Acyclicity of Switching Classes

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Abstract

For a finite undirected graph $G = (V, E)$ and a subset $A \subseteq V$, the vertex-switching of G by A is defined as the graph $G_A = (V, E')$, which is obtained from G by removing all edges between A and its complement $\overline{A}$ and adding as edges all nonedges between A and $\overline{A}$. A switching class $[G]$ determined by G consists of all vertex-switchings G_A for subsets $A \subseteq V$. We prove that the trees of a switching class $[G]$ are isomorphic to each other. We also determine the types of trees T that have isomorphic copies in $[G]$. Finally we show that apart from one exceptional type of forests, the real forests in a switching class are isomorphic. Here a forest is real, if it is disconnected.

1 Introduction

We denote by $E(V) = \{xy \mid x, y \in V,\ x \neq y\}$ the set of all unordered pairs of elements from the set V. The graphs in this paper will be finite, undirected and simple, i.e., they contain no loops or multiple edges.

For a (finite) set A we let $|A|$ denote its cardinality.

For a graph $G = (V, E)$ and a subset $A \subseteq V$, the *vertex-switching* of G by A is defined as the graph $G_A = (V, E_A)$, where for each $xy \in E(V)$ with exactly one of x and y in A, we add xy to E if $xy \notin E$, and we remove xy from E if $xy \in E$.

We shall identify each subset $A \subseteq V$ with its characteristic function $A : V \to \mathbf{Z}_2$, where $\mathbf{Z}_2 = \{0, 1\}$ is the cyclic group of order two, and each graph $G = (V, E)$ with the characteristic function $G : E(V) \to \mathbf{Z}_2$ of its set of edges so that $G(xy) = 1$ for $xy \in E$, and $G(xy) = 0$ for $xy \notin E$. With these notations we have for all $xy \in E(V)$,

$$G_A(xy) = A(x) + G(xy) + A(y) \ .$$

Later we shall use both of these notations, $G = (V, E)$ and $G : V(E) \to \mathbf{Z}_2$.

The set $[G] = \{G_A \mid A \subseteq V\}$ is called the *switching class* of G.

In this paper we first prove that every switching class contains at most one tree up to isomorphism. We also characterize the types of trees for which the switching class contains more than one tree up to equality.

We then proceed along the same lines with real forests. A *real forest* is a disconnected graph whose components are trees. A switching class can contain two nonisomorphic real forests, but this happens only for one special type of forests as shown in Section 4.

Related results of vertex-switching have been obtained by Seidel [7], who showed that each switching class contains a unique Euler graph when the graphs have an odd number of vertices. We refer also to Mallows and Sloane [6] for this problem. For a survey of switching classes and their relations to two-graphs we refer to Seidel and Taylor [8]. In this paper we have much adopted the notations of Ellingham [3]. For generalizations of our approach to graphs we refer also to Gross and Tucker [4] and Ehrenfeucht and Rozenberg [2].

Cameron [1] has presented two constructions that yield a switching class generated by a tree, and he has given characterizations of those switching classes in terms of forbidden subgraphs. In particluar, in [1] it is shown that in those switching classes the trees are unique up to isomorphisms.

For a graph G we let $G|X$ denote the *subgraph* of G *induced* by the subset X of the vertices. Hence in the function notation $G|X : E(X) \to \mathbf{Z}_2$.

In the following we use V to denote a (finite) set of vertices and for $A \subseteq V$, we denote the *complement* of A with respect to V with $\overline{A}$.

We end this section with some immediate results on switching classes.

For graphs G and H we define $G + H$ by $(G + H)(e) = G(e) + H(e)$ for $e \in E(V)$. Clearly, the graphs form an abelian group Γ under this operation.

Let $0_V = (V, \emptyset)$ be the *discrete graph* on V. Let K_A denote the *complete bipartite graph* with the partition $(A, \overline{A})$. Then $G|A = G_A|A$ and $G|\overline{A} = G_A|\overline{A}$ for all $A \subseteq V$, and hence $G|A + G_A|A = 0_A$ and $G|\overline{A} + G_A|\overline{A} = 0_{\overline{A}}$. On the other hand, G_A and G differ on edges between A and $\overline{A}$ and thus all those edges exist in $G_A + G$. This means that the result is equal to K_A. Obviously $K_A = K_{\overline{A}}$ and thus also $G_A = G_{\overline{A}}$.

The following lemma is immediate, see also Ellingham [3].

