Axioms for Generalized Graphs, illustrated by a Cantor-Bernstein Proposition

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Abstract

The notion of a graph type $\mathcal{T}$ is introduced by a collection of axioms. A graph of type $\mathcal{T}$ (or $\mathcal{T}$-graph) is defined as a set of edges, of which the structure is specified by $\mathcal{T}$. From this, general notions of subgraph and isomorphism of $\mathcal{T}$-graphs are derived. A Cantor-Bernstein (CB) result for $\mathcal{T}$-graphs is presented as an illustration of a general proof for different types of graphs. By definition, a relation $\mathcal{R}$ on $\mathcal{T}$-graphs satisfies the CB property if $A \mathcal{R} B$ and $B \mathcal{R} A$ imply that A and B are isomorphic. In general, the relation 'isomorphic to a subgraph' does not satisfy the CB property. However, requiring the subgraph to be disconnected from the remainder of the graph, a relation that satisfies the CB property is obtained. A similar result is shown for $\mathcal{T}$-graphs with multiple edges.

Introduction

Graphs are used to model the structure of complex systems. As examples one can think of the flowchart of a program, the state transition diagram of a finite automaton, the flowgraph of a CCS process (see [11, 10] and [12, Section 3.4]), or the net of a Petri net [16]. For different purposes, different types of graphs are used which mainly differ with respect to the notion of an edge. For instance, an edge may be directed or undirected, labeled or unlabeled, or it may be a hyperedge which is incident with any number of nodes rather than just two. Formally, in these respective cases, an edge is an ordered pair (x, y) of nodes, or an unordered pair $\{x, y\}$ (see [7]), or a triple (x, a, y) where a is a label, or a set $\{x_1, \ldots, x_n\}$ of nodes (for undirected hypergraphs [1]) or a sequence $(x_1, \ldots, x_n)$ of nodes (for relational systems [9], or directed hypergraphs [6]).

For a Petri net it is natural (but not so usual) to view the places of the net as nodes and the transitions of the net as edges: in this case an edge is an ordered pair (X, Y) where X and Y are sets of nodes (the pre-set and post-set of the transition, respectively). Each time one needs a new notion of graph, one has to introduce concepts (such as homomorphisms, connectedness) that are very similar to the corresponding concepts for ordinary graphs, and one has to repeat the proofs of similar elementary properties of these concepts. To avoid this, we propose in this paper an axiomatic approach to all such types of graphs, and we give an example of how such elementary properties can be proved in one stroke for every type of graph that satisfies the axioms. The basic intuition, suggested above, is that we wish to capture with our axioms all types of (generalized) graphs of which the edges are arbitrary data structures that are built from the nodes: not just sets or sequences of nodes, but also, e.g., sets of sets of nodes, matrices of nodes, records of nodes, or even graphs of nodes. In this setting, a graph homomorphism is, as usual, a mapping f from the nodes of one graph to the nodes of another, which naturally induces a mapping from edges to edges by replacing each node x that occurs in an edge by the node $f(x)$. Thus, an edge (x, a, y) is mapped to the edge $(f(x), a, f(y))$, and an edge (X, Y) is mapped to $(f(X), f(Y))$ where, as usual, $f(X) = \{f(x) \mid x \in X\}$, and similarly for more complicated data structures. Rather than defining a general notion of data structure, our axioms just formulate some obvious laws that these mappings should satisfy.

Our interest in graphs of a type more general than the usual ones came from our investigation [2, 3, 4] into the structure of object-oriented parallel systems, in particular the processes of the π-calculus (see [14, 13]). In general, a (massively) parallel system can be viewed as a (large) collection of (active) objects that communicate with each other through the use of common resources. The structure of such a system can be modeled by a (generalized) graph, as follows. The available resources are the nodes of the graph and the objects are the edges of the graph, where each object is an edge between the resources that it actually uses. It should be clear that it does not suffice to consider hypergraphs (where each object is just a set of resources), because we do not wish to abstract from the internal structure of an object, such as the order in which it uses its resources and the actions it executes inbetween. As a concrete example, a process of the π-calculus is modeled in [2] as a (multi-)set of objects, where each object is an (unordered, possibly infinite) tree of which the nodes are labeled by channel names. Such an object communicates with other objects through channels, which are the resources; the tree indicates the partial order between the communications that are executed by the object. It should be noted here that a process may consist of infinitely many objects with infinitely many resources. As an example, as a result of the replication operator (!) of the π-calculus, a process term of the form $!(\nu x, y)(P(x, y) \mid Q(x, y))$ stands for an infinite collection of objects $P(x_i, y_i)$ and $Q(x_i, y_i)$, $i \in \mathbb{N}$, that communicate through private channels x_i, y_i. Thus, the corresponding graph has infinitely many nodes and edges.

In [3, 4] we have investigated the concepts of structural equivalence (well known from, e.g., [13]) and structural inclusion of parallel systems. Two parallel systems are structurally equivalent if their (generalized) graphs are isomorphic. This definition is based on the fact that the precise names of the resources are irrelevant; thus, the names x and y in the above example are bound by the restriction operator $(\nu x, y)$, where ν stands for 'new'. One parallel system is structurally included in another if the graph of the first is (isomorphic to) a subgraph of the second. Intuitively, this means that one system is a "part" of the other. A natural question is whether two parallel systems that are structurally included into each other are also structurally equivalent. This is obviously true for finite systems, with finitely many resources and objects, but in general not for infinite systems. In fact, for ordinary graphs it is well known (and easy to see) that there exist infinite non-isomorphic graphs which are subgraphs of each other. It is also easy to prove that if each graph is isomorphic to a collection of connected components of the other, then they must be isomorphic: one can then apply the Cantor-Bernstein proposition to the sets of connected components of both graphs. This is the result that we will prove for generalized graphs, as an illustration of our axiomatic framework. For parallel systems it means that for a given parallel system P we only consider those "parts" of P that cannot communicate with the remainder of P. This is certainly a natural special case of structural inclusion, but also a drastic restriction. We will show that for certain natural weaker restrictions the result does not hold. We also note that none of the other notions of structural inclusion investigated in [4] for the π-calculus satisfies the above property.

Since a (generalized) graph may be viewed as a set of edges, i.e., a set of structured objects, the above result may be viewed as an (elementary) Cantor-Bernstein (CB) proposition for sets of structured objects. In other areas of mathematics it is well known that, in the presence of structure, the CB property sometimes holds, but usually does not. Since this is not meant to be a survey on such CB-like results, we just give a few references. In algebra, divisible abelian groups have the CB property, but abelian groups do not (see [17, Exercise 9.34]). In topology, there exist non-homeomorphic closed sets that are (homeomorphic to) open subsets of each other (see [8]). In computability theory, one-one reducibility satisfies the CB property (see [15]), but one-one polynomial-time reducibility does not (see [5]).

In the first section of this paper we define the notion of a "graph type", through a small number of axioms. Each graph type $\mathcal{T}$ defines a particular type of generalized graph, which we call a "$\mathcal{T}$-graph" for lack of a better name. As described above, a $\mathcal{T}$-graph is a set of (generalized) edges, i.e., a set of structured objects built from the nodes of the $\mathcal{T}$-graph. We also prove some elementary properties of graph types, and define the notions of subgraph and isomorphism of $\mathcal{T}$-graphs. In the second section we define $\mathcal{T}$-graph B to be a "free subgraph" of $\mathcal{T}$-graph A if B is isomorphic to a subgraph of A that is not connected to the remainder of A by any edge. Then we show the CB result discussed above: two $\mathcal{T}$-graphs A, B that are free subgraphs of each other, are isomorphic. In particular, we show that the bijection that results from the application of the usual Cantor-Bernstein proposition to the sets of nodes of A and B, is a $\mathcal{T}$-graph isomorphism. In the last section we extend the CB result to multi-graphs, i.e., graphs with multiple edges, by defining a unary operation on graph types that leads from graphs of type $\mathcal{T}$ to multi-graphs of type $\mathcal{T}$.

1 Axioms for Generalized Graphs

In this section, we present the basic definition of a $\mathcal{T}$-graph as a set of edges of a specific structure. In this view, an edge is composed of nodes, and any (non-trivial) mapping on its nodes results in a change of location of the edge, but not in a change of its structure. Thus, every change of nodes induces a structure preserving change of edges, and hence these mappings can be viewed as $\mathcal{T}$-graph homomorphisms. Note that, in a $\mathcal{T}$-graph, there is structure at two levels: at the "local" level of edges, because each edge is a structured object, and at the "global" level of the graph itself, because a graph represents a structure through the incidence relation between edges and nodes in the usual way. Both kinds of structure are preserved by a global change of nodes.

For any set A, we denote by $\mathcal{P}(A)$ the set of all subsets of A. For a mapping $f : A \to B$ and a set $A' \in \mathcal{P}(A)$, the restriction $f{\restriction}A' : A' \to B$ of f to A' is defined as $(f{\restriction}A')(a) = f(a)$, for all $a \in A'$. The image of A' under f is defined as $f(A') = \{f(a) \mid a \in A'\}$. We denote the identity mapping on A by id_A.

