Complexity Problems in Switching Classes of Graphs

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Abstract

For a graph $G = (V, E)$ and a subset $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^\sigma = (V, E')$, which is obtained from G by removing all edges between σ and its complement $\overline{\sigma}$ and adding as edges all nonedges between σ and $\overline{\sigma}$. The switching class $[G]$ determined by G consists of all switchings G^σ for subsets $\sigma \subseteq V$.

In this paper we compare the complexity of a number of problems for graphs with the complexity of these problems for switching classes. It turns out that every imaginable situation can occur.

We show that every switching class, except the class of all complete bipartite graphs, contains a pancyclic graph. This implies that deciding whether a switching class contains a hamiltonian graph can be done in polynomial time although this problem is NP-complete for graphs.

Properties that are NP-complete both for graphs and for switching classes are obtained by generalizing a result of Yannakakis on hereditary properties. We also prove that the embedding problem and the 3-colourability problem for switching classes are NP-complete.

A graph is equally divided if it consists of two connected components with the same number of vertices. Deciding this property can be done in linear time for graphs, while deciding whether a switching class contains an equally divided graph is NP-complete.

1 Introduction

For a finite undirected graph $G = (V, E)$ and a subset $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^\sigma = (V, E')$, which is obtained from G by removing all edges between σ and its complement $\overline{\sigma}$ and adding as edges all nonedges between σ and $\overline{\sigma}$. The switching class $[G]$ determined by G consists of all switchings G^σ for subsets $\sigma \subseteq V$.

A switching class is an equivalence class of graphs under vertex switching, see the survey papers by Seidel [14] and Seidel and Taylor [15]. Generalizations of this approach can be found in Gross and Tucker [9], Ehrenfeucht and Rozenberg [5], and Zaslavsky [18].

A property $\mathcal{P}$ of graphs can be transformed into an existential property of switching classes as follows: $\mathcal{P}_\exists(G)$ if and only there is a graph $H \in [G]$ such that $\mathcal{P}(H)$.

First we consider hamiltonicity and pancyclicity of graphs. We prove that $hamilton_\exists$ and $pancyclic_\exists$ are in P. On the other hand, deciding whether a graph is hamiltonian is NP-complete. In our results on hamiltonicity we follow the main lines of J. Kratochvíl, J. Nešetřil, and O. Zýka [13] as communicated to us by J. Kratochvíl [12]. We also give a short list of problems that are NP-complete for graphs, but easy for switching classes.

The second part of the paper is devoted to a number of problems that are hard for switching classes. We generalize to switching classes a result of Yannakakis [16] on graphs, which is then used to prove that the independence problem is NP-complete for switching classes. This problem can be polynomially reduced to the embedding problem (given two graphs G and H, does there exist a graph in $[G]$ in which H can be embedded). Hence, the latter problem is also NP-complete for switching classes. It also turns out that deciding whether a switching class contains a 3-colourable graph is NP-complete.

A graph is said to be equally divided if it consists of exactly two connected components with the same number of vertices. This problem is linear for graphs, but $equally\text{-}divided_\exists$ turns out to be NP-complete for switching classes.

2 Preliminaries

For a (finite) set V, let $|V|$ be the *cardinality* of V. We shall often identify a subset $A \subseteq V$ with its characteristic function $A : V \to \mathbf{Z}_2$, where $\mathbf{Z}_2 = \{0, 1\}$ is the cyclic group of order two. We use the convention that for $x \in V$, $A(x) = 1$ if and only if $x \in A$. For $A \subseteq V$, we denote the *complement* of A with respect to V by $\overline{A}$.

The restriction of a function $f : V \to W$ to a subset $A \subseteq V$ is denoted by $f|_A$.

We now turn to notation and terminology for graphs and switching classes.

The set $E(V) = \{xy \mid x, y \in V,\ x \neq y\}$ denotes the set of all unordered pairs of distinct elements of V. The graphs of this paper will be finite, undirected and simple, i.e., they contain no loops or multiple edges. For a graph $G = (V, E)$ we often write $xy \in G$ instead of $xy \in E$. We use $E(G)$ and $V(G)$ to denote the set of edges E and the set of vertices V, respectively, and $|V|$ and $|E|$ are called the *order*, respectively, *size* of G. Analogously to sets, a graph $G = (V, E)$ will be identified with the characteristic function $G : E(V) \to \mathbf{Z}_2$ of its set of edges so that $G(xy) = 1$ for $xy \in E$, and $G(xy) = 0$ for $xy \notin E$. Later we shall use both notations, $G = (V, E)$ and $G : E(V) \to \mathbf{Z}_2$, for graphs.

A *switching* of a graph G by a selector $\sigma : V \to \mathbf{Z}_2$ is the graph G^σ such that for all $xy \in E(V)$,

$$G^\sigma(xy) = \sigma(x) + G(xy) + \sigma(y) \ .$$

Clearly, this definition of switching is equivalent to the one given at the beginning of the introduction. We reserve lower case σ and τ for selectors (subsets) used in switching.

