of M. Let B be an i.c component of $M \cup N$ that contains A. From **Part 1**, B is contained in M and, then, is equal to A. Then, A is an i.c component of $M \cup N$. Let A be an i.c component of $M \cup N$. From **Part 1**, A is contained in M or in N. Let B be an i.c component of M or of N that contains A. B is i.c, verifies $A \subseteq B \subseteq M \cup N$ and, then, is equal to A. Then, A is an i.c component of M or of N. $\square$

The next lemma is a consequence of precedent fact and expresses the conditions that permit for some $X \in \mathcal{T}$ to define thanks a logical-set a "partial" e-discrete-decomposition of $\underline{discr}(X)$.

Lemma 96 *Let $X \in \mathcal{T}_{i.c} \cap Type_{>0}$. If there are some $Y_1, \ldots, Y_m \in \mathcal{T}_{i.c}$ for some $m \geq 1$ and some hypergraphs $H_1, \ldots, H_m$ such that $X = (\ldots(Y_1[Y_2])\ldots)[Y_m]$ and such that for every $i \in [m]$:*

- $\mathbf{g}_X(\mathbf{r}_{Y_i}) \backslash \mathbf{e}_X \subseteq H_i \subseteq \mathbf{g}_X(\mathbf{T}_{Y_i})$.
- *H_i is internally connected in $\mathbf{val}(Y_i)$.*

then, there is a logical-set L with $\underline{discr}(\mathbf{val}(X), L) = \underline{discr}(X, \{\mathbf{r}_{Y_i} \mid i \in [m]\})$.

Proof.
Let $X \in \mathcal{T}_{i.c} \cap Type_{>0}$. Let $m \geq 1$, $Y_1, \ldots, Y_m \in \mathcal{T}_{i.c}$ be m e-tree-decompositions and $H_1, \ldots, H_m$ be m hypergraphs that verify the conditions described above. For every $i \in [m]$, let A^i the sequence where $A^i_1, A^i_2, A^i_3, A^i_4$ contain respectively:

- every tentacle of H_i incident with a source of Y_i.
- every tentacle of H_i not incident with a source of Y_i.
- every tentacle of H_i incident with a vertex of $\mathbf{g}_X(\mathbf{r}_{Y_i})$.
- every tentacle of H_i incident with an edge of $\mathbf{g}_X(\mathbf{r}_{Y_i}) \backslash \mathbf{e}_X$.

By hypothesis $X \in \mathcal{T}_{i.c} \cap Type_{>0}$. It follows $\{Y \mid Y \sqsubseteq X\} \subseteq \mathcal{T}_{i.c} \cap Type_{>0}$. Then, for every $i \in [m]$, $\mathbf{g}_X(\mathbf{r}_{Y_i})$ and H_i contain at least one source of Y_i. As a consequence of Definition 9, for every $i \in [m]$, every element of $\mathbf{V}_{H_i}$ (resp. $\mathbf{E}_{H_i}$) is incident in H_i with some element of $\mathbf{E}_{H_i}$ (resp. $\mathbf{V}_{H_i}$) and then, with a tentacle of H_i. Thus, for every $i \in [m]$, A^i is internally connected (see Definition 92) and verifies $\underline{discr}(\mathbf{val}(X), A^i) = \underline{discr}(X, \mathbf{r}_{Y_i})$.

From Definition 19 and 9, we prove by recurrence on n that for every $l \in [2, m]$ every tentacle $a \in A^l_1 \cup A^l_2$ and every tentacle $b \in \bigcup_{i \in [l-1], j \in [2]} A^i_j$ are distinct, are not edge-adjacent and are not vertex adjacent if $a \in A^l_2$ and $b \in \bigcup_{i \in [l-1]} A^i_2$. By Fact 95, we have: $\underline{discr}(\mathbf{val}(X), \bigcup_{i \in [m]} A^i) = \underline{discr}(X, \{\mathbf{r}_{Y_1}, \ldots, \mathbf{r}_{Y_m}\})$. $\square$

In order to establish Corollary 98, we sate an important fact that permits to partition every set D of "not-needed" edges into a partition of "not-needed" subsets of D.