Next, the notion of a $\mathcal{T}$-graph is defined axiomatically. To stress the fact that we view its $\mathcal{T}$-edges as being composed of its $\mathcal{T}$-nodes, we will sometimes call $\mathcal{T}$-edges structured objects. Mostly however, if there is no danger of confusion, we refer to $\mathcal{T}$-edges simply as edges, and to $\mathcal{T}$-nodes as nodes.

Definition 1.1 A *graph type* $\mathcal{T}$ is a tuple (V, E, ν, ρ), where

- V is a set of $\mathcal{T}$*-nodes*,
- E is a set of $\mathcal{T}$*-edges* (or *structured objects*),
- ν is a mapping $E \to \mathcal{P}(V)$ (for $e \in E$, $\nu(e) \subseteq V$ is the set of *nodes incident with* e),
- ρ is a mapping that assigns to every mapping $f : V \to V$ a mapping $f^\rho : E \to E$ (f^ρ is the *edge relocation induced by* f),

such that, for all $e \in E$ and $f, g : V \to V$,

(1) $\nu(f^\rho(e)) = f(\nu(e))$,

(2) if $f{\restriction}\nu(e) = g{\restriction}\nu(e)$, then $f^\rho(e) = g^\rho(e)$,

(3) $(g \circ f)^\rho = g^\rho \circ f^\rho$, and

(4) $(\mathrm{id}_V)^\rho = \mathrm{id}_E$.

A $\mathcal{T}$-*graph* is a subset of E.

Intuitively, in Definition 1.1, V is a reservoir of ($\mathcal{T}$-)nodes and E is the collection of all possible ($\mathcal{T}$-)edges between these nodes. Thus, if we let $E = V \times V$, then a $\mathcal{T}$-graph is a directed graph in the usual sense, with isolated nodes excluded (this can be by-passed by viewing isolated nodes as a special kind of edges, i.e., by letting $E = (V \times V) \cup V$).

Note that Definition 1.1 is axiomatic rather than constructive: the precise way in which the $\mathcal{T}$-edges in E are constructed from the $\mathcal{T}$-nodes in V is left unspecified. In fact, the only information that one can retrieve from an edge e are the nodes $\nu(e)$ incident with it. However, this information suffices to determine the global structure of the $\mathcal{T}$-graph.

Also, the precise way in which the relocation f^ρ works is left unspecified. Intuitively, f^ρ changes the nodes $\nu(e)$ incident with e according to f, thereby relocating the edge e, but preserving its structure. In this perspective, a graph type acts as a compound datatype; for instance, in a pseudo-Pascal programming language we could think of the following:

```
type V = <BT>

type E = <CTO> of V,
```

where `<BT>` is a basic type, such as `Integer` or `Char`, and `<CTO>` can be any compound type operator such as `Array` or `Record`. Now a relocation that acts on `E` changes the `V`-values in `E`, but keeps the `<CTO>` structure. If we elaborate on this, a database is a natural example of a $\mathcal{T}$-graph. Information in a database is organized as a collection of records. Each record consists of fields, corresponding to related data values, such as a person's name and address. Thus, a change of data values results in changing the contents of some fields in a record, but the structure of the record remains unaltered. Clearly, records in the database are linked to each other by having the same data values in their fields, for instance when two persons have the same address. Thus, a global change of data values (caused, e.g., by a municipal decision to change the names of certain streets) does not change the global structure of the database; in database terminology, a query that is insensitive to such a change is called *generic*. However, the comparison with databases is misleading because we are mainly interested in *infinite* sets of structured objects.

In Definition 1.1, properties (1) and (2) ensure that the nodes incident to a relocated edge are derived from its original nodes, and that the relocation of an edge depends only on the change of the nodes incident with it. Property (4) states that a relocation can affect only the nodes of an edge, not its structure. Finally by property (3), relocation distributes over composition.

Example 1.2 Below we give some examples of graph types $\mathcal{T} = (V, E, \nu, \rho)$. They should suggest the large variety of formalisms that can be expressed as a specific graph type.

(1) Plain sets. This graph type will be denoted $\mathcal{T}_{\mathrm{s}}$.

$E = V$,

$\nu(e) = \{e\}$, for $e \in E$, and

$f^\rho = f$, for $f : V \to V$.

For $\mathcal{T} = \mathcal{T}_{\mathrm{s}}$, $\mathcal{T}$-edges just consist of one $\mathcal{T}$-node and thus graphs of this type $\mathcal{T}$ are just sets of nodes, and any mapping between these nodes is a relocation. In other words, $\mathcal{T}$-graphs are discrete graphs.

(2) Directed graphs. This graph type will be denoted $\mathcal{T}_{\mathrm{g}}$.

$E = V \times V$,

$\nu((v, w)) = \{v, w\}$, and

$f^\rho((v, w)) = (f(v), f(w))$, for $v, w \in V$ and $f : V \to V$.

For this graph type $\mathcal{T} = \mathcal{T}_{\mathrm{g}}$, a $\mathcal{T}$-graph is a directed graph in the usual sense, as observed earlier (with isolated nodes excluded). Note that f, together with the induced relocation f^ρ, forms a graph homomorphism (in the usual sense). More precisely, if $A, B \subseteq V \times V$ are directed graphs then $f \restriction \nu(A)$ is a graph homomorphism from A to B iff $f^\rho(A) \subseteq B$; in particular, $f^\rho(A)$ is the homomorphic image of A under f. This example is easily extended to arbitrary relational systems.

(3) Sets of binary trees of which the nodes are labeled by integers.

$V = \mathbb{Z}$,

$E =$ `Set of tree`, where
`tree = ^Record (val:`$\mathbb{Z}$`; left, right: tree)`,

$\nu(\texttt{t}) = \{\texttt{t\^{}.val}\} \cup \nu(\texttt{t\^{}.left}) \cup \nu(\texttt{t\^{}.right})$, for `t: tree`
($\nu(\texttt{nil}) = \varnothing$), and

$f^\rho(\texttt{t}) = \texttt{u}$, where $\texttt{u\^{}.val} = f(\texttt{t\^{}.val})$, $\texttt{u\^{}.left} = f^\rho(\texttt{t\^{}.left})$ and $\texttt{u\^{}.right} = f^\rho(\texttt{t\^{}.right})$, for `t: tree` and $f : \mathbb{Z} \to \mathbb{Z}$
($f^\rho(\texttt{nil}) = \texttt{nil}$).

In this Pascal-like example, $\mathcal{T}$-graphs are sets of binary node-labeled trees, where each tree is, as usual, a (pointer to a) node-record (its root) which is labeled by an integer and contains pointers to the (roots of the) direct subtrees. Relabeling of a tree `t` is done by a relocation f^ρ that changes the integer values $\nu(\texttt{t})$ of its nodes by $f : \mathbb{Z} \to \mathbb{Z}$. The reader should realize that there are two kinds of nodes in this example: the $\mathcal{T}$-nodes of the $\mathcal{T}$-graph, which are integers, and the nodes of each $\mathcal{T}$-edge of the $\mathcal{T}$-graph, which are tree nodes; the tree nodes are labeled by $\mathcal{T}$-nodes.

(4) Petri nets.

$$E = \mathcal{P}(V) \times \mathcal{P}(V)$$
$$\nu((X,Y)) = X \cup Y, \text{ and}$$
$$f^\rho((X,Y)) = (f(X), f(Y)), \text{ for } X, Y \subseteq V \text{ and } f : V \to V.$$

In this example, V is a reservoir of places, and E is the collection of transitions (X,Y) that have $X \subseteq V$ as pre-set and $Y \subseteq V$ as post-set. So a $\mathcal{T}$-graph $A \subseteq E$ of this type is the net underlying a Petri net, where $\nu(A) = \bigcup\{\nu(e) \mid e \in A\}$ is the set of its places. Note that, again, isolated places are excluded. Note also that there do not exist two transitions with the same pre- and post-set; in Petri net terminology this means that we only consider T-simple nets. Arbitrary Petri nets can be modeled as multisets over E, see Section 3.