The set $[G] = \{G^\sigma \mid \sigma \subseteq V\}$ is called the *switching class* of G. The graph G is called a *generator* of its switching class $[G]$.

For a graph $G = (V, E)$ and $X \subseteq V$, let $G|_X$ denote the *subgraph* of G *induced* by X. Hence, $G|_X : E(X) \to \mathbf{Z}_2$.

The complement of G is $\overline{G} = (V, \overline{E})$ with $\overline{E} = \{xy \mid xy \notin E\}$. For a set $\mathcal{G}$ of graphs, we let $\overline{\mathcal{G}} = \{\overline{G} \mid G \in \mathcal{G}\}$.

Two vertices $x, y \in V$ are *adjacent* (in G) if $xy \in E$. The *degree* of a vertex $x \in V$, denoted by $d_G(x)$, is the number of vertices adjacent to x. The graph G is called *even (odd)* if all vertices are of even (odd) degree. A vertex of degree zero is called *isolated.* A set $U \subseteq V$ is a *clique* if every vertex in U is adjacent to every other vertex in U.

A sequence of vertices $\pi = (v_1, \dots, v_k)$ is a *path* in G if v_i is adjacent to v_{i+1} for $i = 1, \dots, k-1$ and all vertices are distinct. If $\pi = (v_1, \dots, v_k)$ is a path then $(v_1, \dots, v_k, v_1)$ is a *cycle* if $k \geq 2$.

The *complete connection* of two vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is $G = G_1 \oplus G_2$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1), y \in V(G_2)\}$.

Let $\overline{K}_V = (V, \emptyset)$ and $K_V = (V, E(V))$ be the *discrete graph* and the *complete graph* on V respectively, and let $K_{A,\overline{A}}$ denote the *complete bipartite graph* with the partition $\{A, \overline{A}\}$. If the sets of vertices themselves are irrelevant, we write K_n and $K_{k,m}$ where $n = |V|$, $k = |A|$ and $m = |\overline{A}|$. For graphs G and H we define $G + H$ by $(G + H)(e) = G(e) + H(e)$ for $e \in E(V)$. Clearly, the graphs form an abelian group under this operation; we use Γ to denote this group.

The following lemma is immediate, see also Ellingham [6].

Lemma 2.1

i. $[\overline{K}_V]$ consists of the complete bipartite graphs on V, and it is a subgroup of Γ.

ii. For all $\sigma \subseteq V$ and graphs G on V, $G^\sigma = G^{\overline{\sigma}}$.

iii. For all $\sigma, \tau \subseteq V$, $(G^\sigma)^\tau = G^{\sigma+\tau}$. $\square$

In particular, $(G^\sigma)^\sigma = G$, and $[G] = [G^\sigma]$ for all σ.

Lemma 2.2
For a graph $G = (V, E)$, $[\overline{G}] = \overline{[G]}$. Furthermore, if $|V| \geq 3$ then $[G] \cap [\overline{G}] = \emptyset$.
Proof:
We show first that for a graph $G = (V, E)$ and $\sigma \subseteq V$: $\overline{G^\sigma} = \overline{G}^\sigma$. Indeed, let $x, y \in V$. Then $\overline{G^\sigma}(xy) = 1 - (\sigma(x) + G(xy) + \sigma(y)) = \sigma(x) + (1 - G(xy)) + \sigma(y) = \overline{G}^\sigma(xy)$, because $(1 - a) + (1 - b) = a + b$ for $a, b \in \mathbf{Z}_2$.

The additional claim clearly holds. $\square$

3 General problem setting

In this section we first introduce some notions for transforming properties of graphs into properties of switching classes. By way of introduction, we review some known results in this area.

Recall from the introduction that for a property $\mathcal{P}$ of graphs the *existential lifting* of $\mathcal{P}$, denoted $\mathcal{P}_\exists$, is defined by:

$$\mathcal{P}_\exists(G) \text{ if and only if there exists an } H \in [G] \text{ such that } \mathcal{P}(H) \ .$$

We write $\overline{\mathcal{P}}(G)$ if $\mathcal{P}(G)$ does not hold.

Clearly, if $\mathcal{P}$ is in NP, then so is $\mathcal{P}_\exists$, because one can guess a selector σ, and then check whether $\mathcal{P}(G^\sigma)$ holds in nondeterministic polynomial time.

Lemma 3.1
If deciding a property $\mathcal{P}$ of graphs is in NP, then deciding $\mathcal{P}_\exists$ is also in NP. □

Recall that a graph is eulerian if there exists a cycle that traverses each edge exactly once. It was proved by Seidel [14] that if the number of vertices of a graph G is odd, then the switching class $[G]$ contains a unique graph with eulerian connected components, that is,

Theorem 3.2
If G is a graph of odd order, then $[G]$ contains a unique even graph G^σ. □

For graphs of even order, a switching class $[G]$ can contain noneulerian graphs. However, we have

Theorem 3.3
Let G be a graph of even order. Then either $[G]$ has no even and no odd graphs, or exactly half of its graphs are even while the other half are odd.
Proof:
Let G be a graph. Define $u \sim_G v$, if $d_G(u) \equiv d_G(v) \pmod 2$, that is, if the degrees of u and v have the same parity. This relation is an equivalence relation on $V(G)$.