Fact 97 *Let H be an e-hypergraph of type some integer k and let $D \subseteq \mathbf{E}_H \backslash \mathbf{e}_H$ be a nonempty set with no critical edge of H. There is a partition $\mathcal{D}$ of D of cardinality at most 2^k such that for every $C \in \mathcal{D}$, the e-hypergraph $H \backslash C$ contains an internally connected component that contains every source of H.*

Proof.
Let k be an integer. This proof comports two parts. A first one, we treat the case $D = \mathbf{E}_G \backslash \mathbf{e}_H$. A second one, we treat general case. The

sentence "internally connected" is abbreviated in "i.c". For every $H \in \mathcal{G}$, we denote by $\|H\|$ the cardinality of $(\mathbf{V}_H \backslash \mathbf{vert}_H(\mathbf{e}_H)) \cup (\mathbf{E}_H \backslash \mathbf{e}_H)$. Clearly, every e-hypergraph of the form $H[K]$ verifies: $\|H[K]\| = \|H\| + \|K\|$. A *good-partition of* H is a partition $\mathcal{E}$ of $\mathbf{E}_G \backslash \mathbf{e}_H$ of cardinality at most 2^k such that for every $D \in \mathcal{E}$, there is an i.c component of $H \backslash D$ that contains every source of H.

Part 1

Suppose there is $n \geq 0$ such that every $H \in \mathcal{G}$ of type k, of size $\|H\| < n$ with $\mathbf{E}_H \backslash \mathbf{e}_H \neq \emptyset$ and with no critical edges, admits a good-partition of cardinality at most 2^k. Let H be an e-hypergraph of type k, of size $\|H\| = n$ with $\mathbf{E}_H \backslash \mathbf{e}_H \neq \emptyset$ and with no critical edge. Denote by R the hypergraph $(H \upharpoonright \mathbf{vert}_H(\mathbf{e}_H)) \backslash \mathbf{e}_H$. Different cases appear:

- two distinct i.c component of H contain R.
  Denote by L and M two distinct i.c component of H that contain R. The set $\{\mathbf{E}_L, \mathbf{E}_H \backslash (\{\mathbf{e}_H\} \cup \mathbf{E}_L)\}$ is a good partition of H.

- one i.c component of H does not contain R.
  Let L be an i.c component of H that does not contain R. Denote by K the e-hypergraph $(H \backslash \mathbf{E}_L) \backslash (\mathbf{V}_L - \mathbf{V}_R)$. For every $d \in \mathbf{E}_H \backslash (\{\mathbf{e}_H\} \cup \mathbf{E}_L)$, the i.c component of $H \backslash d$ that contains R is edge-disjoint with L, and is an i.c component of K'. Then, K has for type k, has a size at most $n-1$, has no critical edge. By induction hypothesis, K admits a good partition $\{E_1, \ldots, E_n\}$. Clearly, $\{\mathbf{E}_L \cup E_1, \ldots, E_n\}$ is a good partition of H.

- H is i.c and $H \backslash d$ is not i.c for some $d \in \mathbf{E}_H$.
  Let d be an edge such that $H \backslash d$ not i.c. Denote by $\{H_1, \ldots, H_m\}$ the set of i.c component of $H \backslash d$. It comes $2 \leq m$. By hypothesis, d is not a 1-critical edge of H, then at least one hypergraph of the form H_i with $i \in [m]$ contains R. Without pert of generality, we can suppose: $R \subseteq H_1$. If $R \subseteq H_i$ for some $i \in [2, m]$, then, $\{\mathbf{E}_{H_1}, \mathbf{E}_H \backslash (\{\mathbf{e}_H\} \cup \mathbf{E}_{H_1})\}$ is a good partition of H. Moreover, we suppose $R \not\subseteq H_i$, for every $i \in [2, m]$.