(5) Languages over an alphabet $V \cup C$.

$$E = (V \cup C)^*$$
$$\nu(x_1 \cdots x_k) = \{x_1, \ldots, x_k\} \cap V, \text{ and}$$
$$f^\rho(x_1 \cdots x_k) = x'_1 \cdots x'_k, \text{ where } x'_i = \begin{cases} f(x_i) & \text{if } x_i \in V \\ x_i & \text{if } x_i \in C \end{cases}$$
$$\text{for } k \geq 0,\ x_i \in V \cup C, \text{ and } f : V \to V.$$

In this example, $\mathcal{T}$-graphs are languages over a (possibly infinite) alphabet $V \cup C$, where V and C are assumed disjoint. For a string $e \in E$, only its symbols in V are regarded as nodes (hence if $e \in C^*$, $\nu(e) = \varnothing$), and f^ρ only changes the symbols in V. A $\mathcal{T}$-graph of this type can also be viewed as a set of arrays (of unbounded length) of type $V \cup C$. In the database example explained above, C represents constant data, and hence objects in C^* can be viewed as 'facts'. □

For a $\mathcal{T}$-graph A, we define $\nu(A) = \bigcup\{\nu(e) \mid e \in A\}$ (as we did in the Petri net example above) and for $f : V \to V$, we let $f^\rho(A) = \{f^\rho(e) \mid e \in A\}$ (i.e., $f^\rho(A)$ is the image of A under f^ρ). Intuitively, $\nu(A)$ is the set of nodes of A and $f^\rho(A)$ is the homomorphic image of A under f. The next lemma shows that these mappings ν and ρ satisfy property (1)–(4) of Definition 1.1. Intuitively, this means that `Set of` is also a `CTO`. Note that applying this operator to the graph type $\mathcal{T}_\mathrm{s}$ of plain sets (Example 1.2(1)) results in the graph type of (usual) hypergraphs, in which each edge is a set of nodes. Applying it to the graph type $\mathcal{T}_\mathrm{g}$ of directed graphs (Example 1.2(2)) one obtains generalized graphs of which the edges are directed graphs. The routine proof of the lemma is left to the reader.

Lemma 1.3 *Let (V, E, ν, ρ) be a graph type. Then $(V, \mathcal{P}(E), \nu, \rho)$ is a graph type.*

For a $\mathcal{T}_{\mathrm{g}}$-graph A (see Example 1.2(2)), it is obvious that if we take a mapping f that is injective on its nodes $\nu(A)$, then A and $f^\rho(A)$ are isomorphic directed graphs. Moreover, f^ρ is then injective on the edges of A. In general, for every $\mathcal{T}$-graph A and for every mapping $f : V \to V$ that is injective on $\nu(A)$, there exists a mapping $\overline{f}_A : V \to V$ such that $(\overline{f}_A)^\rho(f^\rho(e)) = e$ for all $e \in A$. In fact, define $\overline{f}_A = (f{\restriction}\nu(A))^{-1} \cup \mathrm{id}_{V - f(\nu(A))}$. Then the next lemma shows that, restricted to A, $(\overline{f}_A)^\rho$ is the inverse of f^ρ.

Lemma 1.4 *For a $\mathcal{T}$-graph A, if $f : V \to V$ is injective on $\nu(A)$, then f^ρ is injective on A. In particular, $(\overline{f}_A)^\rho(f^\rho(e)) = e$ for all $e \in A$.*
Proof Let $e \in A$. Since $(\overline{f}_A \circ f){\restriction}\nu(e) = \mathrm{id}_V{\restriction}\nu(e)$, we have

$$(\overline{f}_A)^\rho(f^\rho(e)) = (\overline{f}_A \circ f)^\rho(e) = (\mathrm{id}_V)^\rho(e) = \mathrm{id}_E(e) = e,$$

by Definition 1.1(3), (2), and (4), respectively. □

Next, we define isomorphism of $\mathcal{T}$-graphs and the notion of a subgraph of a $\mathcal{T}$-graph.

Definition 1.5 For $\mathcal{T}$-graphs A and B, an *isomorphism between A and B* is a mapping $f : V \to V$ such that f is injective on $\nu(A)$ and $f^\rho(A) = B$. If such an isomorphism exists, A and B are *isomorphic*, denoted $A \simeq B$.

Note that if $f^\rho(A) = B$ then $f(\nu(A)) = \nu(B)$ by Lemma 1.3 and Definition 1.1(1). Thus an isomorphism is a bijection (between nodes) that relocates $\mathcal{T}$-edges, but preserves their structure ánd the global structure of the $\mathcal{T}$-graph, as discussed in the beginning of this section. Notice that in the set case ($\mathcal{T} = \mathcal{T}_{\mathrm{s}}$), an isomorphism is nothing more than a bijection, i.e., two sets are isomorphic if they are equipotent. In the directed graph case ($\mathcal{T} = \mathcal{T}_{\mathrm{g}}$), it corresponds to the usual definition of isomorphism of directed graphs (and, more generally, to the definition of isomorphism of relational systems).

Observe that $\simeq$ is an equivalence relation: let $A \simeq B$ and let f be the isomorphism between A and B. Then also $B \simeq A$, by the isomorphism $\overline{f}_A$ (cf. Lemma 1.4). Transitivity follows from Definition 1.1(3), and reflexivity from Definition 1.1(4).

For $\mathcal{T}$-graphs A and B, we define B to be a *concrete subgraph* of A, if $B \subseteq A$. Furthermore, B is an *(abstract) subgraph* of A, if B is isomorphic to a concrete subgraph of A, i.e., if $f^\rho(B) \subseteq A$ for some $f : V \to V$ that is injective on $\nu(B)$. Note that for $\mathcal{T}_{\mathrm{g}}$, this corresponds to the usual definitions of concrete subgraph and subgraph, respectively.

Although this will not really be needed in what follows, we now define a $\mathcal{T}$-graph to be *connected* if there do not exist nonempty concrete subgraphs B and

C of A such that $A = B \cup C$ and $\nu(B) \cap \nu(C) = \varnothing$. A *connected component* of a $\mathcal{T}$-graph is a maximal connected concrete subgraph of A. Clearly, the connected components A_i of A form a partition of A such that $\nu(A_i) \cap \nu(A_j) = \varnothing$ for $i \neq j$. It should be clear that these definitions are the appropriate generalizations of the ones for $\mathcal{T}_{\mathrm{g}}$.

2 The Cantor-Bernstein Proposition for Graph Types

It is well known from set theory that two sets Γ and Λ are equipotent if Γ is equipotent to a subset of Λ and vice versa, i.e., if there exist injections $\phi_1 : \Gamma \to \Lambda$ and $\phi_2 : \Lambda \to \Gamma$. This is the Cantor-Bernstein proposition (see for instance [9] among numerous other works on set theory). The central idea in this proposition lies in the construction of a bijection between any two such sets. Observe that since in the above case ϕ_2 is a bijection between Λ and $\Delta = \phi_2(\Lambda)$, it suffices to show the existence of a bijection between Γ and Δ. For completeness sake we state its construction below, as well as the proof that it is a bijection. We will consider functions $\phi : \Sigma \to \Sigma$ where Σ is a set containing Γ, because this is uniform with the functions $f : V \to V$ and $f^\rho : E \to E$ of a graph type.

Definition 2.1 Let $\phi : \Sigma \to \Sigma$ be injective on $\Gamma \subseteq \Sigma$ and let $\phi(\Gamma) \subseteq \Delta \subseteq \Gamma$. The *Bernstein modification of* ϕ *with respect to* (Γ, Δ), denoted $\phi^{\mathbf{B}} : \Sigma \to \Sigma$, is defined as

$$\phi^{\mathbf{B}}(x) = \begin{cases} \phi(x) & \text{if } x \in \bigcup_{i \geq 0} \phi^i(\Gamma - \Delta) \\ x & \text{otherwise.} \end{cases}$$

The Bernstein modification $\phi^{\mathbf{B}}$ of ϕ, with respect to (Γ, Δ), is depicted in Fig. 1. The dark areas inside Γ show the set $\bigcup_{i \geq 0} \phi^i(\Gamma - \Delta)$.

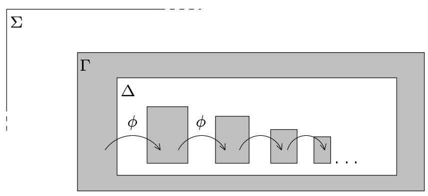

Figure 1: The Bernstein modification of ϕ

Proposition 2.2 *For every mapping* $\phi : \Sigma \to \Sigma$ *and all sets* $\Gamma, \Delta \subseteq \Sigma$ *such that* ϕ *is injective on* Γ *and* $\phi(\Gamma) \subseteq \Delta \subseteq \Gamma$, *the Bernstein modification* $\phi^{\mathbf{B}} : \Sigma \to \Sigma$ *of* ϕ *with respect to* (Γ, Δ) *is injective on* Γ *and moreover,* $\phi^{\mathbf{B}}(\Gamma) = \Delta$.

Proof Let $\Omega = \bigcup_{i \geq 0} \phi^i(\Gamma - \Delta)$. Note that $\Omega \subseteq \Gamma$.

To prove injectivity of $\phi^{\mathbf{B}}$ on Γ, assume $x, y \in \Gamma$ with $x \neq y$. We consider four cases. If $x \notin \Omega$ and $y \notin \Omega$, then $\phi^{\mathbf{B}}(x) = x$ and $\phi^{\mathbf{B}}(y) = y$. Hence $\phi^{\mathbf{B}}(x) \neq \phi^{\mathbf{B}}(y)$. If $x \in \Omega$ and $y \notin \Omega$, then $\phi^{\mathbf{B}}(x) \in \Omega$ and since $\phi^{\mathbf{B}}(y) = y \notin \Omega$, we have $\phi^{\mathbf{B}}(x) \neq \phi^{\mathbf{B}}(y)$. The case in which $x \notin \Omega$ and $y \in \Omega$ is proven similarly. If both $x \in \Omega$ and $y \in \Omega$, then $\phi^{\mathbf{B}}(x) = \phi(x)$ and $\phi^{\mathbf{B}}(y) = \phi(y)$. Since, by assumption, ϕ is injective on Γ, we have $\phi^{\mathbf{B}}(x) \neq \phi^{\mathbf{B}}(y)$.

