A Cantor-Bernstein Result for Structured Objects

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Abstract

The notion of a structure system — sets of structured objects that are composed of atomic objects — is introduced by a collection of axioms. By a uniform change of the atomic objects, the relationship, induced by the atomic objects, between the structured objects is preserved, resulting in a notion of isomorphism of sets of structured objects. By definition, a relation $\mathcal{R}$ on such structured sets satisfies the Cantor-Bernstein property if $A \mathcal{R} B$ and $B \mathcal{R} A$ imply that A and B are isomorphic. It is shown that 'isomorphic to a subset' does not satisfy the Cantor-Bernstein property. However, a restricted version of this relation does, resulting in an extension of the Cantor-Bernstein proposition for sets of structured objects. Similar results are shown for multisets of structured objects.

Introduction

In set theory, the Cantor-Bernstein proposition expresses that two sets are of equal size, if a subset of the one is of equal size to the other and vice versa. Here, 'equal size' means that the elements of one set can be put in a one to one correspondence with the elements of another set. Another way to view this correspondence is as an isomorphism. Indeed, although the structure of a plain set is the simplest one could imagine, this correspondence is structure preserving. The goal of this paper is to investigate whether the Cantor-Bernstein proposition holds for isomorphisms between sets of structured objects.

In many fields of mathematics and computer science we encounter structured objects that are composed of atomic objects. Often the latter act as information carriers and are organized in some way to form the former, providing them with their structure. Immediately the object-oriented paradigm springs to mind. Indeed, an object — or more precisely: an object class — has a collection of methods or attributes which, according to a set of actions, enable it to interact with other objects. Hence these methods, among other entities that describe the object class, give structure to the objects to which they belong. Moreover, they also induce a — perhaps more important — structure on a collection of objects, relating objects that are capable of interaction.

Often though, this second, global interaction structure is the sole objective of the atomic objects, i.e., their identity is of no importance. For instance, if we view a graph as a collection of structured objects — viz. its edges — that are composed of atomic objects — viz. its nodes —, then the identity of the nodes is unimportant: we could change their identity, as long as the structure they induce, i.e., the graph, is unchanged. As another example, in the λ-calculus (see [2]), bound variables impose a structure on a λ-term, that is independent of the names given to them. Thus, both $\lambda x.ux(\lambda y.xy)$ and $\lambda z.uz(\lambda x.zx)$ are instances of the same λ-term; the one is renamed into the other by α-conversion. As a matter of fact, this structure, i.e., the relationship among the occurences of bound variables, is the only role bound variables play in a λ-term. Indeed, in [3], de Bruijn proposes a single notation for α-congruent λ-terms (obtaining the so called *namefree expressions*), in which bound names are eliminated from ordinary λ-terms and replaced by numbers, thereby preserving their interrelationship. In process algebra, bound names resulting from restriction (CCS [10] and π-Calculus [12]) or encapsulation (ACP [1]) impose a similar structure on process terms.

Abstracting away from the identity of atomic objects, it is natural to define an isomorphism relation on structured objects (and hence on sets of structured objects), calling two (sets of) structured objects isomorphic, if, by a uniform change of atomic objects, the one is changed into the other. In the example above this isomorphism corresponds to α-congruence, in the graph case it is equal to graph isomorphism.

In this paper we present a general framework for defining structured objects that are composed of atomic objects. Moreover, we investigate the circumstances under which the Cantor-Bernstein property holds for these structured sets, i.e., when two structured sets A and B are isomorphic if, for an appropriate preorder $\mathcal{R}$, $A \mathcal{R} B$ and $B \mathcal{R} A$. It is shown that the preorder 'A is isomorphic to a subset of B' does not satisfy the Cantor-Bernstein property. However, a restricted version, in which a subset B' of B has no atomic objects in common with its complement $B - B'$, does.

In other branches of mathematics this possible failure of the Cantor-Bernstein proposition has also been noticed. For instance in [8] two closed and bounded sets A and B are defined in a topological space. It is shown that A and B are not homeomorphic — which is the usual notion of isomorphism in a topological context — although A is homeomorphic to an open subset of B and vice versa. A computability result closely related to the Cantor-Bernstein proposition appears in [13] in which it is proven that if a set A is one-one reducible to B and vice versa, A and B are recursively isomorphic. However, restricted to polynomial-time reductions, this result fails to hold: in [6] it is shown that there exist sets that are not p-isomorphic, yet each one is reducible to the other by one-one, polynomial-time invertible reductions (assuming $\mathrm{P} \neq \mathrm{PSPACE}$).

In the first section of this paper, we define sets of structured objects formally by a collection of axioms. This axiomatic approach has the advantage of meeting many formalisms in mathematics and computer science, such as (labeled) graphs or trees, multisets, formal languages, etc. In the second section we study preorders on these sets that satisfy the Cantor-Bernstein property. In the third section, we extend the results of the second section to multisets of structured objects. We define a natural notion of isomorphism on these multisets and we present a multiset relation that satisfies the Cantor-Bernstein property.

1 Basic Definitions

In this section, we present the basic definitions of "structures": sets of structured objects. A structured object is composed of atomic objects; changing its atomic objects changes the object, but not its structure. Thus, every change of atomic objects induces a structure preserving change of objects, and hence these mappings can be viewed as homomorphisms on structures. Also, structures themselves are not just plain sets; there is a relationship among structured objects that have atomic objects in common. We could think of such structured objects as being akin. Hence, the atomic objects induce a second, global structure on these sets. We will show that there exist non-isomorphic sets (i.e., sets that differ in their global structure), the one being isomorphic to a subset of the other and vice versa. The goal of this paper is to show that this deficiency is removed if we take a restricted inclusion relation, the proof of which results in an extension of the Cantor-Bernstein proposition to sets of structured objects.

For any set A, we denote by $\mathcal{P}(A)$ the set of all subsets of A. For a mapping $f : A \to B$ and a set $A' \in \mathcal{P}(A)$, the restriction $f{\restriction}A' : A' \to B$ of f to A' is defined as $(f{\restriction}A')(a) = f(a)$, for all $a \in A'$. The identity mapping on A is denoted by id_A.

Next, sets of structured objects are defined. To stress the fact that the atomic objects are viewed as unstructured objects we also call them labels. For structured objects we will use the letter m (which stands for "molecule").

Definition 1.1 A *structure system* $\mathcal{S}$ is a tuple (M, L, α, μ), where

- M is a set of *(structured) objects*,
- L is a set of *atomic objects* or *labels*,
- α is a mapping $M \to \mathcal{P}(L)$ (for $m \in L$, $\alpha(m) \subseteq L$ is the set of *labels used in* m),
- μ is a mapping that assigns to every mapping $f : L \to L$ a mapping $f^\mu : M \to M$ (f^μ is the *relabeling induced by* f),

such that, for all $m \in M$ and $f, g : L \to L$,

(1) $\alpha(f^\mu(m)) = f(\alpha(m))$,

(2) if $f\restriction\alpha(m) = g\restriction\alpha(m)$, then $f^\mu(m) = g^\mu(m)$,

(3) $(g \circ f)^\mu = g^\mu \circ f^\mu$, and

(4) $(\mathrm{id}_L)^\mu = \mathrm{id}_M$.