Lemma 1.1

i. $[0_V]$ consists of the complete bipartite graphs on V, and it is a subgroup of Γ.

ii. For all $A \subseteq V$ and $G \in \Gamma$: $G + G_A \in [G_0]$.

iii. For all $A, B \subseteq V$, $(G_A)_B = G_{A+B}$. $\square$

In particular, $(G_A)_A = G$, and $[G] = [G_A]$ for all A.

2 Trees

In this section we prove that every switching class $[G]$ for a graph G contains at most one tree up to isomorphism. Moreover, we indicate the types of trees T for which there exists a subset A so that T and T_A are isomorphic trees.

Define a *star at* x as a tree $K_{1,n-1} = (V, E)$, where $n = |V|$, $x \in V$ and $zy \in E$ if and only if $z = x$ or $y = x$.

We denote by $P_t(m,k)$ the tree that is obtained from the path P_t of t vertices when the leaves are substituted by $K_{1,m}$ and $K_{1,k}$, see Figure 1(b) for $P_2(m,k)$ and Figure 1(d) for $P_4(m,k)$. Similarly, $K^*_{1,m}$ denotes the tree, where the leaves of $K_{1,m}$ are substituted by edges $K_{1,1}$, see Figure 1(a). Further, $K_{1,3}(m,k)$ denotes the tree, where two of the leaves of $K_{1,3}$ are substituted by the stars $K_{1,m}$ and $K_{1,k}$, see Figure 1(c). Note that $K_{1,m} = P_2(1, m-1)$ for all stars $K_{1,m}$ with $m \geq 1$.

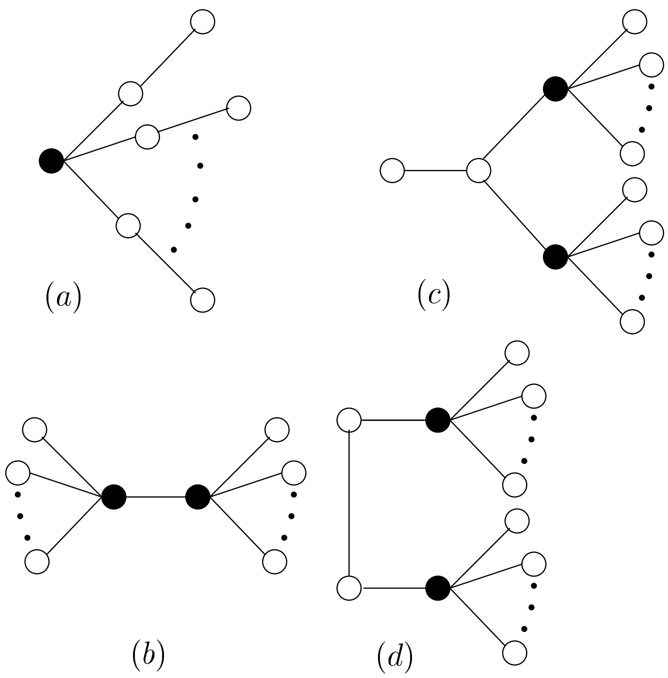

Figure 1: Types of trees having self isomorphic vertex-switchings

We begin with a general result on forests.

Lemma 2.1
Let $F = (V, E)$ and F_A be forests for a subset $A \subseteq V$.

i. If C is a connected component of $F|A$, then C consists of at most two vertices. Moreover, if $C = \{x, y\}$ with $x \neq y$, then $xy \in E$, and

$$\forall z \in \overline{A}: \quad \text{either} \quad zx \in E \quad \text{or} \quad zy \in E \quad \text{but not both} \ . \tag{1}$$

ii. If $F|A$ has an edge, then $F|\overline{A}$ is discrete.

iii. For any $x, y \in A$ with $x \neq y$ there exists at most one $z \in \overline{A}$ such that $zx, zy \in E$.

Proof:
Clearly, $F_A|A$ and $F|A$ have the same connected components. Let $C \subseteq A$ be a connected component of $F|A$. By acyclicity for each $z \in \overline{A}$ there can be at most one edge zx of F

and similarly of F_A such that $x \in C$. On the other hand, each zx with $x \in C$ is an edge either in F or in F_A. This shows the first claim. The second claim follows immediately from this.

The third claim is clear, since if $z_1, z_2 \in \overline{A}$ with $z_i x, z_i y \in E$, then (x, z_1, y, z_2, x) would be a cycle in F. □

Suppose then that $T = (V, E)$ is a tree for which there exists a subset $A \subseteq V$ such that T_A is also a tree. We may suppose that $A \neq \emptyset$ and $A \neq V$, since otherwise $T_A = T$. Further, by Lemma 2.1, we may assume that $T|\overline{A}$ is discrete, since $T_A = T_{\overline{A}}$.