Since obviously $\phi^{\mathbf{B}}(\Gamma) \subseteq \Delta$, it remains to show that $\Delta \subseteq \phi^{\mathbf{B}}(\Gamma)$. Assume $x \in \Delta$. If $x \in \Omega$, then there exists $p \geq 1$ such that $x \in \phi^p(\Gamma - \Delta)$. Hence there exists $y \in \phi^{p-1}(\Gamma - \Delta)$ with $x = \phi(y)$, and thus $\phi^{\mathbf{B}}(y) = x$, by definition of $\phi^{\mathbf{B}}$. If $x \notin \Omega$, we immediately derive $\phi^{\mathbf{B}}(x) = x \in \Gamma$. $\square$

Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type. For any pre-order $\mathcal{R} \subseteq \mathcal{P}(E) \times \mathcal{P}(E)$, we will say that $\mathcal{R}$ satisfies the *Cantor-Bernstein (CB) property*, if $A \mathcal{R} B$ and $B \mathcal{R} A$ imply $A \simeq B$, for every pair A, B of $\mathcal{T}$-graphs. Now if we view sets Γ and Λ as $\mathcal{T}_{\mathrm{s}}$-graphs (as in Example 1.2(1)), then indeed Proposition 2.2 proves that Γ and Λ are isomorphic (in the sense of Definition 1.5), if Γ is a subgraph of Λ and vice versa. Thus, for sets, the pre-order 'subgraph of' (as defined at the end of Section 1) satisfies the CB property. For $\mathcal{T}_{\mathrm{g}}$ however, this does not hold, as the following example shows.

Example 2.3 Let $\mathcal{T} = \mathcal{T}_{\mathrm{g}}$ (see Example 1.2(2)) with $V = \mathbb{N} = \{0, 1, 2, \ldots\}$, and let $A = \{(3n+i, 3n+j) \mid n \geq 0 \text{ and } 0 \leq i, j \leq 2\}$, $B' = \{(3n+i, 3n+j) \mid n \geq 1 \text{ and } 0 \leq i, j \leq 2\}$, and $B = B' \cup \{(1,2), (2,1)\}$, as depicted in Fig. 2 (where a double arrow represents two edges pointing in opposite directions, and loops are omitted). Observe that A and B are equivalence relations on V. Clearly, B is a (concrete) subgraph of A. Also, A is a subgraph of B, since A and B' are isomorphic (by the isomorphism $f(k) = k+3$). However, A and B are evidently not isomorphic. This example is easily generalized to arbitrary V and arbitrary equivalence relations A and B, of which all equivalence classes have the same cardinality, except one which has a smaller cardinality. Thus, the CB property fails to hold for very simple graphs already. $\square$

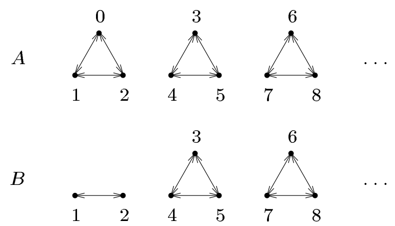

Figure 2: A and B are not isomorphic

The example above suggests that, if we want the subgraph relation to satisfy the CB property, we should strengthen the concrete subgraph relation $B \subseteq A$ in such a way that B is not connected to $A - B$. This means that B and $A - B$ should not have any nodes in common.

Definition 2.4 For $\mathcal{T}$-graphs A and B, B is a *concrete free subgraph* of A, denoted $B \subseteq^\nu A$, if $B \subseteq A$ and $\nu(B) \cap \nu(A - B) = \varnothing$.

In other words, $B \subseteq^\nu A$ iff $B \subseteq A$ and $\nu(A - B) = \nu(A) - \nu(B)$. Note that $\subseteq^\nu$ is a partial order on $\mathcal{P}(E)$. To show transitivity, let $C \subseteq^\nu B \subseteq^\nu A$. Then $\nu(A - C) = \nu((A - B) \cup (B - C)) = \nu(A - B) \cup \nu(B - C) = (\nu(A) - \nu(B)) \cup (\nu(B) - \nu(C)) = \nu(A) - \nu(C)$. Reflexivity and antisymmetry are obvious.

Definition 2.5 For $\mathcal{T}$-graphs A and B, B is an *(abstract) free subgraph* of A, denoted $B \lesssim A$, if B is isomorphic to a concrete free subgraph of A, i.e., $f^\rho(B) \subseteq^\nu A$ for some $f : V \to V$ that is injective on $\nu(B)$.

Clearly (cf. the end of Section 1), a free subgraph B of A is a graph that is isomorphic to a collection of connected components of A. Note that in the set case, the notion of concrete free subgraph coïncides with 'subset', since for any pair of sets A and B, we trivially have $B \cap (A - B) = \varnothing$.

We need the following lemma to show that $\lesssim$ is a pre-order on $\mathcal{P}(E)$. It expresses that free subgraphs are preserved under injective relocations; in other words, an isomorphism preserves connected components.

Lemma 2.6 *For $\mathcal{T}$-graphs A and B, if $B \subseteq^\nu A$ and $f : V \to V$ is injective on $\nu(A)$, then $f^\rho(B) \subseteq^\nu f^\rho(A)$.*
Proof Clearly, $f^\rho(B) \subseteq f^\rho(A)$. Note that by Lemma 1.4, f^ρ is injective on A. Hence $f^\rho(A) - f^\rho(B) = f^\rho(A - B)$ and thus

$$\begin{aligned} \nu(f^\rho(B)) \cap \nu(f^\rho(A) - f^\rho(B)) &= f(\nu(B)) \cap f(\nu(A - B)) \\ &= f(\nu(B) \cap \nu(A - B)) \\ &= \varnothing, \end{aligned}$$

by Definition 1.1(1) (see Lemma 1.3), since f is injective on $\nu(A)$, and since $\nu(B) \cap \nu(A - B) = \varnothing$, respectively. Hence $f^\rho(B) \subseteq^\nu f^\rho(A)$. □

To show transitivity of $\lesssim$, let $C \lesssim B \lesssim A$, or, equivalently, let $f^\rho(C) \subseteq^\nu B$ and $g^\rho(B) \subseteq^\nu A$, for $f, g : V \to V$, injective on $\nu(C)$ and $\nu(B)$, respectively. By Definition 1.1(3) and Lemma 2.6, we have $(g \circ f)^\rho(C) = g^\rho(f^\rho(C)) \subseteq^\nu g^\rho(B) \subseteq^\nu A$. Hence $C \lesssim A$, by transitivity of $\subseteq^\nu$ and since $g \circ f$ is injective on $\nu(C)$. Consequently, $\lesssim$ is a pre-order.

In Theorem 2.10 we will prove that indeed $\lesssim$ satisfies the CB property, i.e., if $g^\rho(A) \subseteq^\nu C$ and $h^\rho(C) \subseteq^\nu A$, for mappings $g, h : V \to V$, injective on $\nu(A)$ and $\nu(C)$, respectively, then $A \simeq C$. Before starting the formal proof, we sketch its outlines. By an argument similar to the one in the first paragraph of this section, the CB property can be reduced to a statement involving only one mapping $f = h \circ g$: if we let $B = h^\rho(C)$, then we obtain $f^\rho(A) \subseteq^\nu B \subseteq^\nu A$, and hence it suffices to show that A and B are isomorphic. It is this form of the CB property that will be proved in Theorem 2.9, i.e., if $f^\rho(A) \subseteq^\nu B \subseteq^\nu A$, for two $\mathcal{T}$-graphs A and B, and a mapping $f : V \to V$, injective on the nodes of A, then $A \simeq B$. Moreover, we will show that, in particular, the Bernstein modification $f^{\mathbf{B}}$ of f with respect to $(\nu(A), \nu(B))$ is an isomorphism between A and B. In order to prove this, we will show in Lemma 2.8 that $f^{\mathbf{B}\rho}$ is equal to $f^{\rho\mathbf{B}}$, where the latter Bernstein modification is taken with respect to (A, B). Note that by Lemma 1.4, f^ρ is injective on A, so $f^{\rho\mathbf{B}}$ exists. Observe that by Definition 2.1

$$f^{\rho\mathbf{B}}(e) = \begin{cases} f^\rho(e) & \text{if } e \in \bigcup_{i\geq 0} f^{\rho i}(A - B) \\ e & \text{otherwise,} \end{cases}$$

and

$$f^{\mathbf{B}}(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_{i\geq 0} f^{i}(\nu(A) - \nu(B)) \\ x & \text{otherwise.} \end{cases}$$

To prove that $f^{\mathbf{B}\rho}(e) = f^{\rho\mathbf{B}}(e)$, we will show that for every $x \in \nu(e)$, $e \in \bigcup_{i\geq 0} f^{\rho i}(A - B)$ iff $x \in \bigcup_{i\geq 0} f^{i}(\nu(A) - \nu(B))$. In fact, we will prove in Lemma 2.7 a claim that is stronger than we need: for an edge $e \in A$ and a node $x \in \nu(e)$, we show that $e \in f^{\rho i}(A - B)$ iff $x \in f^{i}(\nu(A) - \nu(B))$, or, as depicted in Fig. 3, both A and $\nu(A)$ are partitioned into dark areas, the iterations of f^ρ and f, respectively, and one white area. Each 'dark set' in A can be assigned a number by its iteration; the same holds for 'dark sets' in $\nu(A)$. Now a node incident with an edge in the ith dark set in A, must be an element of the ith dark set in $\nu(A)$, and furthermore, the white area in $\nu(A)$ represents *exactly* the set of nodes incident with the edges in the white area in A.