Assume then that the order n of G is even. If we consider singleton selectors σ only (hence switching with respect to one vertex only), then it is easy to see that $\sim_G$ and $\sim_{G^\sigma}$ coincide for all selectors σ. In other words, if G has even order, then the relation $\sim_G$ is an invariant of the switching class $[G]$.

This means that if $[G]$ contains an even graph, then all graphs in $[G]$ are either even or odd. Further, if G is even, and $\sigma : V(G) \to \mathbf{Z}_2$ is a singleton selector, then for each $v \in V(G)$, $d_G(v)$ and $d_{G^\sigma}(v)$ have different parity. From this the theorem follows. □

From the above theorems it follows that *euler*$_\exists$ can be decided in time quadratic in the order of the graph. A general uniqueness result such as Theorem 3.2 is not possible without restrictions on the vertex set. Indeed, if $\mathcal{P}$ is any graph property, which is preserved under isomorphisms, then there exists a switching class that either has no graphs with property $\mathcal{P}$ or it has at least two graphs with property $\mathcal{P}$. This statement follows from a result on automorphisms of switching classes, see Cameron [3]: there exist switching classes for which the group of automorphisms is strictly larger than the group of automorphisms of its graphs.

In [10] it was shown that the property *tree*$_\exists$ satisfies the uniqueness property up to isomorphism:

Theorem 3.4 ([10])
All trees in a switching class are isomorphic. □

Clearly, not all switching classes contain trees. For example, if G contains a complete graph of five vertices as a subgraph then $[G]$ has no trees (in fact, no triangle-free graphs).

Below we show that each switching class, apart from the ones that are generated by the discrete graphs, contains a pancyclic graph.

4 Pancyclic and hamiltonian switching classes

In this section $G = (V, E)$ is a graph of order n. Recall that a path (cycle) is a *hamilton path (cycle)* of G if it contains all vertices in V. A graph with such a cycle is called hamiltonian. Also, G is called *pancyclic*, if it has a cycle of length i for each $i = 3, \ldots, n$. In particular, each pancyclic graph is hamiltonian.

We show that *hamilton*$_\exists$ can be characterized in such a way that the property can be checked in polynomial time. This can be constrasted with the fact that the hamiltonian problem is NP-complete for graphs. We prove, following the main lines of [13] as communicated to us in [12], that a switching class has a hamiltonian graph if and only if the class is different from a switching class of all complete bipartite graphs of odd order. We actually prove a stronger result, which states that all switching classes different from $[\overline{K}_V]$ contain a pancyclic graph. This result is in accordance with Bondy's metaconjecture in [1] which declares that almost all nontrivial general graph properties that imply hamiltonicity imply also pancyclicity. In our result there is only one (trivial) exception: the switching classes of the complete bipartite graphs of even orders contain hamiltonian graphs but do not contain any pancyclic graphs.

Seidel [14] proved that the parity of edges in triangles of a graph determines its switching class:

Theorem 4.1
The following conditions are equivalent for graphs G and H on a common vertex set V.

i. $[G] = [H]$.

ii. The subgraphs induced by the subsets $T \subseteq V$ of size three have the same parity of edges in G and H.

iii. All cycles (of K_V) have the same parity of edges in G and H. □

The closure of a graph G is defined inductively as the graph G_k obtained from a sequence of graphs $G = G_0, G_1, \ldots, G_k$, where $G_{i+1} = G_i + u_i v_i$, $d_{G_i}(u_i) + d_{G_i}(v_i) \geq n$ with $u_i v_i \notin E(G_i)$, and $d_{G_k}(u) + d_{G_k}(v) < n$ for all $uv \notin E(G_k)$, see [2].

The first case of the following lemma is due to Bondy [1], and the second to Bondy and Chvátal, see [2].

Lemma 4.2

i. If G is hamiltonian and $|E(G)| \geq n^2/4$, where $n = |V|$, then G is pancyclic or $K_{n/2,n/2}$.

ii. A graph G is hamiltonian if and only if $G + uv$ is hamiltonian, whenever $d_G(u) + d_G(v) \geq n$ for $uv \notin E(G)$. Moreover, G is hamiltonian if and only if the closure of G is hamiltonian. $\square$

Theorem 4.3
For each graph $G = (V, E)$ of order $n \geq 3$, $[G]$ contains a pancyclic graph if and only if $[G] \neq [\overline{K}_V]$.
Proof:
Let $G = (V, E)$ be a maximum size graph in its switching class, i.e. G has the maximum number of edges. From this we find that for all $A \subseteq V$, there are at least

$$\frac{|A|(n - |A|)}{2} \tag{1}$$

edges leaving A, for, otherwise, switching with respect to A would yield a graph of greater size.