Let $n = |V|$, $p = |A|$ and suppose $T|A$ contains r edges, $x_i y_i$ for $i = 1, 2, \dots, r$ with $x_i, y_i \in A$, where $\{x_i, y_i\}$ are the nonsingleton connected components of $T|A$. Since T is a tree, it has $n-1$ edges, and so there are $n-1-r$ edges of T in $A \times \overline{A}$. Also, T_A has $n-1$ edges, and there are $p(n-p)-(n-1-r)$ edges of T_A in $A \times \overline{A}$. Therefore, the number of edges of T_A is $n-1 = p(n-p)-(n-1-r)+r$, that is,

$$(p-2)n = (p-2)(p+1) + (p-2r) \ . \tag{2}$$

If $p = 1$, then $r = 0$, and $n = 1$, which is a trivial case.

If $p = 2$, then $p = 2r$ and hence $r = 1$, and in this case $A = \{x_1, y_1\}$ with $x_1 y_1 \in E$, and, by (1), $\overline{A} = B_1 \cup B_2$, where $B_1 = \{z \in \overline{A} \mid zx_1 \in E\}$ and $B_2 = \{z \in \overline{A} \mid zy_1 \in E\}$ form a partition of $\overline{A}$. Therefore T is a $P_2(m,k)$ with $m \geq 0$ and $k \geq 1$ of Figure 1(b), where the black vertices are in A. Here T_A is also a $P_2(m,k)$, and thus isomorphic to T.

Assume then that $p > 2$. Now equation (2) becomes

$$n = p + 1 + \frac{p-2r}{p-2} \ . \tag{3}$$

It is immediate that either $2r = p$, or $r = 1$, or $r = 0$. These cases give us the following solutions.

If $2r = p$, then $n = p+1$. Now, $T|A$ consists of r edges and it has no singleton connected components, and $T|\overline{A}$ is a singleton graph. Therefore T is a $K^*_{1,r}$ (with $r \geq 1$) of Figure 1(a), where the black vertex is in $\overline{A}$. Clearly, also in this case T_A is isomorphic to T.

If $r = 1$, then $n = p+2$, and thus $T|A$ has one edge $x_1 y_1$ and $p-2$ isolated vertices, and $T|\overline{A}$ is a discrete graph of two vertices, say z_1, z_2. By (1), there are now two choices: z_1 and z_2 are connected to the same or different vertices of $\{x_1, y_1\}$. From these we obtain that T is either $K_{1,3}(m,k)$ or $P_4(m,k)$ with $m, k \geq 0$ of Figure 1(c) and 1(d), respectively. Again, as is easy to see, T_A is isomorphic to T in both of these cases.

If $r = 0$, then $p = 3$ or $p = 4$. In this case $n = 7$, and there are 11 nonisomorphic trees on seven vertices, see Harary [5]. Of these trees seven are 3-by-4-bipartite. Now, if T and T_A are both trees, then clearly T contains no independent set with two vertices in A and two vertices in $\overline{A}$. Further, T cannot contain a vertex of degree four, and we are then left with only two trees T of seven vertices. These are the trees of Figure 2(a) and 2(b), the first one of which is a path P_7, the second one will be referred to as T_7. For both of these trees T_A is isomorphic to T.

We have thus proved our main theorem.

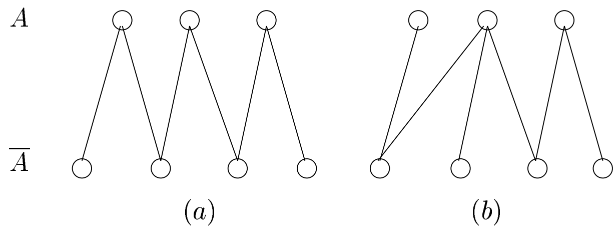

Figure 2: The only two 3-by-4-bipartite trees on seven vertices yielding a tree

Theorem 2.2
Every switching class contains at most one tree up to isomorphism. If it contains more than one tree up to equality, then the tree is one of $K^*_{1,m+1}$, $P_2(m, k+1)$, $K_{1,3}(m, k)$, $P_4(m, k)$ for $m, k \geq 0$, or one of the two special trees P_7 or T_7 of Figure 2 (with the set A as indicated). $\square$

3 Trees into real forests

Let $k, m \geq 0$, and let $S_{k,m}$ denote the tree, which is obtained from a star $K_{1,k+m}$ by substituting m leaves by an edge, see Figure 3(b).

We consider now the case where a tree T produces a real forest T_A.