We now start the formal proof of the CB property of $\lesssim$.

Lemma 2.7 *Let, for $\mathcal{T}$-graphs A and B, $B \subseteq^\nu A$, and let $f : V \to V$ be injective on $\nu(A)$ with $f^\rho(A) \subseteq^\nu B$. Let, furthermore, $e \in A$ and $x \in \nu(e)$. Then for every $i \geq 0$, $e \in f^{\rho i}(A - B)$ if and only if $x \in f^{i}(\nu(A) - \nu(B))$.*

Proof We will use the following property of concrete free subgraphs: for all $\mathcal{T}$-graphs C, if $C \subseteq^\nu A$, then for all $e \in A$ and $x \in \nu(e)$ the following holds:

$$e \in C \iff x \in \nu(C).$$

Since $f^\rho(A) \subseteq^\nu B \subseteq^\nu A$ and $A - B \subseteq^\nu A$, by i applications of Lemma 2.6 we have $f^{\rho i}(A - B) \subseteq^\nu A$, for all $i \geq 0$. Hence by the above property, $e \in f^{\rho i}(A - B)$ iff $x \in \nu(f^{\rho i}(A - B)) = f^{i}(\nu(A - B)) = f^{i}(\nu(A) - \nu(B))$, using the remark below Definition 2.4. □

Lemma 2.8 *For $\mathcal{T}$-graphs A and B, if $B \subseteq^\nu A$ and $f^\rho(A) \subseteq^\nu B$, with $f : V \to V$ injective on $\nu(A)$, then $f^{\mathbf{B}\rho} = f^{\rho\mathbf{B}}$, where the Bernstein modifications are taken with respect to $(\nu(A), \nu(B))$ and (A, B), respectively.*

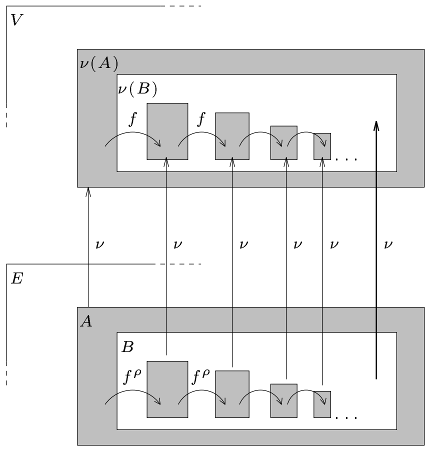

Figure 3: $\forall x \in \nu(e)$: $e \in {f^{\rho}}^{i}(A - B)$ iff $x \in f^{i}(\nu(A) - \nu(B))$

Proof First observe that $f(\nu(A)) \subseteq \nu(B) \subseteq \nu(A)$, and hence f, $\nu(A)$, and $\nu(B)$ satisfy the requirements of Definition 2.1. By Lemma 1.4, this also holds for f^{ρ}, A, and B. Let $C = \bigcup_{i \geq 0} {f^{\rho}}^{i}(A - B)$. Let $e \in E$. By Lemma 2.7, $e \in C$ iff $x \in \bigcup_{i \geq 0} f^{i}(\nu(A) - \nu(B))$ for all $x \in \nu(e)$. Hence

$$f^{\mathbf{B}} \restriction \nu(e) = \begin{cases} f \restriction \nu(e) & \text{if } e \in C \\ \mathrm{id}_V \restriction \nu(e) & \text{otherwise.} \end{cases}$$

Consequently, by Definition 1.1(2) we obtain

$$f^{\mathbf{B}\rho}(e) = \begin{cases} f^{\rho}(e) & \text{if } e \in C \\ (\mathrm{id}_V)^{\rho}(e) & \text{otherwise.} \end{cases}$$

Since by Definition 1.1(4) $(\mathrm{id}_V)^{\rho} = \mathrm{id}_E$, we conclude $f^{\mathbf{B}\rho} = f^{\rho\mathbf{B}}$. $\square$

Next, we state the main CB results of this section; as mentioned earlier, Theorem 2.10 extends the classical CB proposition to graph types.

Theorem 2.9 *For $\mathcal{T}$-graphs A and B, if $B \subseteq^{\nu} A$ and $f^{\rho}(A) \subseteq^{\nu} B$, with $f : V \to V$ injective on $\nu(A)$, then $f^{\mathbf{B}}$ is an isomorphism between A and B, where the Bernstein modification is taken with respect to $(\nu(A), \nu(B))$.*

Proof It has to be shown that $f^{\mathbf{B}}$ is injective on $\nu(A)$ and $f^{\mathbf{B}\rho}(A) = B$. By Lemma 2.8 we have $f^{\mathbf{B}\rho} = f^{\rho\mathbf{B}}$, where the latter Bernstein modification is taken with respect to (A, B). As observed in the proof of Lemma 2.8, the triples $(f, \nu(A), \nu(B))$ and (f^{ρ}, A, B) satisfy the requirements of Proposition 2.2. Thus, by two applications of Proposition 2.2, $f^{\mathbf{B}}$ is injective on $\nu(A)$ and $f^{\rho\mathbf{B}}(A) = B$. $\square$

Thus, if $A \lesssim B \subseteq^\nu A$, then $A \simeq B$.

Theorem 2.10 *For $\mathcal{T}$-graphs A and B, if A is a free subgraph of B and vice versa, then A and B are isomorphic.*

Proof Let $A \lesssim B \lesssim A$. Since $B \lesssim A$, $B \simeq B' \subseteq^\nu A$ for some $\mathcal{T}$-graph B'. Since $\lesssim$ is a pre-order, $A \lesssim B \simeq B'$ implies $A \lesssim B'$. Hence $A \lesssim B' \subseteq^\nu A$. By Theorem 2.9, we have $A \simeq B'$ and hence $A \simeq B$. □

Observe that if $A \lesssim B$ and $B \lesssim A$ by the mappings $f, g : V \to V$ (injective on $\nu(A)$ and $\nu(B)$, respectively), i.e., $f^\rho(A) \subseteq^\nu B$ and $g^\rho(B) \subseteq^\nu A$, then $(g \circ f)^\rho(A) \subseteq^\nu g^\rho(B)$ by Definition 1.1(3) and Lemma 2.6. Hence by Theorem 2.9, $(g \circ f)^{\mathbf{B}}$ is an isomorphism between A and $g^\rho(B)$. Since by Lemma 1.4, $(\overline{g}_B)^\rho(g^\rho(B)) = B$, we have that $\overline{g}_B \circ (g \circ f)^{\mathbf{B}}$ is an isomorphism between A and B. This is the usual bijection between their sets of nodes, cf. the introduction of this section.

Example 2.11 Let $\mathcal{T} = \mathcal{T}_{\mathrm{g}}$ with $V = \mathbb{N}$, and let $A = \{(5n, 5n{+}1), (5n{+}1, 5n{+}2), (5n{+}3, 5n{+}4) \mid n \geq 0\}$ and $B = \{(5n, 5n{+}1), (5n{+}2, 5n{+}3), (5n{+}3, 5n{+}4) \mid n \geq 0\}$ be the directed graphs shown in Fig. 4. Let $f, g : \mathbb{N} \to \mathbb{N}$ be defined by

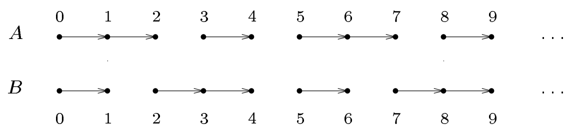

Figure 4: A and B are isomorphic by h

$f(k) = k{+}2$ and $g(k) = k{+}3$. Now $f^\rho(A) \subseteq^\nu B$ and $g^\rho(B) \subseteq^\nu A$, as the reader easily verifies. Clearly A and B are isomorphic. Furthermore, $h = \overline{g}_B \circ (g \circ f)^{\mathbf{B}}$ is an isomorphism between A and B. In fact, as the reader can check,

$$(g \circ f)^{\mathbf{B}}(k) = \begin{cases} k+5 & \text{if } k = 5n, 5n+1, 5n+2 \\ k & \text{if } k = 5n+3, 5n+4, \end{cases}$$

where the Bernstein modificaton is taken with respect to $(\mathbb{N}, \mathbb{N}+3)$, and

$$\overline{g}_B(k) = \begin{cases} k-3 & \text{if } k \geq 3 \\ k & \text{otherwise.} \end{cases}$$

Hence

$$h(k) = \begin{cases} k+2 & \text{if } k = 5n, 5n+1, 5n+2 \\ k-3 & \text{if } k = 5n+3, 5n+4. \end{cases}$$

□

Of course, the CB property is only manifest for infinite $\mathcal{T}$-graphs, i.e., $\mathcal{T}$-graphs with an infinite number of edges. The next result states that, restricted to $\mathcal{T}$-graphs having a finite number of nodes, the CB property even holds for the subgraph relation.