An $\mathcal{S}$-*structure* is a subset of M.

Note that Definition 1.1 is axiomatic rather than constructive: the precise way in which the (structured) objects in M are built from the labels in L is left unspecified. In fact, the only information that one can retrieve from a structured object m are its labels $\alpha(m)$. However, this information suffices to induce a relationship among structured objects: two structured objects m_1 and m_2 are globally related if they share a common label, i.e., if $\alpha(m_1) \cap \alpha(m_2) \neq \varnothing$. Hence we can think of an $\mathcal{S}$-structure as being connected if the graph this global relationship induces is connected.

Also, the precise way in which the relabeling f^μ works is left unspecified. Intuitively, f^μ changes the labels $\alpha(m)$ of an object m according to f, preserving the structure of m. In this perspective, a structure system acts as a compound datatype; for instance, in a pseudo-Pascal programming language we could think of the following:

```
type L = <BT>

type M = <CTO> of L,
```

where `<BT>` is a basic type, such as `Integer` or `Char`, and `<CTO>` can be any compound type operator such as `Array` or `Record`. Now a relabeling that acts on `M` changes the `L`-values in `M`, but keeps the `<CTO>` structure. If we elaborate on this, a database is perhaps the most explanatory example of an $\mathcal{S}$-structure. Information in a database is organized as a collection of records. Each record consists of fields, corresponding to related data values, such as a persons name and address. Thus, a change of data values results in changing the contents of some fields in a record, but the structure of the record remains unaltered. Clearly, records in the database are akin if they have the same data value in some field, for instance when two persons have the same address. Thus, a global change of data values (caused, e.g., by a municipal decision to change the names of certain streets) does not change the global structure of the database. However, the comparison with databases is slightly misleading: we are mainly interested in *infinite* sets of objects.

In Definition 1.1, properties (1) and (2) ensure that any label of a relabeled object is derived from one of its original labels, and that two relabelings act upon an object in the same way if they change its labels in the same way.

Property (4) states that a relabeling can affect only the labels of an object, not its structure. Finally by property (3), μ distributes over composition.

Example 1.2 Some examples of structure systems $\mathcal{S} = (M, L, \alpha, \mu)$.

(1) Plain sets.

$M = L$,

$\alpha(m) = \{m\}$, for $m \in M$, and

$f^\mu = f$, for $f : L \to L$.

Here, objects just consist of one atomic object and thus $\mathcal{S}$-structures are just sets of unstructured objects. Hence a relabeling is just a mapping between (atomic) objects.

(2) Sets of binary trees of which the nodes are labeled by integers.

$L = \mathbb{Z}$,

$M =$ `Set of tree`, where

`tree = Record (val:`$\mathbb{Z}$`; left, right:^tree)`,

$\alpha(\mathtt{t}) = \{\mathtt{t.val}\} \cup \alpha(\mathtt{t.left\^{}}) \cup \alpha(\mathtt{t.right\^{}})$, for `t: tree`, and

$f^\mu(\mathtt{t}) = \mathtt{u}$, where $\mathtt{u.val} = f(\mathtt{t.val})$, $\mathtt{u.left\^{}} = f^\mu(\mathtt{t.left\^{}})$ and $\mathtt{u.right\^{}} = f^\mu(\mathtt{t.right\^{}})$, for $f : \mathbb{Z} \to \mathbb{Z}$.

In this Pascal-like example, $\mathcal{S}$-structures are sets of binary node-labeled trees, where each tree is, as usual, a node-record (its root) which is labeled by an integer and contains pointers to the (roots of the) direct subtrees. Relabeling of a tree `t` is done by changing the integer values $\alpha(\mathtt{t})$ of its nodes by f.

(3) Languages over an alphabet $L \cup L'$.

$M = (L \cup L')^*$

$\alpha(x_1 \cdots x_k) = \{x_1, \ldots, x_k\} \cap L$, and

$f^\mu(x_1 \cdots x_k) = x'_1 \cdots x'_k$, where $x'_i = \begin{cases} f(x_i) & \text{if } x_i \in L \\ x_i & \text{if } x_i \in L' \end{cases}$

with $k \geq 0$, $x_i \in L \cup L'$ and $f : L \to L$. In this example, $\mathcal{S}$-structures are languages over a (possibly infinite) alphabet $L \cup L'$, where L and L' are assumed disjoint. For a word $m \in M$, only its symbols in L are regarded as atomic objects (hence if $m \in L^*$, $\alpha(m) = \varnothing$), and f^μ only changes symbols in L. An $\mathcal{S}$-structure can also be viewed as a set of arrays (of unbounded length) of type $L \cup L'$. In the database example explained above, L' represents data that remain unchanged, and hence objects in $(L')^*$ can be viewed as 'facts'.

(4) Directed graphs.

$$M = L \times L,$$
$$\alpha((v,w)) = \{v,w\}, \text{ and}$$
$$f^\mu((v,w)) = (f(v), f(w)),$$

where $v, w \in L$ and $f : L \to L$. Intuitively, L is a reservoir of nodes and M is the collection of all possible edges between these nodes. Hence an $\mathcal{S}$-structure in this sense is a directed graph (with isolated nodes excluded). The set of nodes incident to an edge $e \in M$ is now denoted by $\alpha(e)$, and relabeling of an edge is done by changing the nodes incident to it. This example is easily extended to arbitrary relational algebras. □

For an $\mathcal{S}$-structure A, we define $\alpha(A) = \bigcup\{\alpha(m) \mid m \in A\}$ and for $f : L \to L$, we let $f^\mu(A) = \{f^\mu(m) \mid m \in A\}$ as customary. The next lemma shows that these mappings satisfy property (1)–(4) of Definition 1.1. Intuitively, this means that `Set of` is also a `CTO`. The routine proof is left to the reader.

Lemma 1.3 *Let (M, L, α, μ) be a structure system. Then $(\mathcal{P}(M), L, \alpha, \mu)$ is a structure system.*

To show the effect of a relabeling of an $\mathcal{S}$-structure, let $L = \mathbb{N}$ and let $A = \{(1,2),(2,3),(3,1)\}$ be an $\mathcal{S}$-structure in the sense of Example 1.2(4). Let $f : \mathbb{N} \to \mathbb{N}$ be a mapping such that $f(1) = 1$, $f(2) = 3$ and $f(3) = 1$. Then $f^\mu(A) = \{(1,3),(3,1),(1,1)\}$, and the directed graphs A and $f^\mu(A)$ are shown pictorially in Fig. 1.