If n is even, then G is hamiltonian, because by (1), $d_G(v) \geq n/2$ for all $v \in V$. In this case, the graph has at least $n^2/4$ edges and so by Lemma 4.2.i we have that G is either pancyclic or $K_{n/2,n/2}$.

Suppose that n is odd. Define $A_G = \{v \mid d_G(v) = (n-1)/2\}$. If $A_G = \emptyset$, then as above we conclude that G is pancyclic. Assume then that $A_G \neq \emptyset$.

Claim 1: A_G is independent.
Indeed, let $B \subseteq A_G$ be a clique of G. For each $v \in B$, there are exactly $(n-1)/2 - (|B| - 1)$ edges that leave B, and hence by (1)

$$|B|(\frac{n-1}{2} - (|B| - 1)) \geq |B|\frac{n - |B|}{2}$$

which is possible if and only if $|B| = 1$.

Claim 2: Every switching class contains a maximum graph G such that $|A_G| \leq (n-1)/2$ or $G = K_{(n-1)/2,(n+1)/2}$.
Indeed, since for $v \in A_G$, $d_G(v) = (n-1)/2$, it follows that $|A_G| \leq (n+1)/2$. If $|A_G| = (n+1)/2$, let $v \in A_G$, and switch with respect to v. We get a maximum graph G^σ with $|A_{G^\sigma}| \geq 1$, since $d_{G^\sigma}(v) = (n-1)/2$. By above, we know that $|A_{G^\sigma}| \leq (n+1)/2$.
We show that if $|A_{G^\sigma}| = (n+1)/2$, then $G = K_{(n+1)/2,(n-1)/2}$. If $G|_{\overline{A_G}}$ contains an edge, then so does $G^\sigma|_{A_{G^\sigma}}$, because $A_G \cap A_{G^\sigma} = \{v\}$ and $|\overline{A_{G^\sigma}}| = |\overline{A_G}| = (n-1)/2$ and so $A_{G^\sigma} = \{v\} \cup \overline{A_G}$, but, by Claim 1, the latter is independent. So $\overline{A_G}$ is independent in G and hence $G = K_{(n-1)/2,(n+1)/2}$.

Assume then that G is a maximum graph in its switching class such that $|A_G| \leq (n-1)/2$, and thus that G is not complete bipartite. We prove that G is hamiltonian. Because $|\overline{A_G}| > (n-1)/2$, for each $v \in A_G$ there exists a $u \in \overline{A_G}$ such that $vu \notin E(G)$, and

$$d_G(v) + d_G(u) \geq (n-1)/2 + (n+1)/2 = n.$$

Now $d_{G+uv}(v)$ equals $(n+1)/2$ and by Lemma 4.2.ii, G is hamiltonian since its closure is the complete K_n.

Knowing that G is hamiltonian, we can prove that it is, in fact, pancyclic:

$$\begin{aligned} 2|E(G)| = \sum d_G(v) &\geq |A_G|\frac{n-1}{2} + (n-|A_G|)\frac{n+1}{2} = \frac{(n+1)n}{2} - |A_G| \\ &\geq \frac{(n+1)n}{2} - \frac{n-1}{2} = \frac{n^2+1}{2} \end{aligned}$$

and thus $|E(G)| \geq (n^2+1)/4$.

By Lemma 4.2.i we conclude that G is pancyclic. □

Note that Claim 2 in the above proof is necessary. The crown graph, $K_{2,3}+e$, where the added edge is between the two vertices in the first part of the partition is an example of a graph that has maximum size among the graphs in its switching class, but which is not hamiltonian.

Corollary 4.4
A switching class $[G]$ contains a hamiltonian graph if and only if G is not a complete bipartite graph of odd order. □

Corollary 4.5
Let G be a graph with $n = |V(G)|$. Then either G is a complete bipartite graph or for each $i = 3, \ldots, n$, there is a cycle C_i (of $K_{V(G)}$) on which the parity of edges of G is the same as the parity of i.
Proof:
Clearly, we may suppose that the order n of G is at least three. Suppose that G is not complete bipartite, and thus that $G \notin [\overline{K}]$. By Theorem 4.3, there exists a pancyclic graph $H \in [G]$, and thus H has a subgraph C_i for each $3 \leq i \leq n$. By Theorem 4.1, the parity of edges of G and H on C_i is the same, which proves the claim. □

When the above corollary is applied to the complement graph of G we obtain

Corollary 4.6
Let G be a noncomplete graph that is not a disjoint union of two cliques. Then for each $i = 3, \ldots, |V(G)|$, there is a cycle C_i (of $K_{V(G)}$) such that G has an even number of edges in C_i. □

5 Easy problems for switching classes

Let G be a graph on V of order n. There are 2^{n-1} graphs in $[G]$, and so checking whether there exists a graph $H \in [G]$ satisfying a given property $\mathcal{P}$ requires exponentially much time, if each graph is required to be checked separately. However, *e.g.*, although the hamiltonian problem for graphs is NP-complete, see [8], *hamiltonian*$_\exists$ can be done in time $O(n^2)$ by Corollary 4.4, since one needs only to check that a given graph is not complete bipartite of odd order.