Theorem 2.12 *For $\mathcal{T}$-graphs A and B with finite $\nu(A)$ and finite $\nu(B)$, if A is a subgraph of B and vice versa, then A and B are isomorphic.*
Proof As before, it suffices to prove this for the case that B is a concrete subgraph of A (cf. the proof of Theorem 2.10). Thus, let $B \subseteq A$ and $f^\rho(A) \subseteq B$, where $f : V \to V$ is a mapping that is injective on $\nu(A)$. We will show that $f^\rho(A) = B$, i.e., f is an isomorphism between A and B. First observe that $\nu(B) \subseteq \nu(A)$ and $f(\nu(A)) \subseteq \nu(B)$. Since $\nu(A)$ and $\nu(B)$ are finite and f is injective on $\nu(A)$, we have in fact $\nu(B) = \nu(A) = f(\nu(A))$, so f is a permutation on $\nu(A)$. Now let $e \in B$. We prove that $e \in f^\rho(A)$. Since f is a permutation on $\nu(B)$, and since $\nu(B)$ is finite, there exists a number $k \geq 1$ such that $f^k \restriction \nu(B) = \mathrm{id}_{\nu(B)}$ and hence $f^k \restriction \nu(e) = \mathrm{id}_V \restriction \nu(e)$. By Definition 1.1(2,3,4), we have $(f^\rho)^k(e) = (f^k)^\rho(e) = (\mathrm{id}_V)^\rho(e) = \mathrm{id}_E(e) = e$. Since $k \geq 1$, we conclude that $e \in f^\rho(A)$, and hence $f^\rho(A) = B$. $\square$

The final part of this section is devoted to one remaining question: how restrictive is the notion of free subgraph? Upto this point, there is no guarantee that there does not exist another, weaker relation, of which the abstract version $\lesssim$ also satisfies the CB property. In the remainder of this section we will show that, provided it satisfies some "natural" assumptions, such an extension of 'free subgraph' does not exist. We need some new terminology to explain these assumptions.

A relation $\sqsubseteq\ \subseteq \mathcal{P}(E) \times \mathcal{P}(E)$ is called an *inclusion relation*, whenever $B \sqsubseteq A$ implies that $B \subseteq A$. For an inclusion relation $\sqsubseteq$ we define the *abstract version of* $\sqsubseteq$ to be $\mathrm{Abstr}(\sqsubseteq) = \{(B, A) \mid f^\rho(B) \sqsubseteq A$ for some $f : V \to V$ that is injective on $\nu(B)\}$. Thus, $\lesssim\ = \mathrm{Abstr}(\subseteq^\nu)$. We say that $\sqsubseteq$ is *closed under intersection*, whenever $B \sqsubseteq A$ implies $B \cap C \sqsubseteq A \cap C$; $\sqsubseteq$ is *closed under ν-disjoint union*, whenever $B \sqsubseteq A$ and $\nu(A) \cap \nu(C) = \varnothing$ imply that $B \cup C \sqsubseteq A \cup C$; $\sqsubseteq$ is *closed under isomorphism*, whenever $B \sqsubseteq A$ implies $f^\rho(B) \sqsubseteq f^\rho(A)$, for every $f : V \to V$ that is injective on $\nu(A)$. Note that $\subseteq^\nu$ is closed under intersection, ν-disjoint union, and isomorphism (cf. Lemma 2.6).

The next result expresses that whenever the free subgraph relation is augmented with pairs $B \subseteq A$ such that $\nu(B) \cap \nu(A - B) \neq \varnothing$, and the resulting inclusion relation is closed under intersection, ν-disjoint union, and isomorphism, then its abstract version does not satisfy the CB property. It assumes a graph type with infinite V, which is a neccesary requirement by Theorem 2.12.

Theorem 2.13 *Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type with infinite V. For any inclusion relation $\sqsubseteq\ \subseteq \mathcal{P}(E) \times \mathcal{P}(E)$ that strictly contains $\subseteq^\nu$ and that is closed under intersection, ν-disjoint union, and isomorphism,* $\mathrm{Abstr}(\sqsubseteq)$ *does not satisfy the CB property.*

Proof Assume that $\sqsubseteq$ satisfies the conditions in the theorem. Let A and B be $\mathcal{T}$-graphs with $B \sqsubseteq A$, but $B \not\subseteq^{\nu} A$. Since $\sqsubseteq$ is an inclusion relation, $B \subseteq A$. Let $e \in B$ and $e' \in A - B$ such that $\nu(e) \cap \nu(e') \neq \varnothing$. We will show that there exist non-isomorphic $\mathcal{T}$-graphs A' and B' such that both $(B', A') \in \mathrm{Abstr}(\sqsubseteq)$ and $(A', B') \in \mathrm{Abstr}(\sqsubseteq)$. Since $\sqsubseteq$ is closed under intersection, we have $\{e\} = B \cap \{e, e'\} \sqsubseteq A \cap \{e, e'\} = \{e, e'\}$. Let $V = \bigcup_{i \in \mathbb{N}} V_i$ with mutually disjoint V_i, such that there exist bijections $f_i : V \to V_i$. Note that this can be accomplished by the infiniteness of V (and by assuming the axiom of choice from set theory). Let $e_i = f_i^{\rho}(e)$ and $e'_i = f_i^{\rho}(e')$. Observe that $\nu(\{e_i, e'_i\}) \cap \nu(\{e_j, e'_j\}) = \varnothing$ iff $i \neq j$. Moreover, $\nu(e_i) \cap \nu(e'_i) \neq \varnothing$, since $\nu(e) \cap \nu(e') \neq \varnothing$. Since $\sqsubseteq$ is closed under isomorphism, we have $\{e_0\} = \{f_0^{\rho}(e)\} \sqsubseteq \{f_0^{\rho}(e), f_0^{\rho}(e')\} = \{e_0, e'_0\}$. Now let $C = \bigcup_{i \in \mathbb{N} - \{0\}} \{e_i, e'_i\}$, and let $A' = \{e_0, e'_0\} \cup C$ and $B' = \{e_0\} \cup C$. Since $\{e_0\} \sqsubseteq \{e_0, e'_0\}$ and $\sqsubseteq$ is closed under ν-disjoint union, we have $B' \sqsubseteq A'$ and hence $(B', A') \in \mathrm{Abstr}(\sqsubseteq)$. Conversely, let $f = \bigcup_{i \in \mathbb{N}} (f_{i+1} \circ f_i^{-1})$. Note that f is a mapping $V \to V$ that is injective on V with $f^{\rho}(e_i) = e_{i+1}$ and $f^{\rho}(e'_i) = e'_{i+1}$ for all $i \in \mathbb{N}$. Hence we have $f^{\rho}(A') = C \subseteq^{\nu} B'$ and thus $(A', B') \in \mathrm{Abstr}(\sqsubseteq)$. However, A' and B' are not isomorphic: B' has an edge (viz., e_0) that has no nodes in common with any other edge, but A' does not have such an edge. $\square$

Example 2.14 To show that there exist numerous inclusion relations that are closed under intersection, ν-disjoint union, and isomorphism, let A and B be $\mathcal{T}$-graphs for an arbitrary graph type $\mathcal{T}$. If $B \subseteq A$, then we call an edge $e \in A - B$ a (B, A)-*crossing*, if $\nu(B) \cap \nu(e) \neq \varnothing$, and we denote by $\mathrm{cross}(B, A)$ the set of all (B, A)-crossings. Let $P : \mathcal{P}(E) \to \{\mathrm{true}, \mathrm{false}\}$ be an arbitrary hereditary property of $\mathcal{T}$-graphs that is preserved by isomorphisms, i.e., if $P(A)$ holds for a $\mathcal{T}$-graph A, then $P(B)$ holds for every subset B of A and $P(f^{\rho}(A))$ holds for every mapping $f : V \to V$ that is injective on $\nu(A)$. Intuitively, $P(\mathrm{cross}(B, A))$ can be viewed as a measure of the way in which B is connected to $A - B$. Then, as the reader can check, the relation $\sqsubseteq_P$ defined by

$$B \sqsubseteq_P A \text{ iff } B \subseteq A \text{ and } P(\mathrm{cross}(B, A))$$

is closed under intersection, ν-disjoint union, and isomorphism, since

$$\mathrm{cross}(B \cap C, A \cap C) \subseteq \mathrm{cross}(B, A),$$
$$\mathrm{cross}(B \cup D, A \cup D) = \mathrm{cross}(B, A), \text{ and}$$
$$\mathrm{cross}(f^{\rho}(B), f^{\rho}(A)) = f^{\rho}(\mathrm{cross}(B, A)),$$

for every $\mathcal{T}$-graph C, every $\mathcal{T}$-graph D with $\nu(D) \cap \nu(A) = \varnothing$, and every mapping $f : V \to V$ that is injective on $\nu(A)$. In particular, if $P(\mathrm{cross}(B, A))$ is the property "$\mathrm{cross}(B, A) = \varnothing$", then $\sqsubseteq_P = \subseteq^{\nu}$. Other examples of such inclusion relations $\sqsubseteq_P$ are the ones generated by the properties below

- $\mathrm{cross}(B, A)$ is finite,
- $\nu(\mathrm{cross}(B, A))$ is finite,

- $\#(\text{cross}(B, A)) \leq k$, for some fixed number $k \in \mathbb{N}$,
- $\text{cross}(B, A)$ consists of edges labeled by a fixed $c \in C$ (for the graph type of Example 1.2(5)).