  Denote by Q the hypergraph $(H \upharpoonright d) \cup H_2 \cup \ldots \cup H_m$. Denote by P the e-hypergraph obtained from Q by adding a new edge, noted f, of extremities every vertex of $H_1 \cap Q$ and by considering f as its source-edge. Every element of $\mathbf{V}_H \cup (\mathbf{E}_H \backslash f)$ is the initial extremity of an internal path of P of terminal extremity d. Then, P is i.c. Denote by N the e-hypergraph $(\mathbf{e}_H, (H \restriction \mathbf{e}_H) \cup H_1 \cup (K \restriction f))$. It comes: $H = N[P]$ with $0 < \|P\|$ and $\|N\| < n$. For every $c \in \mathbf{E}_Q$, H_1 is an i.c subhypergraph of $H \backslash c$, c is not critical in H. Then, every edge of $\mathbf{E}_N \backslash \mathbf{e}_H$ is not critical in N (Fact 46). K verifies the induction hypothesis, admits a good-partition $\mathcal{E}$. The set $\{(E_1 \backslash f) \cup \mathbf{E}_{H_m}, E_2, \ldots, E_l\}$ with $\mathcal{E} = \{E_1, \ldots, E_l\}$ and $f \in E_1$, is a good partition of H.

- H and $H \backslash d$ are i.c for every $d \in \mathbf{E}_H \backslash \mathbf{e}_H$.
  If every vertex of H is a source, H contains an unique edge distinct with $\mathbf{e}_H$. Contradiction. Let $x \in \mathbf{V}_H - \mathbf{V}_R$. Denote by $\{r_1, \ldots, r_k\}$ the set $\mathbf{V}_R$ and for every $i \in [k]$, denote by H_i the e-hypergraph $(H \backslash\backslash (\mathbf{V}_R \backslash r_i))$. For every $i \in [k]$ and every $d \in \mathbf{E}_H \backslash \mathbf{e}_H$, the hypergraph $H_i \backslash d$ contains x, is i.c and then is connected. Then, for every $i \in [k]$, H_i is 2-edge-connected (Lemma 7). For every $i \in [k]$, there are two edge-disjoint path $p_{i,0}$ and $p_{i,1}$ of H_i from x to r_i with no internal-vertex in $\mathbf{V}_R$. For every $i \in [k]$ and every $j \in \{0,1\}$, we denote by $E_{i,j}$ the set of edges of p_i^j, augmented if $(i,j) = (1,0)$ with the set of edges that belong to any path of the form $p_{i,j}$ with $(i,j) \in [k] \times [0,1]$. The set $\{E_{i,j} \neq \emptyset \mid i \in [k], j \in [0,1]\}$ is a good-partition of H of size at most 2^k.

Part 2
Let $H \in \mathcal{G}$ of type k, D a nonempty subset of $\mathbf{E}_H \backslash \mathbf{e}_H$ with no critical edge of H and S the set of sources of H. For every $d \in D$, $H \backslash d$ contains an i.c component that contains S. Denote by K the e-hypergraph obtained from H by adding, for every edge $c \in \mathbf{E}_G \backslash D$ a new edge of extremities whose of c. Clearly, for every edge $c \in \mathbf{E}_K \backslash \mathbf{e}_K$, the e-hypergraph $K \backslash c$ contains an i.c component that contains S. Then, there is a good-partition $\mathcal{E} = \{E_1, \ldots, E_n\}$ of K (see result of **Part 1**). The set $\mathcal{D} = \{E_i \cap D \mid E_i \cap D \neq \emptyset\}$ is a partition of D such that for every $C \in \mathcal{D}$, the e-hypergraph $H \backslash C$ contains an i.c component that contains S. □

Corollary 98 explains, in a "technical" way, why we can encode every e-tree-decomposition $X \in \mathcal{T}_{i.c}^{\mathbf{nc}} \cap Rank_k$ "inside" its value $\mathbf{val}(X)$. Consequence of this corollary and Lemma 96 , there is a partition $\{N_1, \ldots, N_{2^{k+1}}\}$ and a sequence of 2^{k+1} logical-sets that define respectively $\underline{discr}(X, N_1), ldots, \underline{discr}(X, N_{2^{k+1}})$ and then $\underline{discr}$ (see proof of Lemma 99).