We state first a uniqueness result for switching classes, see, e.g., Zaslavsky [17].

Recall that a *star* of a graph G is a vertex of degree $|V(G)|-1$.

Lemma 5.1
Let G be a graph. For each acyclic graph T on $V(G)$ there exists a unique graph $H \in [G]$ having T as its spanning subgraph. In particular

i. For each subset $A \subseteq V(G) - \{v\}$ there exists a unique $H \in [G]$ such that A consists of the neighbours of v in H

ii. There exists a unique $H \in [G]$ such that v is isolated in H.

iii. There exists a unique $H \in [G]$ such that v is a star of H. □

By Lemma 5.1 some problems that are NP-complete for graphs become easy or even trivial for switching classes. As an example we have

Example 5.2
Every switching class $[G]$ contains a graph that has a
(a) a hamilton path,
(b) a spanning tree with maximum degree $\leq k$,
(c) a spanning tree with at least k leaves (for $2 \leq k \leq n-1$),
(d) a subgraph that is (noninduced) complete bipartite of order k (for $2 \leq k \leq n$).
The existence problems (a) – (d) for graphs are all NP-complete, see [8], but easy for switching classes. □

We say that a family $\mathcal{G}$ of graphs has *bounded minimum degree* k, if each $G \in \mathcal{G}$ has a vertex v of degree $\leq k$.

For a nonnegative integer $k < n$, $|V| = n$, and $v \in V$, there are $\binom{n-1}{k}$ k-subsets of $V - \{v\}$, and therefore we have

Lemma 5.3
Let $v \in V$. There are exactly $\binom{n-1}{k}$ graphs $H \in [G]$ such that $d_H(v) = k$. In particular, there are at most n^k graphs $H \in [G]$ with $d_H(v) \leq k$. □

Because for a fixed k, the bound on the number of graphs is polynomial in n, we obtain the following theorem.

Theorem 5.4
Let $\mathcal{P}$ be a property of graphs that has bounded minimum degree k and such that deciding $\mathcal{P}$ is in P. Then deciding $\mathcal{P}_\exists$ is in P. □

An analogous result can be proved for graphs that always have a vertex of degree at least $n - k$, and we get polynomial algorithms for "sparse" and "dense" graphs.

Corollary 5.5
Let $\mathcal{P}$ be a property of graphs such that $\overline{\mathcal{P}}$ has bounded minimum degree k and such that deciding $\mathcal{P}$ (or equivalently $\overline{\mathcal{P}}$) is in P. Then deciding $\mathcal{P}_\exists$ is in P. □

Example 5.6
(1) It is well known that planarity of a graph can be checked in time linear in the number of vertices (see, e.g., Even [7]). Because planar graphs have bounded minimum degree 5 one can decide in polynomial time whether a switching class contains a planar graph.
(2) By doing a breadth-first search on a graph we can determine in linear time whether a graph is acyclic. Every acyclic graph has a vertex of degree at most 1, and so we can apply Theorem 5.4.
(3) Along the same lines we can conclude from Theorem 5.4 that for a fixed number k there is a polynomial algorithm that verifies whether a switching class contains a k-regular graph. However, as is stated in [13], it is an NP-complete problem to determine whether a switching class contains a k-regular graph for some k. □

6 NP-completeness in switching classes

Let $\mathcal{P}$ be a property of graphs that is preserved under isomorphisms.
We say that $\mathcal{P}$ is

(i) *nontrivial*, if there exists a graph G such that $\overline{\mathcal{P}}(G)$ and there are arbitrarily large graphs G such that $\mathcal{P}(G)$;

(ii) *switch-nontrivial*, if $\mathcal{P}$ is nontrivial and there exists a switching class $[G]$ such that $\overline{\mathcal{P}}(H)$ for all $H \in [G]$;

(iii) *hereditary*, if $\mathcal{P}(G|_A)$ for all $A \subseteq V(G)$ whenever $\mathcal{P}(G)$.

Example 6.1
The following are examples of nontrivial hereditary properties of graphs that are also switch-nontrivial: G is discrete, G is complete, G is bipartite, G is complete bipartite, G is acyclic, G is planar, G has chromatic number $\chi(G) \leq k$ where k is a fixed integer, G is chordal, and G is a comparability graph. □

Yannakakis proved in [16] (see also [8]) the following general completeness result.

Theorem 6.2
Let $\mathcal{P}$ be a nontrivial hereditary property of graphs. Then the problem for instances (G, k) with $k \leq |V(G)|$ whether G has an induced subgraph $G|_A$ such that $|A| \geq k$ and $\mathcal{P}(G|_A)$, is NP-hard. Moreover, if $\mathcal{P}$ is in NP, then the corresponding problem is NP-complete. □

We shall transform this result to a corresponding result for switching classes. For this let $\mathcal{P}$ be a switch-nontrivial hereditary property.