By Theorem 2.13, none of them satisfies the CB property. □

3 An Extension to Multi-graphs

In this section we will extend the results of Section 2 to multi-graphs, i.e., to graphs with multiple edges. A multi-graph of an arbitrary graph type $\mathcal{T} = (V, E, \nu, \rho)$ is a multiset over E. We present a relation on these multisets that satisfies the CB property. In fact, it turns out that such multisets can be represented by the graphs of a special graph type $\mathcal{M}_\mathcal{T}$ called the *multi-graph type of* $\mathcal{T}$. Thus, the proof is by an application of the CB results in the previous section.

As discussed in the Introduction, sets and multisets can be used to formalize the semantics of object-oriented parallel systems. In [2, 3, 4] for instance, the (Petri net) semantics of a π-calculus process term is a multiset of structured objects composed of names. The reason that multisets are needed (instead of just sets) is the replication operation of the π-calculus: if P denotes a process with, say, one object, then $!P$ denotes a process consisting of infinitely many copies of the same object.

As usual, a *multiset* S *(with countable multiplicities)* is defined as a set D_S together with a mapping $\psi_S : D_S \to \mathbb{N}_+ \cup \{\omega\}$, that defines the multiplicity of the elements in S (where $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ and $\omega = \aleph_0$ stands for countably infinite multiplicity). By convention, we define $\psi_S(x) = 0$ for every object x not in S. For convenience sake, we sometimes denote multisets by set notation; e.g., $\{a, b^2, c^\omega\}$ denotes the multiset S defined by $D_S = \{a, b, c\}$ and $\psi_S(a) = 1$, $\psi_S(b) = 2$, and $\psi_S(c) = \omega$. In order to relate multiplicities, the linear order $\leq$ on $\mathbb{N}$ is extended to a linear order on $\mathbb{N} \cup \{\omega\}$, defining $k \leq \omega$ for every $k \in \mathbb{N} \cup \{\omega\}$. We call T a *submultiset* of S, denoted $T \leq S$, if $\psi_T(d) \leq \psi_S(d)$, for all $d \in D_T$. For a set X, S is a *multiset over* X if $D_S \subseteq X$. If S is a multiset over X and $f : X \to Y$ is an arbitrary mapping, then the *multiset image* $f(S)$ of S under f is defined by $D_{f(S)} = f(D_S)$ and $\psi_{f(S)}(e) = \sum_{f(d)=e} \psi_S(d)$ (where summation is extended to ω in a straightforward way). Obviously this corresponds to $\psi_{f(S)}(f(d)) = \psi_S(d)$ if f is injective on D_S, i.e., d and $f(d)$ have the same multiplicity (in S and $f(S)$, respectively).

Next, we define multisets of edges and extend to them the basic definitions of Sections 1 and 2.

Definition 3.1 Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type. Multisets over E are called *multi-graphs of type* $\mathcal{T}$, or just multi-graphs if $\mathcal{T}$ is understood. For multi-graphs S and T, if there exists a mapping $f : V \to V$ such that f is injective on

$\nu(D_S)$ and $f^\rho(S) = T$, then S and T are *isomorphic*, denoted $S \simeq T$. If $T \le S$ and for all $e \in E$, $\psi_T(e) < \psi_S(e)$ implies that $\nu(D_T) \cap \nu(e) = \varnothing$, then T is a *concrete free subgraph* of S, denoted $T \le^\nu S$. T is an *(abstract) free subgraph* of S, denoted $T \lesssim S$, if T is isomorphic to a concrete free subgraph of S, i.e., $f^\rho(T) \le^\nu S$ for some $f : V \to V$ that is injective on $\nu(D_T)$.

By realizing that every set is also a multiset, the reader can easily check that Definition 3.1 is consistent with Definitions 1.5, 2.4, and 2.5. Observe that the notion of isomorphism in Definition 3.1 is a natural one: by Lemma 1.4, f^ρ is injective on D_S if $f : V \to V$ is injective on $\nu(D_S)$. Hence f^ρ preserves the multiplicity of edges in S as well as their internal structure. Also, the notion of free subgraph in Definition 3.1 naturally extends Definition 2.4; if $T \le^\nu S$, then for any $e \in D_T$, either $\psi_T(e) = \psi_S(e)$, or $\psi_T(e) < \psi_S(e)$ and $\nu(e) = \varnothing$.

Example 3.2 Some examples of multi-graphs for graph types $\mathcal{T} = (V, E, \nu, \rho)$.

(1) Plain multisets.

Let $\mathcal{T} = \mathcal{T}_\mathrm{s}$, cf. Example 1.2(1). Then multi-graphs are just multisets over V. Note that T is a concrete free subgraph of S if and only if $\psi_T(e) = \psi_S(e)$ for all $e \in D_T$. Also note that S and T are isomorphic if there exists a bijection $f : D_S \to D_T$ that preserves the multiplicity of elements in S and T, i.e., $\#_S(e) = \#_T(f(e))$, for all $e \in D_S$.

(2) Directed multi-graphs.

Let $\mathcal{T} = \mathcal{T}_\mathrm{g}$, cf. Example 1.2(4). Then multi-graphs are directed graphs with multiple edges: a pair of nodes can be joined by more than one edge. Isomorphism of such multi-graphs corresponds to the usual isomorphism of graphs with multiple edges.

(3) Solutions in the Multiset π-Calculus.

A typical example of multisets of structured objects can be found in [2, 3, 4], where a multiset semantics is given of the π-calculus of [13]. For $\mathcal{T} = (\mathrm{New}, \mathrm{Mol}, \mathrm{new}, \mu)$, multi-graphs are called *solutions*. A solution is a multiset of *molecules* (from the set Mol) and each molecule is a structured object that can be viewed as a tree of which the nodes are labeled by (channel) names, cf. the Introduction. New names (from the set New) are names that are bound by the restriction operator of the π-calculus. The mapping $\mathrm{new}(m)$ collects all the new names occurring in a molecule m. The semantics of a process term of the π-calculus is defined to be such a solution, modulo an injective renaming of its new names, i.e., modulo isomorphism (see Lemma 5 of [2]; in [3] isomorphic solutions are said to be a "copy" of each other). For a renaming $f : \mathrm{New} \to \mathrm{New}$, $f^\mu : \mathrm{Mol} \to \mathrm{Mol}$ denotes the induced relabeling of molecules. The relation $\le^{\mathrm{new}}$ (defined in Definition 3.1) is denoted $\subseteq^{\mathrm{n}}$ in [4] and called 'strong containment'. □

It follows immediately from Example 2.3 that the relation 'isomorphic to a submultiset' does not satisfy the CB property for multi-graphs (since every set is a multiset). In fact, even for plain multisets (as in Example 3.2(1)), it fails to hold that S and T are isomorphic if S is isomorphic to a submultiset of T and vice versa. This is shown by the next example.

Example 3.3 Consider the following multi-graphs: $S = \{0, 1^\omega, 2^\omega, 3^\omega, \ldots\}$ and $T = \{0^\omega, 1^\omega, 2^\omega, \ldots\}$ of type $\mathcal{T}_\mathrm{s}$ with $V = \mathbb{N}$. Clearly, $S \leq T$, so trivially S is isomorphic to a submultiset of T. With the mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(k) = k + 1$, $f(T) = \{1^\omega, 2^\omega, 3^\omega, \ldots\}$, and hence T is isomorphic to a submultiset of S. However, S and T are not isomorphic, since S has an element of multiplicity 1, which T has not. Note that S is not a free subgraph of T. $\square$

In the remainder of this section we will show that the free subgraph relation satisfies the CB property for multi-graphs. In order to do so, we define a graph type $\mathcal{M}_\mathcal{T}$ for every graph type $\mathcal{T}$, such that graphs of type $\mathcal{M}_\mathcal{T}$ represent multi-graphs of type $\mathcal{T}$. We can represent any multiset S uniquely by the set $[S] = \{(d, k) \mid d \in D_S \text{ and } 0 \leq k < \psi_S(d)\}$, since $D_S = \{d \mid (d, k) \in [S] \text{ for some } k \in \mathbb{N}\}$ and $\psi_S(d) = \#\{k \mid (d, k) \in [S]\}$ (where $\#\mathbb{N} = \omega$). Moreover, subsets $[T]$ of $[S]$ represent exactly the submultisets T of S, i.e., $T \leq S$ iff $[T] \subseteq [S]$. Note that this is in contrast with subsets of $\{(d, \psi_S(d)) \mid d \in D_S\}$, which is the usual representation of S. We will call $[S]$ a *multiset representation*, and in particular the *representation of* S.