Corollary 98 *Let $X \in \mathcal{T}_{i.c}^{\mathbf{nc}} \cap Rank_k$ for some $k \geq 0$. The set $\mathbf{N}_X$ admits a partition $\{N_1, \ldots, N_m\}$ of cardinality at most 2^{k+2} such that for every $i \in [m]$ and every $t \in N_i$, there is a hypergraph H verifying:*

- $\mathbf{g}_X(t)\backslash\mathbf{e}_X \subseteq H \subseteq \mathbf{g}_X(T)$.
- *H is internally connected in* $\mathbf{val}(X|T)$.

  where T designs the tree of $\mathbf{T}_X\backslash Q$ that contains t with Q the difference $N_i\backslash t$ augmented with the parent of t in X if $t \neq \mathbf{r}_X$.

Proof.
Let $X \in \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k$ for some $k \geq 0$. Let $m = 2^{k+1}$. A *m-partition* (resp. $2m$-partition) of some set N is a sequence $(N_1, \ldots, N_l)$ of subsets of N of length m (resp. $2m$) such that $N = \bigcup_{i\in[l]} N_i$ and $N_i \cap N_j = \emptyset$ for all $1 \leq i < j \leq l$. We denote by φ the bijective mapping $\mathbf{N}_X \to \{\mathbf{e}_X\} \cup \mathbf{A}_X$ that associates with $\mathbf{r}_X$ the source-edge $\mathbf{e}_X$ and to every $t \in \mathbf{N}_X\backslash\mathbf{r}_X$ the unique arc of $\mathbf{A}_X$ incident with t and with its parent. For every $t \in \mathbf{N}_X$, we denote by A_t the set of all arcs of $\mathbf{A}_X\backslash\varphi(t)$ incident with t.

Clearly, $\{A_t \mid t \in \mathbf{N}_X\}$ is a partition of $\mathbf{A}_X$. From Definition 45, for every $t \in \mathbf{N}_X$, A_t contains no critical edge of $\mathbf{val}(X|t)$. As a consequence of Fact 97, for every $t \in \mathbf{N}_X$, there is a m-partition $(A_{t,1}, \ldots, A_{t,m})$ of A_t such that for every $i \in [m]$, the e-hypergraph $\mathbf{val}(X|t)\backslash A_{t,i}$ contains an internally connected component that contains every source of $X|t$. Let $(A_1, \ldots, A_m)$ the m-partition of $\mathbf{A}_X$ where for every $i \in [m]$ the term A_i denotes $\bigcup_{t\in\mathbf{N}_X} A_{t,i}$. Let f the mapping $\mathbf{N}_X \to [2m]$ that associates with every $t \in \mathbf{N}_X$:

- 1 if $t = \mathbf{r}_X$.
- $j(t)$ if $t \neq \mathbf{r}_X$ and if the distance in $\mathbf{T}_X$ between t and $\mathbf{r}_X$ is even.
- $m + j(t)$, otherwise.

  where for every $t \in \mathbf{N}_X\backslash\mathbf{r}_X$, $j(t)$ is the unique integer i such that $\varphi(t) \in A_i$.

Let $(N_1, \ldots, N_{2m}) = (f^{-1}(1), \ldots, f^{-1}(2m))$ be the $2m$-partition of $\mathbf{N}_X$. Let $i \in [2m]$ and D_i the set of all arcs $d \in \mathbf{A}_X$ such that $\varphi^{-1}(d) \in N_i$. For every $d \in \mathbf{A}_X$ (resp. $d = \mathbf{e}_X$), we denote by T_d the maximal subtree of $\mathbf{T}_{X\downarrow d}\backslash D_i$ (resp. $\mathbf{T}_X\backslash D_i$) that contains $\varphi^{-1}(d)$. Observing that $T_c \subseteq T_d$ for all $d \in \{\mathbf{e}_X\} \cup \mathbf{A}_X$ and $c \in A_{\varphi^{-1}(d)} \cap D_i$, we prove by recurrence on $\mathbf{card}(\mathbf{V}_{T_d})$ that for every $d \in \{\mathbf{e}_X\} \cup \mathbf{A}_X$, there is a hypergraph H_d such that:

- $\mathbf{s}_X(\varphi^{-1}(d)) \subseteq H_d \subseteq \mathbf{g}_X(T_d)$.
- H_d is internally connected in $\mathbf{val}(X|T_d)$.