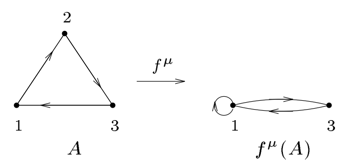

Figure 1: Relabeling of A

Obviously, if we take a mapping f that is injective on the labels $\alpha(A) = \{1,2,3\}$ of A, say $f(n) = n+1$ for all $n \in \mathbb{N}$, then A and $f^\mu(A) = \{(2,3),(3,4),(4,2)\}$ are isomorphic directed graphs. Moreover, f^μ is then injective on the elements of A. In general, for every mapping $f : L \to L$ that is injective on $\alpha(A)$, there exists a mapping $\overline{f}_A : L \to L$ such that $(\overline{f}_A)^\mu(f^\mu(m)) = m$ for all $m \in A$. In fact, define $\overline{f}_A = (f \restriction \alpha(A))^{-1} \cup \mathrm{id}_{L - f(\alpha(A))}$. Then the next lemma shows that, restricted to A, $(\overline{f}_A)^\mu$ is the inverse of f.

Lemma 1.4 *For an $\mathcal{S}$-structure A, if $f : L \to L$ is injective on $\alpha(A)$, then f^μ is injective on A. In particular, $(\overline{f}_A)^\mu(f^\mu(m)) = m$ for all $m \in A$.*

Proof Let $m \in A$. Since $(\overline{f}_A \circ f)\restriction\alpha(m) = \mathrm{id}_L\restriction\alpha(m)$, by Definition 1.1(3), (2) and (4), respectively we have $(\overline{f}_A)^\mu(f^\mu(m)) = (\overline{f}_A \circ f)^\mu(m) = (\mathrm{id}_L)^\mu(m) = \mathrm{id}_M(m) = m$. $\square$

Next, we define isomorphism of $\mathcal{S}$-structures.

Definition 1.5 $\mathcal{S}$-structures A and B are *isomorphic*, denoted $A \simeq B$, if there is a mapping $f : L \to L$ such that f is injective on $\alpha(A)$ and $f^\mu(A) = B$.

A mapping $F : M \to M$, such that $F(A) = B$ and $F = f^\mu$ for some $f : L \to L$ that is injective on $\alpha(A)$, is defined to be an isomorphism between A and B. Thus an isomorphism is a mapping that preserves the global structure of an $\mathcal{S}$-structure, as observed in the beginning of this section. Notice that in the set case (see Example 1.2(1)), an isomorphism is nothing more than a bijection, i.e., two sets are isomorphic if they are equipotent. In the graph case (see Example 1.2(4)), it corresponds to the usual definition of isomorphism of directed graphs (and, more generally, to the definition of isomorphism of relational algebras).

Observe that $\simeq$ is an equivalence relation: let $A \simeq B$ and let f^μ be the isomorphism between A and B. Then also $B \simeq A$ by the existence of $\overline{f}_A$ (cf. Lemma 1.4) and by Lemma 1.3. Transitivity follows from Definition 1.1(1,3), and reflexivity from Definition 1.1(4).

For $\mathcal{S}$-structures A and B, we define B to be a *concrete substructure* of A, if $B \subseteq A$. Furthermore, B is a *substructure* of A, if B is isomorphic to a concrete substructure of A, i.e., if $f^\mu(B) \subseteq A$ for some $f : L \to L$ that is injective on $\alpha(B)$. Note that in the graph case, these notions correspond to the definitions of concrete subgraph, and (isomorphic to a concrete) subgraph, respectively.

2 The Cantor-Bernstein Proposition for Structure Systems

It is a well-known fact from set theory that two sets Γ and Λ are equipotent if Γ is equipotent to a subset of Λ and vice versa, i.e., if there exist injections $\phi_1 : \Gamma \to \Lambda$ and $\phi_2 : \Lambda \to \Gamma$. This is the Cantor-Bernstein proposition (see for instance [9] among numerous other works on set theory). The central idea in this proposition lies in the construction of a bijection between any two such sets. Observe that since in the above case $\Delta = \phi_2(\Lambda) \subseteq \Gamma$ and $\phi = (\phi_2 \circ \phi_1) : \Gamma \to \Delta$ is injective, it suffices to show the existence of a bijection between Γ and Δ. For completeness sake we state its construction below, as well as the proof that it is a bijection. For technical reasons we assume a universe Σ that contains Γ and Δ.

Definition 2.1 Let $\phi : \Sigma \to \Sigma$ be injective on $\Gamma \subseteq \Sigma$ and let $\phi(\Gamma) \subseteq \Delta \subseteq \Gamma$. The *Bernstein modification of* ϕ *with respect to* (Γ, Δ), denoted $\phi^{\mathbf{B}} : \Sigma \to \Sigma$, is defined as

$$\phi^{\mathbf{B}}(x) = \begin{cases} \phi(x) & \text{if } x \in \bigcup_{i \geq 0} \phi^i(\Gamma - \Delta) \\ x & \text{otherwise.} \end{cases}$$

The Bernstein modification $\phi^{\mathbf{B}}$ of ϕ, with respect to (Γ, Δ), is depicted in Fig. 2. The dark areas inside Γ show the set $\bigcup_{i \geq 0} \phi^i(\Gamma - \Delta)$.

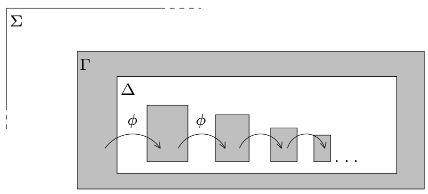

Figure 2: The Bernstein modification of ϕ

Proposition 2.2 *For every mapping* $\phi : \Sigma \to \Sigma$ *and all sets* $\Gamma, \Delta \subseteq \Sigma$ *such that* ϕ *is injective on* Γ *and* $\phi(\Gamma) \subseteq \Delta \subseteq \Gamma$*, the Bernstein modification* $\phi^{\mathbf{B}} : \Sigma \to \Sigma$ *of* ϕ *with respect to* (Γ, Δ) *is injective on* Γ *and moreover,* $\phi^{\mathbf{B}}(\Gamma) = \Delta$.

Proof Let $\Omega = \bigcup_{i \geq 0} \phi^i(\Gamma - \Delta)$. Note that $\Omega \subseteq \Gamma$.

To prove injectivity of $\phi^{\mathbf{B}}$ on Γ, assume $x, y \in \Gamma$ with $x \neq y$. We consider four cases. If $x \notin \Omega$ and $y \notin \Omega$, then $\phi^{\mathbf{B}}(x) = x$ and $\phi^{\mathbf{B}}(y) = y$. Hence $\phi^{\mathbf{B}}(x) \neq \phi^{\mathbf{B}}(y)$. If $x \in \Omega$ and $y \notin \Omega$, then $\phi^{\mathbf{B}}(x) \in \Omega$ and since $\phi^{\mathbf{B}}(y) = y \notin \Omega$, we have $\phi^{\mathbf{B}}(x) \neq \phi^{\mathbf{B}}(y)$. The case in which $x \notin \Omega$ and $y \in \Omega$ is proven similarly. If both $x \in \Omega$ and $y \in \Omega$, then $\phi^{\mathbf{B}}(x) = \phi(x)$ and $\phi^{\mathbf{B}}(y) = \phi(y)$. Since, by assumption, ϕ is injective on Γ, we have $\phi^{\mathbf{B}}(x) \neq \phi^{\mathbf{B}}(y)$.