The property $\mathcal{P}$ is nontrivial, because it is switch-nontrivial, and $\mathcal{P}_\exists$ is hereditary, since

$$(G|_A)^\sigma = G^\sigma|_A \tag{2}$$

for all $A \subseteq V(G)$ and $\sigma\colon V(G) \to \mathbb{Z}_2$.

Theorem 6.3
Let $\mathcal{P}$ be a switch-nontrivial hereditary property. Then the following problem for instances (G, k) with $k \le |V(G)|$, is NP-hard: does the switching class $[G]$ contain a graph H that has an induced subgraph $H|_A$ with $|A| \ge k$ and $\mathcal{P}(H|_A)$? If $\mathcal{P} \in$ NP then the corresponding problem is NP-complete.
Proof:
Since $\mathcal{P}_\exists$ is a nontrivial hereditary property, we have by Theorem 6.2 that the problem for instances (G, k) whether G contains an induced subgraph of order at least k satisfying $\mathcal{P}_\exists$, is NP-hard. This problem is equivalent to the problem stated in the theorem, since by (2), for all subsets $A \subseteq V(G)$, $\mathcal{P}_\exists(G|_A)$ if and only if there exists a selector σ such that $\mathcal{P}((G|_A)^\sigma)$. Equation (2) completes the proof.

If $\mathcal{P}$ is in NP then, by Lemma 3.1, the problem is NP-complete. □

6.1 Embedding problems for switching classes

We consider now the *embedding problem* for switching classes. Recall that a graph H can be embedded into a graph G, denoted $H \hookrightarrow G$, if H is isomorphic to a subgraph M of G, that is, there exists an injective function $\alpha\colon V(H) \to V(G)$ such that

$$M(\alpha(u)\alpha(v)) = H(uv)$$

for all $u \neq v$. Note that we do not require that M should be an induced subgraph of G. We write $H \hookrightarrow [G]$, if $H \hookrightarrow G^\sigma$ for some selector σ. The embedding problem for graphs is known to be NP-complete, see [8], and below we show that it remains NP-complete for switching classes.

For a subset $A \subseteq V(G)$ and a selector $\sigma\colon V(G) \to \mathbb{Z}_2$ we have by (2) that $[G|_A] = [G]|_A$, where

$$[G]|_A = \{G^\sigma|_A \mid \sigma\colon V(G) \to \mathbb{Z}_2\}$$

is called the *subclass* of G induced by A.

Hence the switching class $[G]$ contains a graph H which has an independent subset A if and only if the induced subgraph $G|_A$ generates the switching class $[\overline{K}_A]$ of the complete bipartite graphs on A.

An instance of the *independence problem* consists of a graph G and an integer $k \le |V(G)|$, and we ask whether there exists a graph $H \in [G]$ containing an independent set A with k or more vertices. This problem is NP-complete for graphs (that is, the problem whether a graph G contains an independent subset of size $\ge k$, see [8]) and, by Theorem 6.3, it stays NP-complete for switching classes.

Theorem 6.4
The independence problem is NP-complete for switching classes. In particular, the problem whether a switching class $[G]$ has a subclass $[\overline{K}_m]$ with $m \ge k$, is NP-complete. □

Since for all graphs $[\overline{G}] = \overline{[G]}$, Theorem 6.4 yields the following corollary.

Corollary 6.5
For an instance (G, k), where G is a graph and k an integer such that $k \le |V(G)|$, the problem whether $[G]$ contains a graph with clique size $\ge k$, is NP-complete. □

Also, if a complete graph K embeds into a graph G, then K is isomorphic with an induced subgraph of G. From this simple observation we obtain

Corollary 6.6
The embedding problem, $H \hookrightarrow [G]$, for switching classes is NP-complete for the instances (H, G) of graphs. $\square$

Since we can instantiate H with the complete graph on D and then use it to solve the clique problem of Corollary 6.5 using the same value for k, we can conclude the following.

Corollary 6.7
For an instance (G, H, k) for graphs G and H on the same domain D of size n and k an integer with $3 \leq k \leq n-1$, the problem whether there is a set $X \subseteq D$ with $|X| \geq k$ such that $H|_X \in [G|_X]$ is NP-complete. $\square$

The following lemma, see *e.g.* [11], is needed for showing that the embedding problem is equivalent to the problem of embedding a switching class into another switching class.