Let $t \in N_i$ and $H = (\mathbf{g}_X(t)\backslash\mathbf{e}_X) \cup \bigcup_{c\in A_t} H_c$. It comes $\mathbf{g}_X(t)\backslash\mathbf{e}_X \subseteq H \subseteq \mathbf{g}_X(T_{\varphi(t)})$ and $H = (\mathbf{G}_{X|t} \cup \bigcup_{c\in A_t}(H_c))\backslash(\{\varphi(t)\} \cup A_t)$. The inclusion $A_t \subseteq N_i$ and Lemma 16 imply that H is internally connected in $\mathbf{val}(X|T_{\varphi(t)})$. □

The next lemma establishes the MSO-definability of the converse of the restriction of **val** on $\underline{discr}(\mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k)$. Its proof is obtained by extending the MSO-transduction induced by the precedent corollary to e-tree-decompositions of type null or non-null.

Lemma 99 *For each k, $\{(|\mathbf{val}(X)|, |\underline{discr}(X)|) \mid X \in \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k\}$ is MSO-definable.*

Proof.
For each $k \geq 0$, we denote by p_k (resp. q_k) the transduction $\{(\mathbf{val}(X), \underline{discr}(X)) \mid X \in \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k\}$ (resp. $\in \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k \cap Type_{>0}\}$). Let k be an integer. This proof comports two parts, that concern respectively q_k and p_k.

Part 1
An *e-hypergraph-with-tentacle* H is a e-hypergraph augmented with a set, denoted by $\mathbf{B}_H$, and with two mappings $f : \mathbf{B}_H \to \mathbf{V}_H$ and $g : \mathbf{B}_H \to \mathbf{E}_H$ that describe the set of tentacles of H. We denote by $\mathcal{GB}_k$ the set of all e-hypergraph-with-tentacles of rank at most k and by f_k the mapping that associates with every $H \in \mathcal{GB}_k$ its e-hypergraph induced. As a consequence of Theorem 90, the converse of f_k is MSO-definable.
Clearly, the notions of a logical-set and of an internally connected component of a logical-set are MSO-definable. Clearly, the mapping that associates with every $H \in \mathcal{GB}_k$ and to every sequence of logical-sets $(L^1, \ldots, L^{2^{k+2}})$ of H the sequence $\bigcup_{i\in[2^{k+2}]} \underline{discr}(f_k(H), L^i)$ (without pert of generality, we can suppose L^i_5 disjoint with L^j_5 for every $1 \leq i < j \leq 2^{k+2}$) induces a definition scheme (see Definition 66). By Lemma 96 and Corollary 98, every $H \in f_k^{-1}(\mathbf{val}(\mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_k \cap Type_{>0}))$ admits a sequence of length $5 \cdot 2^{k+2}$ of subsets of $\mathbf{B}_H$ that defines $\underline{discr}(X)$. It follows that q_k is MSO-definable.

Part 2
Obviously, every $X \in \mathcal{T}^{\mathbf{nc}}_{i.c} \cap Rank_0$ is atomic. It follows that p_0 is MSO-definable. Denote by g the transduction that associates with every e-hypergraph H itself and, if H has no source, every e-hypergraph obtained from H by adding a new source r incident with $\mathbf{e}_H$ and a new edge d

incident with r and with at most one vertex of H. We suppose that the new edge and the new vertex are labelled by a special symbol \$. Clearly, g is MSO-definable. Denote by h the transduction that associates with every $X \in Discret$ the e-discrete-decomposition obtained from X by deleting the eventual edge and the eventual vertex of X labelled \$. Clearly, h is MSO-definable. Without difficulty, we prove that p_k is contained in $p_0 \cup (g \circ q_k \circ h)$ and is equal to $p_0 \cup (g \circ q_k \circ h \circ i)$, where i denotes the identity transduction having for domain $\underline{discr}_k(\mathcal{T}_{i.c}^{\mathbf{nc}} \cap Rank_k)$. By Lemma 88 and Proposition 69, i is MSO-definable. By Proposition 68, p_k is MSO-definable. □

Proof of Theorem 70.
Direct consequence of Lemmas 88 and 99 and Proposition 68. □

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