Since obviously $\phi^{\mathbf{B}}(\Gamma) \subseteq \Delta$, it remains to show that $\Delta \subseteq \phi^{\mathbf{B}}(\Gamma)$. Assume $x \in \Delta$. If $x \in \Omega$, then there exists $p \geq 1$ such that $x \in \phi^p(\Gamma - \Delta)$. Hence there exists $y \in \phi^{p-1}(\Gamma - \Delta)$ with $x = \phi(y)$, and thus $\phi^{\mathbf{B}}(y) = x$, by definition of $\phi^{\mathbf{B}}$. If $x \notin \Omega$, we immediately derive $\phi^{\mathbf{B}}(x) = x \in \Gamma$. $\square$

Let $\mathcal{S} = (M, L, \alpha, \mu)$ be a structure system. For any pre-ordering $\mathcal{R} \subseteq \mathcal{P}(M) \times \mathcal{P}(M)$, we will say that $\mathcal{R}$ satisfies the *Cantor-Bernstein property*, if $A \mathrel{\mathcal{R}} B$ and $B \mathrel{\mathcal{R}} A$ imply $A \simeq B$, for every pair A, B of $\mathcal{S}$-structures. Now if we view sets Γ and Λ as $\mathcal{S}$-structures (as in Example 1.2(1)), then indeed Proposition 2.2 proves that Γ and Λ are isomorphic (in the sense of Definition 1.5), if Γ is a substructure of Λ and vice versa. Thus, for sets, the pre-ordering 'substructure of' (as defined at the end of Section 2) satisfies the Cantor-Bernstein property. In the general case however, this does not hold, as the following example shows.

Example 2.3 Consider the $\mathcal{S}$-structures (in the sense of Example 1.2(4)) $A' = \{(x_i, y_j) \mid i \geq 0, j \geq 1\}$ and $B = \{(u_i, v_j) \mid i, j \geq 0\}$ and let $A = A' \cup \{(x_0, y_0)\}$, as depicted in Fig. 3. It is easy to see that A is a substructure of B, since $A \cup \{(x_i, y_0), \mid i \geq 1\}$ is isomorphic to B. Also, B is a substructure of A, since by the bijection $f : \alpha(B) \to \alpha(A')$ with $f(u_i) = x_i$ and $f(v_j) = y_{j+1}$, B and A' are isomorphic. However A and B are not isomorphic, since A has a node of degree one, viz. y_0, which B has not. $\square$

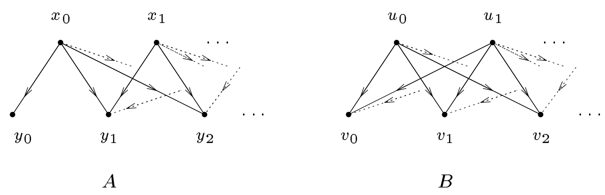

Figure 3: A and B are not isomorphic

In order to satisfy the Cantor-Bernstein property, we adapt the substructure relation $B \subseteq A$ so that it will not affect the degree of any nodes of B. In general, this means for arbitrary $\mathcal{S}$-structures A and B, that no object in B may share a label with an object in $A - B$. Intuitively, this means that there is no global relationship between objects in B and $A - B$. We adapt the definition of concrete substructure accordingly.

Definition 2.4 For $\mathcal{S}$-structures A and B, B is a *concrete component* of A, denoted $B \subseteq^\alpha A$, if $B \subseteq A$ and $\alpha(B) \cap \alpha(A - B) = \varnothing$.

Hence if $B \subseteq^\alpha A$, then $\alpha(A - B) = \alpha(A) - \alpha(B)$. Note that $\subseteq^\alpha$ is a partial order on $\mathcal{P}(M)$; to show transitivity, let $C \subseteq^\alpha B \subseteq^\alpha A$. Then $\alpha(A - C) = \alpha((A-B) \cup (B-C)) = \alpha(A-B) \cup \alpha(B-C) = (\alpha(A) - \alpha(B)) \cup (\alpha(B) - \alpha(C)) = \alpha(A) - \alpha(C)$. Reflexivity and antisymmetry are obvious.

Definition 2.5 For $\mathcal{S}$-structures A and B, B is a *component* of A, denoted $B \lesssim A$, if B is isomorphic to a concrete component of A, i.e., $f^\mu(B) \subseteq^\alpha A$ for some $f : L \to L$ that is injective on $\alpha(B)$.

Observe that in the graph case, B is a component of A, if B is isomorphic to a collection of connected components of A. Also note that in the set case, the notions of (concrete) component and of (concrete) substructure coïncide, since for any pair of sets A and B, we trivially have $B \cap (A - B) = \varnothing$.

We need the following lemma to show that $\lesssim$ is a pre-order on $\mathcal{P}(M)$. It expresses that components are preserved under injective relabelings; in graph terms, this means that a graph isomorphism preserves connected components.

Lemma 2.6 *For $\mathcal{S}$-structures A and B, if $B \subseteq^\alpha A$ and $f : L \to L$ is injective on $\alpha(A)$, then $f^\mu(B) \subseteq^\alpha f^\mu(A)$.*

Proof Clearly, $f^\mu(B) \subseteq f^\mu(A)$. Note that by Lemma 1.4, f^μ is injective on A. Hence $f^\mu(A) - f^\mu(B) = f^\mu(A - B)$ and thus

$$\begin{aligned} \alpha(f^\mu(B)) \cap \alpha(f^\mu(A) - f^\mu(B)) &= f(\alpha(B)) \cap f(\alpha(A - B)) \\ &= f(\alpha(B) \cap \alpha(A - B)) \\ &= \varnothing, \end{aligned}$$

by Definition 1.1(1) (see Lemma 1.3), since f is injective on $\alpha(A)$, and since $\alpha(B) \cap \alpha(A - B) = \varnothing$, respectively. Hence $f^\mu(B) \subseteq^\alpha f^\mu(A)$. $\square$

To show transitivity of $\lesssim$, let $C \lesssim B \lesssim A$, or, equivalently, let $f^\mu(C) \subseteq^\alpha B$ and $g^\mu(B) \subseteq^\alpha A$, for $f, g : L \to L$, injective on $\alpha(C)$ and $\alpha(B)$, respectively. By Definition 1.1(3) and Lemma 2.6, we have $(g \circ f)^\mu(C) = g^\mu(f^\mu(C)) \subseteq^\alpha g^\mu(B) \subseteq^\alpha A$. Hence $C \lesssim A$, by transitivity of $\subseteq^\alpha$ and since $g \circ f$ is injective on $\alpha(C)$. Consequently, $\lesssim$ is a preorder.