Lemma 6.8
Let $\alpha\colon H \to G$ be an isomorphism. Then α is an isomorphism between $H^{\sigma\alpha}$ and G^{σ} for all selectors σ. In particular, α maps $[H]$ bijectively onto $[G]$. In particular, if $\alpha\colon H \to G$ is an embedding, then

$$[\alpha(H)] = \{\alpha(H^{\sigma}) \mid \sigma\colon V(H) \to \mathbb{Z}_2\}.$$

Proof:
Indeed, for all distinct $u, v \in V(H)$, and selectors $\sigma\colon V(G) \to \mathbb{Z}_2$,

$$\begin{aligned} G^{\sigma}(\alpha(u)\alpha(v)) &= \sigma\alpha(u) + G(\alpha(u)\alpha(v)) + \sigma\alpha(v) \\ &= \sigma\alpha(u) + H(uv) + \sigma\alpha(v) = H^{\sigma\alpha}(uv) \end{aligned}$$

as required. $\square$

We write $[H] \hookrightarrow [G]$, if all graphs in the switching class $[H]$ can be embedded into graphs of $[G]$. By Lemma 6.8, the condition $[H] \hookrightarrow [G]$ is equivalent to the condition $H \hookrightarrow [G]$, and thus we have

Corollary 6.9
For instances (H, G) of graphs the switching class embedding problem $[H] \hookrightarrow [G]$ is NP-complete. $\square$

Note, however, that the problem to decide whether a given graph H is a subgraph of a graph in $[G]$ is easy. For this one needs only to apply Lemma 5.1.

6.2 3-colourability for switching classes

We consider in this section the problem of 3-colourability. For a given graph $G = (V, E)$ a function $\alpha : V \to C$ for some set C is a proper colouring of G if for all $uv \in E$, $\alpha(u) \neq \alpha(v)$. The *chromatic number* of G is the minimum cardinality over the ranges of possible colourings of G and it is denoted by $\chi(G)$.

The graph 3-colourability problem (for a graph G, is $\chi(G) \leq 3$?) is NP-complete (see, e.g., [8]). In this section we prove that the 3-colourability problem for graphs can be reduced to the corresponding problem for switching classes, hereby proving that the latter is also NP-complete.

Theorem 6.10
The problem whether a switching class $[G]$ contains a graph H with chromatic number 3, is NP-complete.
Proof:
Let $G = (V, E)$ be any graph, and let $G_9 = G + 3C_3$ be the graph which is a disjoint union of G and three disjoint triangles. Let A be the set of the nine vertices of the added triangles.

We claim that $\chi(G) \leq 3$ if and only if $[G_9]$ contains a graph H such that $\chi(H) = 3$. Since the tranformation $G \mapsto G_9$ is in polynomial time, the claim follows.

It is clear that if $\chi(G) \leq 3$ then $\chi(G_9) = 3$.

Suppose then that there exists a selector σ such that $\chi(G_9^\sigma) = 3$, and let $\alpha \colon V \cup A \to \{1, 2, 3\}$ be a proper 3-colouring of G_9^σ.

If σ is constant (either 0 or 1) on V, then G is a subgraph of G_9^σ, and, in this case, $\chi(G) \leq 3$.

Assume that σ is not a constant on V. Since G_9^σ does not contain K_4 as a subgraph, it follows that σ is not a constant selector on any of the added triangles. Further, each of these triangles contains equally many selections of 1 (and of 0, of course), since otherwise the subgraph $G_9^\sigma|_A$ would contain K_4 as its subgraph.

We may assume that each of the added triangles contains exactly one vertex v with $\sigma(v) = 1$ (otherwise consider the complement of σ). Let these three vertices constitute the subset $A_1 \subset A$.

In the 3-colouring α the vertices of A_1 obtain the same colour, say $\alpha(v) = 1$ for all $v \in A_1$; and in each of the added triangles the other two vertices obtain different colours, 2 and 3, since they are adjacent to each other and to a vertex of A_1 in G_9^σ.

Each $v \in V$ with $\sigma(v) = 1$ is connected to all $u \in A - A_1$; consequently, $\alpha(v) = 1$ for these vertices. Therefore the set $B_1 = \sigma^{-1}(1) \cap V$ is an independent subset of G_9^σ. Since σ is constant on B_1, B_1 is independent also in G. The vertices in $V - B_1$ (for which $\sigma(v) = 0$) are all adjacent to the vertices in A_1 in G_9^σ, and therefore these vertices are coloured by 2 or 3. The subsets $B_2 = \alpha^{-1}(2) \cap V$ and $B_3 = \alpha^{-1}(3) \cap V$ are independent in G_9^σ. Again, since σ is constant on both B_2 and B_3, these are independent subsets of G. This shows that $\chi(G) \leq 3$.

7 NP switching problems that are easy for graphs

A graph G is said to be *equally divided* if it consists of two connected components of order $|V(G)|/2$.

Determining whether a graph is equally divided can easily be done in time linear in the number of vertices of the graph. However, we show that for switching classes this problem is NP-complete (already for rather simple graphs).

Theorem 7.1
It is an NP-complete problem to determine whether for a graph G its switching class contains an equally divided graph.
Proof:
By Lemma 3.1 the problem is in NP for switching classes. We now reduce the partition problem (see [8]) to the equal division problem for switching classes. The partition problem asks for a finite set A and its valuation $s : A \to \mathbf{Z}^+$ (positive integers) whether there exists a subset $A' \subseteq A$ such that $\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$.

It is clear that the partition problem remains NP-complete if we require that a solution A' must satisfy the condition: $1 < |A'| < |A| - 1$.