Suppose that $T = (V, E)$ is a tree such that T_A is a real forest for $A \neq \emptyset$ and $A \neq V$. As above we may assume that $T|\overline{A}$ is discrete. Let again $n = |V|$, $p = |A|$ and suppose $T|A$ contains r edges. Now, T_A has less than $n-1$ edges, and (2) is transformed into

$$(p-2)n < (p-2)(p+1) + (p-2r) \ . \tag{4}$$

If $p = 1$, then clearly $T = K_{1,n-1}$, and hence T_A is the discrete graph.

If $p = 2$, say $A = \{x, y\}$, then $2(1-r) = p - 2r > 0$, and therefore $r = 0$. Since T is connected there exists a vertex $z \in \overline{A}$ such that $zx, zy \in E$, and by Lemma 2.1, the vertex z is unique. Consequently, $\overline{A} = N(x) \cup N(y)$ with $N(x) \cap N(y) = \{z\}$ for the sets $N(x)$ and $N(y)$ of neighbors of x and y. Hence T is a $P_3(m, k)$, where the middle vertex of the P_3 is z, see Figure 3(a) and A is the set of black vertices. In this case T_A is a real forest, where z is isolated, and the edges are xu and yv for all $u \in N(y)$ and $v \in N(x)$.

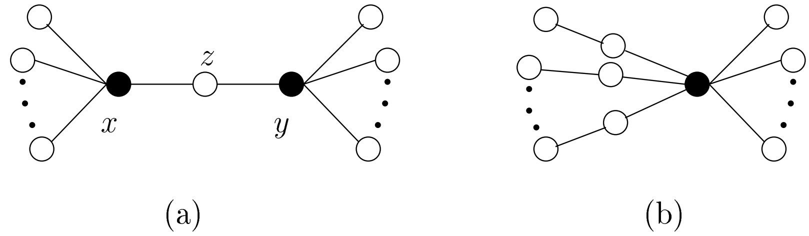

Figure 3: Two types of trees yielding a real forest

If $p > 2$, then (4) gives

$$n < p + 1 + \frac{p - 2r}{p - 2} \ ,$$

which is possible only if $r < p/2$; if $r = p/2$ then we would have $p \geq n$, which cannot happen. Assume first that $p = n - 1$, i.e., $|\bar{A}| = 1$. This case holds always if $r > 0$. The corresponding tree is $T = S_{k,m}$ with $k > 0$, see Figure 3(b), where the black vertex is in $\bar{A}$. Note that this also includes the star graph, namely when $m = 0$.

If $r = 0$ we get

$$n < p + 1 + \frac{p}{p - 2} \ ,$$

in which case $p = 3$ yields $n < 7$ and $p = 4$ also yields $n < 7$. Larger values for p yield $n = p + 1$.

For $p = 3$ only $n = 5$ and $n = 6$ remain and for $p = 4$ only $n = 6$, because the case that $p = n - 1$ has already been taken care of.

If $p = 3$ and $n = 5$ there are three possible trees, a star which is not 3-by-2-bipartite and two others, $P_3(1,1)$ and $S_{2,1}$.

If $p = 3$ and $n = 6$ only the trees P_6, $P_3(1,2)$ and $S_{1,2}$ are 3-by-3-bipartite and yield a real forest.

If $p = 4$ and $n = 6$ there is only one tree, $S_{3,1}$, which is 4-by-2-bipartite.

Theorem 3.1 Let $T = (V, E)$ be a tree such that T_A is a real forest. Then T is $P_3(m,k)$, $S_{k+1,m}$, or P_6. for some $m, k \geq 0$.

4 Real forests

In this section we prove a result analogous to the one in the previous section on trees: every switching class contains at most one real forest up to isomorphism excepting one special kind of forests.

The counterexample is the real forest $S_{k,\ell,m}$ which is formed by adding ℓ isolated nodes to $S_{k,m}$ (see Figure 4(b)). Of course, for $S_{k,\ell,m}$ to be a real forest it is necessary that $\ell > 0$.

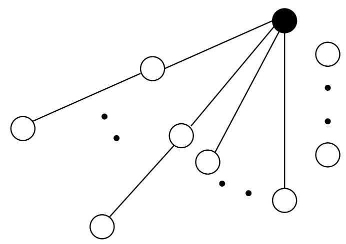

Figure 4: Generic counterexample

If $S = S_{k,\ell,m}$ and we take A to be the one black vertex in the figure, then $S_A = S_{\ell,k,m}$ is a forest of the same type, but S_A is isomorphic to S if and only if $k = \ell$.