In the remainder of this section we will prove that indeed $\lesssim$ satisfies the Cantor-Bernstein property, i.e., if $f^\mu(A) \subseteq^\alpha B \subseteq^\alpha A$, for two $\mathcal{S}$-structures A and B, and a mapping $f : L \to L$, injective on the labels of A, then $A \simeq B$. Moreover, we will show that the relabeling induced by the Bernstein modification of f with respect to $(\alpha(A), \alpha(B))$, i.e., $f^{\mathbf{B}\mu}$, always yields an isomorphism of A and B. In order to prove this, we will show that $f^{\mathbf{B}\mu}$ is in fact equal to $f^{\mu\mathbf{B}}$, where the latter Bernstein modification is taken with respect to (A, B). Note that by Lemma 1.4, f^μ is injective on A, so $f^{\mu\mathbf{B}}$ exists. Observe that by Definition 2.1

$$f^{\mu\mathbf{B}}(m) = \begin{cases} f^\mu(m) & \text{if } m \in \bigcup_{i \geq 0} f^{\mu i}(A - B) \\ m & \text{otherwise,} \end{cases}$$

and

$$f^{\mathbf{B}}(x) = \begin{cases} f(x) & \text{if } x \in \bigcup_{i \geq 0} f^i(\alpha(A) - \alpha(B)) \\ x & \text{otherwise.} \end{cases}$$

To prove that $f^{\mathbf{B}\mu}(m) = f^{\mu\mathbf{B}}(m)$, we will show that for every $x \in \alpha(m)$, $m \in \bigcup_{i \geq 0} f^{\mu i}(A - B)$ iff $x \in \bigcup_{i \geq 0} f^i(\alpha(A) - \alpha(B))$. In fact, we will prove a claim that is stronger than we need: for an object $m \in A$ and a label $x \in \alpha(m)$, we show that $m \in f^{\mu i}(A - B)$ iff $x \in f^i(\alpha(A) - \alpha(B))$, or, as depicted in Fig. 4, both A and $\alpha(A)$ are partitioned in dark areas, the iterations of f^μ and f, respectively, and one white area. Each 'dark set' in A can be assigned a number by its iteration; the same holds for 'dark sets' in $\alpha(A)$. Now an object in the ith dark set in A must have its labels in the ith dark set in $\alpha(A)$, and furthermore, the white area in $\alpha(A)$ represents exactly the set of labels of objects in the white area in A.

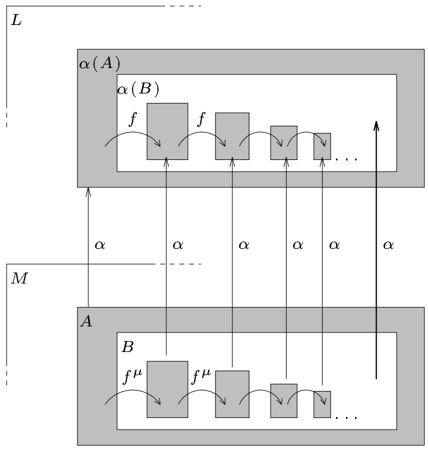

Figure 4: $\forall x \in \alpha(m)$: $m \in {f^\mu}^i(A - B)$ iff $x \in f^i(\alpha(A) - \alpha(B))$

Lemma 2.7 *Let, for $\mathcal{S}$-structures A and B, $B \subseteq^\alpha A$, and let $f : L \to L$ be injective on $\alpha(A)$ with $f^\mu(A) \subseteq^\alpha B$. Let, furthermore, $m \in A$ and $x \in \alpha(m)$. Then for every $i \geq 0$, $m \in {f^\mu}^i(A - B)$ if and only if $x \in f^i(\alpha(A) - \alpha(B))$.*

Proof We will use the following property of concrete components: for all $\mathcal{S}$-structures C, if $C \subseteq^\alpha A$, then for all $m \in A$ and $x \in \alpha(m)$ the following holds

$$m \in C \iff x \in \alpha(C).$$

Since $f^\mu(A) \subseteq^\alpha B \subseteq^\alpha A$ and $A - B \subseteq^\alpha A$, by i applications of Lemma 2.6 we have ${f^\mu}^i(A - B) \subseteq^\alpha A$, for all $i \geq 0$. Hence $m \in {f^\mu}^i(A - B)$ iff $x \in \alpha({f^\mu}^i(A - B)) = f^i(\alpha(A - B)) = f^i(\alpha(A) - \alpha(B))$, by the above claim. $\square$

Lemma 2.8 *For $\mathcal{S}$-structures A and B, if $B \subseteq^\alpha A$ and $f^\mu(A) \subseteq^\alpha B$, where $f : L \to L$ is injective on $\alpha(A)$, then ${f^{\mathbf{B}}}^\mu = {f^\mu}^{\mathbf{B}}$, where the Bernstein modifications are taken with respect to $(\alpha(A), \alpha(B))$ and (A, B), respectively.*

Proof First observe that $f(\alpha(A)) \subseteq \alpha(B) \subseteq \alpha(A)$, and hence f, $\alpha(A)$ and $\alpha(B)$ satisfy the requirements of Definition 2.1. By Lemma 1.4, this also holds for f^μ, A and B. Let $C = \bigcup_{i \geq 0} {f^\mu}^i(A - B)$. Let $m \in M$. By Lemma 2.7, $m \in C$ iff $x \in \bigcup_{i \geq 0} f^i(\alpha(A) - \alpha(B))$ for all $x \in \alpha(m)$. Hence

$$f^{\mathbf{B}} \restriction \alpha(m) = \begin{cases} f \restriction \alpha(m) & \text{if } m \in C \\ \mathrm{id}_L \restriction \alpha(m) & \text{otherwise.} \end{cases}$$

Consequently, by Definition 1.1(2) we obtain

$${f^{\mathbf{B}}}^\mu(m) = \begin{cases} f^\mu(m) & \text{if } m \in C \\ (\mathrm{id}_L)^\mu(m) & \text{otherwise.} \end{cases}$$

Since by Definition 1.1(4) $(\mathrm{id}_L)^\mu = \mathrm{id}_M$, we conclude $f^{\mathbf{B}\mu} = f^{\mu\mathbf{B}}$. $\square$

Finally we state the two main results of this section; as observed earlier in this section, Theorem 2.10 extends the Cantor-Bernstein proposition to structure systems.