Let (A, s) be any instance of the partition problem. For each $a \in A$, let $G_a = \overline{K}_{s(a)}$ be a discrete graph on $s(a)$ vertices, say on $V_a = \{a_1, \ldots, a_{s(a)}\}$.

Define a graph $G = G(A, s) = \bigoplus_{a \in A} G_a$, the complete connection of the graphs G_a.

If the instance (A, s) has a solution $A' \subseteq A$ with $1 < |A'| < |A| - 1$, then the graph G^σ is equally divided where the selector $\sigma : V(G) \to \mathbf{Z}_2$ is defined by

$$\sigma(a_i) = \begin{cases} 1 \text{ if } a \in A' \\ 0 \text{ if } a \notin A' \end{cases}$$

for all $a \in A$ and $i = 1, \ldots, s(a)$.

On the other hand, suppose that G^σ is equally divided for a selector σ. We show that for all $a \in A$, we have $\sigma(a_i) = \sigma(a_j)$ for all $a_i, a_j \in V_a$. From this it then follows that $A' = \{v \mid \sigma(v) = 1\}$ is a solution for the instance (A, s), and this will prove the claim.

Indeed, assume there exists an $a \in A$ and $a_i, a_j \in V_a$ such that $\sigma(a_i) \neq \sigma(a_j)$. The subgraph G_a^σ of G^σ induced by V_a is a complete bipartite graph, because it is a switch of the discrete graph G_a. By assumption $\sigma(a_i) \neq \sigma(a_j)$, G_a^σ is not discrete, and thus it is connected. Finally, each $b_i \in V_b$ with $b \neq a$ is connected in G^σ to either a_i or a_j (the one with same value as b_i). Therefore G^σ is connected, and this contradicts its being equally divided. $\square$

8 Related problems

Consider an abelian group $(\Gamma, +)$, and interpret 0 as 'definitely an edge', and other elements $a \in \Gamma$ as stating certain doubts about being an edge. (Or, if you wish, let $a \in \Gamma$ measure the nonreliability of an edge.) Then the 'definite graph' P_G of G consisting of the edges e with $G(e) = 0$ can be asked the same questions as before: does there exist a hamiltonian definite graph (eulerian graph, tree) in the switching class $[G]$? If so, is it unique? These questions might be interesting already for Z_3.

References

[1] J.A. Bondy. Pancyclic graphs I. *J. of Combin. Theory, Ser. B*, 11:80 – 84, 1971.

[2] J.A. Bondy and U.S.R. Murty. *Graph Theory with Applications.* North Holland, 1976.

[3] P.J. Cameron. Cohomological aspects of two-graphs. *Math. Z.*, 157:101 – 119, 1977.

[4] D.G. Corneil and R.A. Mathon. *Geometry and Combinatorics: Selected Works of J.J. Seidel.* Academic Press, Boston, 1991.

[5] A. Ehrenfeucht and G. Rozenberg. An introduction to dynamic labeled 2-structures. In A.M. Borzyszkowski and S. Sokolowski, editors, *Mathematical Foundations of Computer Science 1993*, volume 711 of *Lecture Notes in Computer Science*, pages 156–173, 1993.

[6] M.N. Ellingham. Vertex-switching, isomorphisms, and pseudosimilarity. *J. of Graph Theory*, 15:563 – 572, 1991.

[7] S. Even. *Graph Algorithms.* Computer Science Press, Potomac, 1979.

[8] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Freeman, 1979.

[9] J.L. Gross and T.W. Tucker. *Topological Graph Theory.* Wiley, New York, 1987.

[10] J. Hage and T. Harju. Acyclicity of switching classes. Technical Report 12, Leiden University, Department of Computer Science, 1996.

[11] D. Harris and H. Liebeck. Isomorphisms in switching classes. *J. Austral. Math. Soc.Ser. A*, 26:475–486, 19??

[12] J. Kratochvíl. Private communication.

[13] J. Kratochvíl, J. Nešetřil, and O. Zýka. On the computational complexity of Seidel's switching. In *Combinatorics Graphs Complexity*, pages 161 – 166, Prague, 1990. Prachatice.

[14] J.J. Seidel. A survey of two-graphs. In *Proc. Internat. Colloq. Theorie Combinatorie*, Rome, 1973. Acc. Naz. Lincei. See also [4].

[15] J.J. Seidel and D.E. Taylor. Two-graphs, a second survey. In *Proc. Internat. Colloq. on Algebraic Methods in Graph Theory*, Szeged, 1978. See also [4].

[16] M. Yannakakis. Node- and edge-deletion NP-complete problems. In *Proc. 10th Ann. ACM Symp. on theory of Computing*, pages 253 – 264, New York, 1978. ACM.

[17] T. Zaslavsky. Signed graphs. *Discrete Applied Math.*, 4:47–74, 1982. Erratum on p. 248 of volume 5.

[18] T. Zaslavsky. Biased graphs. I. Bias, balance, and gains. *J. of Combin. Theory, Ser. A*, 47:32–52, 1989.