Let $F = (V, E)$ be a real forest, $A \subseteq V$ and $p = |A| > 0$, and assume that F_A is a real forest. We suppose that $|A| \le |\overline{A}|$, since $F_A = F_{\overline{A}}$. We prove that if F_A is a real forest, it is isomorphic to F, unless $F = S_{k,\ell,m}$ with $k \ne \ell$.

By Lemma 2.1, we know that either $F|A$ or $F|\overline{A}$ or both are discrete, and that the connected components of $F|A$ and $F|\overline{A}$ are either singletons or edges.

Because F and F_A both have at least two components and no cycles, together they contain at most $2(n-2)$ edges. Therefore, $p(n-p) \le 2(n-2)$, since there are $p(n-p)$ edges in K_A and all of these are in either F or F_A (but not both). We find that $n(p-2) \le p^2 - 4 = (p-2)(p+2)$. Now, either $p = 1$ or $p = 2$, since if $n \le p+2$, then $n \le 4$, because we can assume that $p \le n/2$.

We have thus obtained so far

Lemma 4.1
If $F = (V, E)$ is a real forest with $n = |V|$, and $2 < |A| < n-2$, then F_A is not a real forest. □

Consider first the case $p = 1$, and let $A = \{x\}$. If $|\overline{A}| = 1$, then $n = 2$, and hence $F = S_{0,1,0}$ is discrete, and $F_A = S_{1,0,0}$ is a tree.

Assume then that $|\overline{A}| \ge 2$. By Lemma 2.1(iii), $F = S_{k,\ell,m}$ for some $k, m \ge 0$ and $\ell \ge 1$, and, consequently, $F_A = S_{\ell,k,m}$. Hence, in this case, F_A is a tree if and only if $k = 0$, and otherwise F_A is a real forest. In the latter case, F_A is not isomorphic to F if and only if $k \ne \ell$.

The case that $p = 2$ can be treated as follows. Let $A = \{x, y\}$.

In this case K_A contains exactly $2(n-2)$ edges, while F and F_A both contain at most $n-2$ edges. This implies that F and F_A contain exactly $n-2$ edges and all these edges are between A and $\overline{A}$. Therefore $F|\overline{A}$ is discrete.

Suppose first that x and y belong to different connected components. Now, K_A becomes decomposed into two stars, the leaves of which are the neighbors of x and y, respectively. In this case, F and F_A are clearly isomorphic, see Figure 5(a).

On the other hand, if x and y belong to the same connected component, then, by Lemma 2.1, there exists a unique vertex $v \in \overline{A}$ such that $vx \in E$ and $vy \in E$. Since F is disconnected, there is a vertex $z \in \overline{A}$ that such that $xz, yz \notin E$. Further, because F_A is acyclic and xz, yz are edges of F_A, this vertex z must, like v, be unique. This implies that F and F_A are isomorphic, the isomorphism is the permutation $(x, y)(v, z)$, which leaves all other vertices intact, see Figure 5(b).

All these cases together yield the following main result of this section.

Theorem 4.2
Every switching class contains at most one real forest up to isomorphism, unless it is a class containing $S_{k,\ell,m}$ with $k \ne \ell$ and $k, \ell > 0$. If it contains more than one real forest up to equality, then the forests are of the form given in Figure 5 or is $S_{k,\ell,m}$ with $k \ne \ell$ and $k, \ell > 0$. □

The class containing $S_{1,2,0}$ contains P_3 and $S_{2,1,0}$. Hence there is a class containing three forests up to isomorphism.

From the above we obtain also the following corollary.

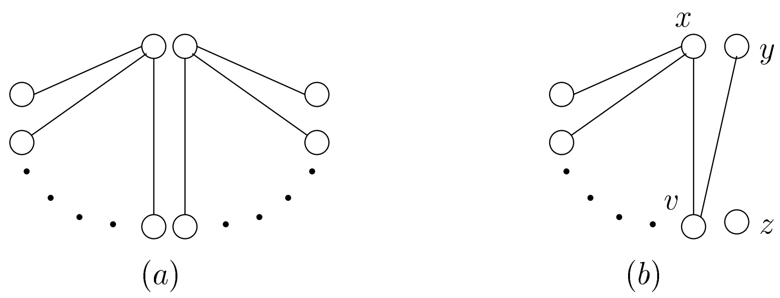

Figure 5: Two cases of self isomorphism

Corollary 4.3
Every switching class contains

- at most two real forests up to isomorphism and the upperbound is reached if and only if it contains $S_{k,\ell,m}$ with $k \neq \ell$ and $k, \ell > 0$.
- at most three forests up to isomorphism. The upper bound is optimal and can only be reached if it contains two real forests up to isomorphism. $\square$

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