Theorem 2.9 *For $\mathcal{S}$-structures A and B, if $B \subseteq^\alpha A$ and $f^\mu(A) \subseteq^\alpha B$, where $f : L \to L$ is injective on $\alpha(A)$, then $f^{\mathbf{B}\mu}$ is an isomorphism between A and B, where the Bernstein modification is taken with respect to $(\alpha(A), \alpha(B))$.*

Proof By Lemma 2.8 we have $f^{\mathbf{B}\mu} = f^{\mu\mathbf{B}}$, where the latter Bernstein modification is taken with respect to (A, B). By two instances of Proposition 2.2, $f^{\mathbf{B}}$ is injective on $\alpha(A)$ and $f^{\mu\mathbf{B}}(A) = B$. $\square$

Theorem 2.10 *For $\mathcal{S}$-structures A and B, if A is a component of B and vice versa, then A and B are isomorphic.*

Proof Let $A \lesssim B \lesssim A$. Then $B \simeq B' \subseteq^\alpha A$, for some $\mathcal{S}$-structure B'. Since $\lesssim$ is a pre-order, we have $A \lesssim B' \subseteq^\alpha A$. Hence $A \simeq A' \subseteq^\alpha B' \subseteq^\alpha A$, for some $\mathcal{S}$-structure A'. By Theorem 2.9, we have $A \simeq B'$ and hence $A \simeq B$. $\square$

Observe that if $A \lesssim B$ and $B \lesssim A$ by the mappings $f, g : L \to L$ (injective on $\alpha(A)$ and $\alpha(B)$, respectively), i.e., $f^\mu(A) \subseteq^\alpha B$ and $g^\mu(B) \subseteq^\alpha A$, then $(g \circ f)^\mu(A) \subseteq^\alpha g^\mu(B)$ by Definition 1.1(3) and Lemma 2.6. Hence by Theorem 2.9, $(g \circ f)^{\mathbf{B}\mu}$ is an isomorphism between A and $g^\mu(B)$. Since by Lemma 1.4, $(\overline{g}_B)^\mu(g^\mu(B)) = B$, we have that $(\overline{g}_B \circ (g \circ f)^{\mathbf{B}})^\mu$ is an isomorphism between A and B.

Example 2.11 Let $\mathcal{S}$ be the structure system of Example 1.2(4) with $L = \mathbb{N}$, and let $A = \{(5n, 5n+1), (5n+1, 5n+2), (5n+3, 5n+4) \mid n \geq 0\}$ and $B = \{(5n, 5n+1), (5n+2, 5n+3), (5n+3, 5n+4) \mid n \geq 0\}$ be the graphs shown in Fig. 5. Let $f, g : \mathbb{N} \to \mathbb{N}$ be defined by $f(k) = k+2$ and $g(k) = k+3$. Now

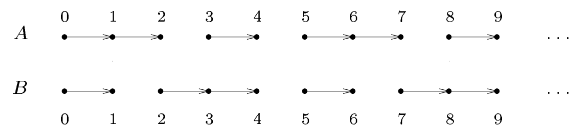

Figure 5: A and B are isomorphic by h^μ

$f^\mu(A) \subseteq^\alpha B$ and $g^\mu(B) \subseteq^\alpha A$, as the reader easily verifies. Clearly A and B are isomorphic. Furthermore, the relabeling induced by $h = \overline{g}_B \circ (g \circ f)^{\mathbf{B}}$ is an isomorphism between A and B. As the reader can check,

$$(g \circ f)^{\mathbf{B}}(k) = \begin{cases} k+5 & \text{if } k = 5n, 5n+1, 5n+2 \\ k & \text{if } k = 5n+3, 5n+4, \end{cases}$$

where the Bernstein modificaton is taken with respect to $(\mathbb{N}, \mathbb{N}+3)$, and

$$\overline{g}_B(k) = \begin{cases} k-3 & \text{if } k \geq 3 \\ k & \text{otherwise.} \end{cases}$$

Hence

$$h(k) = \begin{cases} k+2 & \text{if } k = 5n, 5n+1, 5n+2 \\ k-3 & \text{if } k = 5n+3, 5n+4. \end{cases}$$

3 Multisets of Structured Objects

In this section we will extend the results of Section 2 to multisets of structured objects in an arbitrary structure system $\mathcal{S} = (M, L, \alpha, \mu)$, i.e., to the multisets over M. We present a relation on these multisets that satisfies the Cantor-Bernstein property. In fact, it turns out that such multisets can be represented by the structures of a special structure system called the *multi structure system of* $\mathcal{S}$. Thus, the proof is by an application of the results in the previous section.

In a Petri net (or multiset transition system) a multiset is exactly the mathematical concept that fits the notion of concurrency. In [4, 5] for instance, the semantics of a π-calculus term is a multiset of structured objects composed of names. In other fields of computer science multisets are used to model databases for example. In [7] an algebra for nested multisets is presented to model hierarchical data structures.

As usual, a *multiset* S *(with countable multiplicities)* is defined as a set D_S together with a mapping $\psi_S : D_S \to \mathbb{N}_+ \cup \{\omega\}$, that defines the multiplicity of the elements in S (where $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ and $\omega = \aleph_0$ stands for countably infinite multiplicity). By convention, we define $\psi_S(x) = 0$ for every object x not in S. For convenience sake, we sometimes denote multisets by set notation; e.g., $\{a, b^2, c^\omega\}$ denotes the multiset S defined by $D_S = \{a, b, c\}$ and $\psi_S(a) = 1$, $\psi_S(b) = 2$, and $\psi_S(c) = \omega$. In order to relate multiplicities, the partial order $\leq$ on $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$ is extended to a partial order on $\mathbb{N} \cup \{\omega\}$, defining $k \leq \omega$ for every $k \in \mathbb{N} \cup \{\omega\}$. We call T a *submultiset* of S, denoted $T \leq S$, if $\psi_T(d) \leq \psi_S(d)$, for all $d \in D_T$. For a set X, S is a *multiset over* X if $D_S \subseteq X$. If S is a multiset over X and $f : X \to Y$ is an arbitrary mapping, then the *multiset image* $f(S)$ of S under f is defined by $D_{f(S)} = f(D_S)$ and $\psi_{f(S)}(e) = \sum_{f(d)=e} \psi_S(d)$ (where summation is extended to ω in a straightforward way). Obviously this corresponds to $\psi_{f(S)}(f(d)) = \psi_S(d)$ if f is injective on D_S, i.e., d and $f(d)$ have the same multiplicity (in S and $f(S)$, respectively).

Next, we define multisets of structured objects and extend to them the basic definitions of Sections 1 and 2.

Definition 3.1 Let $\mathcal{S} = (M, L, \alpha, \mu)$ be a structure system. Multisets over M are called *multi* $\mathcal{S}$*-structures.* For multi $\mathcal{S}$-structures S and T, if there exists a mapping $f : L \to L$ such that f is injective on $\alpha(D_S)$ and $f^\mu(S) = T$, then S and T are *isomorphic*, denoted $S \simeq T$. If $T \leq S$ and for all $m \in M$, $\psi_T(m) < \psi_S(m)$ implies that $\alpha(D_T) \cap \alpha(m) = \varnothing$, then T is a *concrete component* of S, denoted $T \leq^\alpha S$. T is a *component* of S, denoted $T \lesssim S$, if T is isomorphic to a concrete component of S, i.e., $f^\mu(T) \leq^\alpha S$ for some $f : L \to L$ that is injective on $\alpha(D_T)$.

By realizing that every set is also a multiset, the reader can easily check that Definition 3.1 is consistent with Definitions 1.5, 2.4, and 2.5. Observe that the notion of isomorphism in Definition 3.1 is a natural one: by Lemma 1.4, f^μ is injective on D_S if $f : N \to N$ is injective on $\alpha(D_S)$. Hence f^μ preserves the multiplicity of objects in S as well as their internal structure. Also, the notion of component in Definition 3.1 naturally extends Definition 2.4; if $T \leq^\alpha S$, then for any $m \in D_T$, either $\psi_T(m) = \psi_S(m)$, or $\psi_T(m) < \psi_S(m)$ and $\alpha(m) = \varnothing$.