Definition 3.4 Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type. The *multi-graph type of* $\mathcal{T}$ is the graph type $\mathcal{M}_\mathcal{T} = (V, E \times \mathbb{N}, \xi, \sigma)$, defined by

(1) $\xi((e, k)) = \nu(e)$, and

(2) $f^\sigma((e, k)) = (f^\rho(e), k)$,

for all $e \in E$, $k \in \mathbb{N}$, and $f : V \to V$.

The reader easily verifies that Definition 3.4 indeed defines a graph type: properties (1)–(4) of Definition 1.1 are all consequences of the fact that $\mathcal{T}$ is a graph type. For instance, $\xi(f^\sigma((e, k))) = \xi((f^\rho(e), k)) = \nu(f^\rho(e)) = f(\nu(e)) = f(\xi((e, k)))$. The other properties are shown in a similar way.

Clearly, the collection of representations of multi-graphs of type $\mathcal{T}$ is a proper subset of the collection of $\mathcal{M}_\mathcal{T}$-graphs. Only if for all $(e, k) \in A$, we have $(e, l) \in A$ for all $0 \leq l < k$, then A represents a multiset.

By (1) of Definition 3.4, we have $\xi([S]) = \nu(D_S)$, for a multi-graph S, and hence the mapping ξ retrieves all the nodes of S. For $\mathcal{T} = \mathcal{T}_\mathrm{s}$, we just have $\xi([S]) = D_S$. By (2) of Definition 3.4, relocations do not affect the multiplicities of the edges. This means that although multiset representations are not closed under arbitrary relocations f^σ, they are closed under injective ones, as expressed in the following lemma.

Lemma 3.5 *Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type and S a multi-graph of type $\mathcal{T}$. Let $f : V \to V$ be injective on $\nu(D_S)$. Then $[f^\rho(S)] = f^\sigma([S])$.*
Proof By Lemma 1.4, f^ρ is injective on D_S. Hence $\psi_{f^\rho(S)}(f^\rho(e)) = \psi_S(e)$, for all $e \in D_S$. And so $[f^\rho(S)] = \{(e, k) \mid e \in f^\rho(D_S), 0 \leq k < \psi_{f^\rho(S)}(e)\} = \{(f^\rho(e), k) \mid e \in D_S, 0 \leq k < \psi_{f^\rho(S)}(f^\rho(e))\} = f^\sigma([S])$. □

Recall that multi-graphs S and T are isomorphic if $f^\rho(S) = T$, for some $f : V \to V$ that is injective on $\nu(D_S)$. Hence by the uniqueness of multiset representations, this is equivalent with $[f^\rho(S)] = [T]$ and thus with $f^\sigma([S]) = [T]$ by Lemma 3.5. Moreover, $\nu(D_S) = \xi([S])$ by Definition 3.4. Consequently, we have the following lemma.

Lemma 3.6 *Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type and let S and T be multi-graphs of type $\mathcal{T}$. Then S is isomorphic with T in $\mathcal{T}$ if and only if $[S]$ is isomorphic with $[T]$ in $\mathcal{M}_\mathcal{T}$.*

Intuitively, Lemma 3.6 holds because, as expressed in Definition 3.4(2), f^σ preserves the multiplicity as well as the internal structure of the edges in S.

Finally, we show that any concrete free subgraph of a multi-graph of type $\mathcal{T}$ defines a concrete free subgraph of its representation. More precisely, we prove that $T \leq^\nu S$ iff $[T] \subseteq^\xi [S]$. Since we already established that $T \simeq S$ iff $[T] \simeq [S]$, this allows us to conclude that the free subgraph relations in $\mathcal{T}$ and $\mathcal{M}_\mathcal{T}$ are basically equivalent.

Lemma 3.7 *Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type and let S and T be multi-graphs of type $\mathcal{T}$. Then $T \leq^\nu S$ if and only if $[T] \subseteq^\xi [S]$, and $T \lesssim S$ if and only if $[T] \lesssim [S]$.*
Proof First, recall that $\xi([T]) = \nu(D_T)$ and that $T \leq S$ iff $[T] \subseteq [S]$. Moreover, for all $e \in E$, there exists $k \in \mathbb{N}$ such that $(e, k) \in [S] - [T]$, if and only if $\psi_T(e) < \psi_S(e)$. Hence $\nu(D_T) \cap \xi([S] - [T]) \neq \varnothing$ if and only if there exists $e \in E$ such that $\psi_T(e) < \psi_S(e)$ and $\nu(D_T) \cap \nu(e) \neq \varnothing$. □

From Theorem 2.10 (applied to $\mathcal{M}_\mathcal{T}$) the following result can now be inferred, which generalizes Theorem 2.10.

Theorem 3.8 *Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type and let S and T be multi-graphs of type $\mathcal{T}$. If S is a free subgraph of T and vice versa, then S and T are isomorphic.*

The following example shows two isomorphic multi-graphs. The construction of an isomorphism between the two is similar to the construction of h of Example 2.11.

Example 3.9 Consider the following multi-graphs of type $\mathcal{T}_\mathrm{s}$ with $V = \mathbb{N}$: $S = \{0^\omega, 1, 2^\omega, 3, \ldots\}$ and $T = \{0, 1^\omega, 2, 3^\omega, \ldots\}$. By the mapping $f : \mathbb{N} \to \mathbb{N}$

with $f(k) = k+1$, which is injective on both D_S and D_T, we have $f(S) \leq^\nu T$ and $f(T) \leq^\nu S$. Hence by Theorem 3.8, S and T are isomorphic. Moreover, the mapping $h = f^{-1} \circ (f \circ f)^{\mathbf{B}}$ (where the Bernstein modification is taken with respect to $(\mathbb{N}, \mathbb{N}_+)$), is an isomorphism between S and T. In this particular case, it yields

$$h(k) = \begin{cases} k+1 & \text{for even } k \\ k-1 & \text{for odd } k. \end{cases}$$

□

The next result is inferred from Theorem 2.12 (applied to $\mathcal{M}_\mathcal{T}$).

Theorem 3.10 *Let $\mathcal{T} = (V, E, \nu, \rho)$ be a graph type and let S and T be multi-graphs of type $\mathcal{T}$ with finite $\nu(D_S)$ and finite $\nu(D_T)$. If S is isomorphic to a submultiset of T and vice versa, then S and T are isomorphic.*

Finally we note that graph types can be naturally composed as follows: we define the *composition* $\mathcal{T}_2 \circ \mathcal{T}_1$ of graph types $\mathcal{T}_1 = (V, E, \nu, \rho)$ and $\mathcal{T}_2 = (E, E', \pi, \tau)$, as the structure system (V, E', ξ, σ), where $\xi(e') = \nu(\pi(e'))$ and $\sigma = \tau \circ \rho$. For instance, the datatype `Array of Record of V` could be modelled in this way, if we let

```
type V = <BT>

type E = Record of V

type E' = Array of E.
```

Also, if we take $E' = E \times \mathbb{N}$ and if we define $\pi((e,k)) = \{e\}$ and $f^\tau((e,k)) = (f(e), k)$, then it is easy to see that the multi-graph type $\mathcal{M}_{\mathcal{T}_1}$ of $\mathcal{T}_1$ is the composition of $\mathcal{T}_1$ and $\mathcal{T}_2$. Note that $\mathcal{T}_2$ is $\mathcal{M}_{\mathcal{T}_0}$ where $\mathcal{T}_0$ is the graph type of plain sets over E; thus $\mathcal{M}_{\mathcal{T}_1} = \mathcal{M}_{\mathcal{T}_0} \circ \mathcal{T}_1$. This means that $\mathcal{M}_{\mathcal{T}_0}$ can be viewed as a `CTO` "`Multi of`", and if $\mathcal{T}_1$ is the type `Record of V` as above, then $\mathcal{M}_{\mathcal{T}_1}$ is the type `Multi of Record of V` (and sets of this type represent multisets of records).

4 Conclusion

In this paper we presented a general axiomatic setting for graphs of any type. Natural notions of subgraph, isomorphism, and connectedness of these graphs were defined and it was shown that the relation 'isomorphic to a subgraph consisting of connected components' satisfies the CB property. It was also shown that the usual bijection between the nodes of the two graphs is in fact a graph isomorphism. For graphs with finitely many nodes (but possibly infinitely many edges) the relation 'isomorphic to a subgraph' also satisfies the CB property, but for arbitrary graphs any "natural" relation inbetween these two does not satisfy the CB property. Finally, similar results were shown for graphs with multiple edges, by an application of the results for graphs.

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