Example 3.2 Some examples of multi $\mathcal{S}$-structures for structure systems $\mathcal{S} = (M, L, \alpha, \mu)$.

(1) Plain multisets.

Let $\mathcal{S}$ be the structure system of Example 1.2(1). Then multi $\mathcal{S}$-structures are unstructured multisets over L. Note that T is a concrete component of S if and only if $\psi_T(m) = \psi_S(m)$ for all $m \in D_T$. Also note that S and T are isomorphic if there exists a bijection $f : D_S \to D_T$ that preserves the multiplicity of elements in S and T, i.e., $\#_S(m) = \#_T(f(m))$, for all $m \in D_S$.

(2) Directed multi-graphs.

Let $\mathcal{S}$ be the structure system of Example 1.2(4). Then multi $\mathcal{S}$-structures are multi-graphs, i.e., directed graphs with multiple edges: a pair of nodes can be joined by more than one edge. Isomorphism of such multi $\mathcal{S}$-structures is the usual isomorphism of multi-graphs.

(3) Solutions in the Multiset π-Calculus.

A typical example of multisets of structured objects can be found in [4, 5]. For $\mathcal{S} = (\mathrm{Mol}, \mathrm{New}, \alpha, \mu)$, multi $\mathcal{S}$-structures are called *solutions*, where New is the set of *new names*, building *molecules* as structured objects which in turn form the set Mol. The semantics of a process term of the π-calculus of [11] is defined to be such a solution, modulo an injective renaming of its new names, i.e., modulo isomorphism (see Lemma 5 of [4]; in [5] isomorphic solutions are said to be a "copy" of each other). $\square$

It follows immediately from Example 2.3 that the relation 'isomorphic to a submultiset' does not satisfy the Cantor-Bernstein property for multi $\mathcal{S}$-structures (since by the remark below Definition 3.1, every set is a multiset). In fact, even for plain multisets (as in Example 3.2(1)), it fails to hold that S and T are isomorphic if S is isomorphic to a submultiset of T and vice versa. This is shown by the next example.

Example 3.3 Consider the following multi $\mathcal{S}$-structures, where $\mathcal{S}$ is the structure system of Example 3.2(1) with $M = L = \mathbb{N}$: $S = \{0, 1^\omega, 2^\omega, 3^\omega, \ldots\}$ and $T = \{0^\omega, 1^\omega, 2^\omega, \ldots\}$. Clearly, $S \leq T$, so trivially S is isomorphic to a submultiset of T. With the mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(k) = k + 1$, $f(T) = \{1^\omega, 2^\omega, 3^\omega, \ldots\}$, and hence T is isomorphic to a submultiset of S. However, S and T are not isomorphic, since S has an element of multiplicity 1, which T has not. Note that S is not a concrete component of T. $\square$

In the remainder of this section we will show that the component inclusion satisfies the Cantor-Bernstein property for multi $\mathcal{S}$-structures. In order to do so, we define a structure system $\mathcal{M}_\mathcal{S}$ for every structure system $\mathcal{S}$, such that $\mathcal{M}_\mathcal{S}$-structures represent multi $\mathcal{S}$-structures. We can represent any multiset S uniquely by the set $[S] = \{(d, k) \mid d \in D_S \text{ and } 0 \leq k < \psi_S(d)\}$, since $D_S = \{d \mid (d, k) \in [S] \text{ for some } k \in \mathbb{N}\}$, and $\psi_S(d) = \#\{k \mid (d, k) \in [S]\}$ (where $\#\mathbb{N} = \omega$). Moreover, subsets $[T]$ of $[S]$ represent exactly the submultisets T of S, i.e., $T \leq S$ iff $[T] \subseteq [S]$. Note that this is in contrast with subsets of $\{(d, \psi_S(d)) \mid d \in D_S\}$, which is the usual representation of S. We will call $[S]$ a *multiset representation*, and in particular the *representation of* S.

Definition 3.4 Let $\mathcal{S} = (M, L, \alpha, \mu)$ be a structure system. The *multi structure system of* $\mathcal{S}$, $\mathcal{M}_\mathcal{S} = (M \times \mathbb{N}, L, \beta, \nu)$, is defined by

(1) $\beta((m, k)) = \alpha(m)$, and

(2) $f^\nu((m, k)) = (f^\mu(m), k)$,

for all $m \in M$, $k \in \mathbb{N}$ and $f : L \to L$.

The reader easily verifies that Definition 3.4 defines a structure system: properties (1)–(4) of Definition 1.1 are all consequences of the fact that $\mathcal{S}$ is a structure system. Let $f : L \to L$. Then, for instance, $\beta(f^\nu((m, k))) = \beta((f^\mu(m), k)) = \alpha(f^\mu(m)) = f(\alpha(m)) = f(\beta((m, k)))$. The other properties are shown in a similar way.

Clearly, the collection of representations of multi $\mathcal{S}$-structures is a proper subset of the collection of $\mathcal{M}_\mathcal{S}$-structures. Only if for all $(m, k) \in A$, we have $(m, l) \in A$ for all $0 \leq l \leq k$, then A represents a multiset. By (1) of Definition 3.4, we have $\beta([S]) = \alpha(D_S)$, for a multi $\mathcal{S}$-structure S, and hence the mapping β retrieves all labels of objects in S. In the case of Example 3.2(1), we just have $\beta([S]) = D_S$. By (2) of Definition 3.4, relabelings do not affect the multiplicities of elements in a multiset. This means that although multiset representations are not closed under arbitrary relabelings f^ν, they are closed under injective ones, as expressed in the following lemma.

Lemma 3.5 *Let $\mathcal{S} = (M, L, \alpha, \mu)$ be a structure system and S a multi $\mathcal{S}$-structure. Let $f : L \to L$ be injective on $\alpha(D_S)$. Then $[f^\mu(S)] = f^\nu([S])$.*

Proof By Lemma 1.4, f^μ is injective on D_S. Hence $\psi_{f^\mu(S)}(f^\mu(m)) = \psi_S(m)$, for all $m \in D_S$. And so $[f^\mu(S)] = \{(m, k) \mid m \in f^\mu(D_S), 0 \leq k < \psi_{f^\mu(S)}(m)\} = \{(f^\mu(m), k) \mid m \in D_S, 0 \leq k < \psi_{f^\mu(S)}(f^\mu(m))\} = f^\nu([S])$. $\square$

Recall that multi $\mathcal{S}$-structures S and T are isomorphic if $f^\mu(S) = T$, for some $f : L \to L$ that is injective on $\alpha(D_S)$. Hence by the uniqueness of multiset representations, this is equivalent with $[f^\mu(S)] = [T]$ and thus with $f^\nu([S]) = [T]$ by Lemma 3.5. Moreover, $\alpha(D_S) = \beta([S])$ by Definition 3.4. Consequently, we have the following lemma.

Lemma 3.6 *Let $\mathcal{S} = (M, L, \alpha, \mu)$ be a structure system and S and T multi $\mathcal{S}$-structures. Then S is isomorphic with T in $\mathcal{S}$ if and only if $[S]$ is isomorphic with $[T]$ in $\mathcal{M}_\mathcal{S}$.*

Intuitively, Lemma 3.6 holds because, as expressed in Definition 3.4(2), f^ν preserves the multiplicity as well as the internal structure of the objects in S.

Finally, we show that any concrete component of a multi $\mathcal{S}$-structure defines a concrete component of its representation. More specifically, we prove that $T \leq^\alpha S$ iff $[T] \subseteq^\beta [S]$. Since we already established that $T \simeq S$ iff $[T] \simeq [S]$, this allows us to conclude that the component relations in $\mathcal{S}$ and $\mathcal{M}_\mathcal{S}$ are basically equivalent.

Lemma 3.7 *Let $\mathcal{S} = (M, L, \alpha, \mu)$ be a structure system and let S and T be multi $\mathcal{S}$-structures. Then $T \leq^\alpha S$ if and only if $[T] \subseteq^\beta [S]$, and $T \lesssim S$ if and only if $[T] \lesssim [S]$.*

Proof First, recall that $\beta([T]) = \alpha(D_T)$ and that $T \leq S$ iff $[T] \subseteq [S]$. Moreover, for all $m \in M$, there exists $k \in \mathbb{N}$ such that $(m, k) \in [S] - [T]$, if and only if $\psi_T(m) < \psi_S(m)$. Hence $\alpha(D_T) \cap \beta([S] - [T]) \neq \varnothing$, if and only if there exists $m \in M$ such that $\psi_T(m) < \psi_S(m)$ and $\alpha(D_T) \cap \alpha(m) \neq \varnothing$. $\square$

From Theorem 2.10 (applied to $\mathcal{M}_\mathcal{S}$) the following result can now be inferred, which generalizes Theorem 2.10.

Theorem 3.8 *Let $\mathcal{S} = (M, L, \alpha, \mu)$ be a structure system and let S and T be multi $\mathcal{S}$-structures. If S is a component of T and vice versa, then S and T are isomorphic.*

The following example shows two isomorphic multi $\mathcal{S}$-structures. The construction of an isomorphism between the two is similar to the construction of h^μ of Example 2.11.

Example 3.9 Consider the following multi $\mathcal{S}$-structures, where $\mathcal{S}$ is the structure system of Example 3.2(1) with $M = L = \mathbb{N}$: $S = \{0^\omega, 1, 2^\omega, 3, \ldots\}$ and $T = \{0, 1^\omega, 2, 3^\omega, \ldots\}$. By the mapping $f : \mathbb{N} \to \mathbb{N}$ with $f(k) = k + 1$, which is injective on both D_S and D_T, we have $f(S) \leq^\alpha T$ and $f(T) \leq^\alpha S$. Hence by Theorem 3.8, S and T are isomorphic. Moreover, the mapping $h = f^{-1} \circ (f \circ f)^{\mathbf{B}}$ (where the Bernstein modification is taken with respect to $(\mathbb{N}, \mathbb{N}_+)$), is an isomorphism between S and T. In this particular case, it yields

$$h(k) = \begin{cases} k+1 & \text{for even } k \\ k-1 & \text{for odd } k. \end{cases}$$

□

Finally we note that structure systems can be naturally composed as follows: we define the *composition* $\mathcal{S}_2 \circ \mathcal{S}_1$ of structure systems $\mathcal{S}_1 = (M, L, \alpha, \mu)$ and $\mathcal{S}_2 = (M', M, \gamma, \rho)$, as the structure system (M', L, β, ν), where $\beta(m') = \alpha(\gamma(m'))$ and $\nu = \rho \circ \mu$. For instance, the datatype `Array of Record of` L could be modelled in this way, if we let

```
type L = <BT>

type M = Record of L

type M' = Array of M.
```

Also, if we take $M' = M \times \mathbb{N}$ and if we define $\gamma((m, k)) = \{m\}$ and $f^\rho((m, k)) = (f(m), k)$, then it is easy to see that the multi structure system $\mathcal{M}_{\mathcal{S}_1}$ of $\mathcal{S}_1$ is the composition of $\mathcal{S}_1$ and $\mathcal{S}_2$. Note that $\mathcal{S}_2$ is $\mathcal{M}_{\mathcal{S}_0}$ where $\mathcal{S}_0$ is the structure system of plain sets (over M); thus $\mathcal{M}_{\mathcal{S}_1} = \mathcal{M}_{\mathcal{S}_0} \circ \mathcal{S}_1$. This means that $\mathcal{M}_{\mathcal{S}_0}$ can be viewed as a `CTO` "`Multi of`", and if $\mathcal{S}_1$ is the type `Record of` L as above, then $\mathcal{M}_{\mathcal{S}_1}$ is the type `Multi of Record of` L (and sets of this type represent multisets of records).

References

[1] J.C.M. Baeten and W.P. Weijland, *Process Algebra*, (Cambridge University Press, Cambridge, 1990).

[2] H.P. Barendregt, *The Lambda Calculus*, (North-Holland, Amsterdam, 1984).

[3] N.G. de Bruijn, Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the Church-Rosser Theorem, *Indagat. Mathem.* **34** (1972) 381-392.

[4] J. Engelfriet, A multiset semantics for the pi-calculus with replication, *Theoret. Comput. Sci.* **153** (1996) 65-94.

[5] J. Engelfriet and T.E. Gelsema, Multisets and structural congruence of the pi-calculus with replication, Tech. Report 95-02, Department of Computer Science, Leiden University, January 1995. To appear in *Theoret. Comput. Sci.*

[6] S.A. Fenner, S.A. Kurtz and J.S. Royer, Every polynomial-time 1-degree collapses iff P = PSPACE, *Proc. of the 30th Ann. IEEE Symp. on Found. of Comput. Sci.* (1989) 624-629.

[7] S. Grumbach and T. Milo, Towards tractable algebras for bags, *Journ. of Comput. and Syst. Sci.* **52** (1996) 570-588.

[8] K. Kuratowski, On a topological problem connected with the Cantor-Bernstein theorem, *Fundam. Mathem.* **37** (1950) 213-216.

[9] K. Kuratowski and A. Mostowski, *Set Theory*, (North-Holland, Amsterdam, 1968).

[10] R. Milner, *Communication and Concurrency*, (Prentice-Hall, Englewood Cliffs, NJ, 1989).

[11] R. Milner, Functions as processes, *Math. Struct. in Comput. Sci.* **2** (1992) 119-141.

[12] R. Milner, J. Parrow and D. Walker, A calculus of mobile processes, *Inform. and Comput.* **100** (1992) 1-77.

[13] J. Myhill, Creative Sets, *Zeitschr. fur mathem. Log. und Grundl. der Mathem.* **1** (1955) 97-108.