Two Categories of Relations

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1 Introduction

Recently there have appeared two notions of 'categories of relations' in the literature. Freyd introduced the notion of *allegory* [FS91], and Carboni and Walters introduced the notion of *cartesian bicategory of relations* [CW87]. On the face of it, the two approaches look rather different. In this paper we show that the latter axiomatization is equal (in a categorical sense) to an enrichment of the former.

The approach by Freyd (see definitions 2.1 and 2.2) is a very smooth axiomatization of the notion of category of relations. But, as Carboni observed [Car93], the theory is rather rigid. In particular, the modular law is not satisfactory from a theoretical point of view: it can not even be stated unless one has an involution which is the "identity on objects", which in nature never occurs unless one is already in a category of relations. Hence Carboni and Walters proposed another axiomatization (definition 2.4 below) which is "more categorical". So both approaches have their strong points, and the result of this paper allows to exploit them both without any penalty at all. This seems to be particular relevant since categories of relations have been used recently in theoretical computer science to model nondeterministic programs [HJ86, She90, Man85].

2 Two categories of relations

In this section we repeat the definition of the two categories. For the composition of two arrows R and S in a category, $R \cdot S$ means first S and then R. Furthermore, we assume that composition binds more tightly than intersection. For further categorical background, consult [FS91, Mac71]. The first definition of a categry of relations we consider in this paper was introduced by Freyd [FS91].

Definition 2.1 *1. A category* **A** *is an* allegory *iff it is a locally ordered 2-category, whose hom-posets have binary meets and an anti-involution $R \mapsto R^\circ$ satisfying the modular law*

$$(R \cdot S) \cap T \leq (R \cap (T \cdot R^\circ)) \cdot R$$

2. An allegory is unitary *iff it has an object U (the unit) such that 1_U is the largest morphism $U \to U$, and for every object A there exists a morphism $t_A : A \to U$ such that $t_A{}^\circ \cdot t_A \geq 1_A$.*

We call a relation $R : X \to Y$ a *partial map* if it is single valued, that is, if $R \cdot R^\circ \leq 1$. The relation is called a *map* if it is moreover total, that is, if $R \cdot R^\circ \leq 1$ and $R^\circ \cdot R \geq 1$. In other words, if R is a map, then R° is its right adjoint ($R \dashv R^\circ$). One can prove that a relation R has a right adjoint iff R° is this right adjoint. A *tabulation* of a relation $R : X \to Y$ is given by two maps $f : Z \to X$ and $g : Z \to Y$ such that $g \cdot f^\circ = R$ and $f^\circ \cdot f \cap g^\circ \cdot g = 1_Z$. The last condition says that f and g are jointly monic. For two objects X and Y in an unitary allegory, the relation $t_Y^\circ \cdot t_X : X \to Y$ can be shown to be the largest relation from X to Y. We denote it by $m_{X,Y}$. In the sequel, for a category of relations $\mathcal{C}$, we let $\mathbf{Map}(\mathcal{C})$ denote the subcategory of $\mathcal{C}$ consisting of all objects and all maps.

Definition 2.2 *A unitary allegory is called* pre-tabular *iff for all objects X and Y, the morphism $m_{X,Y}$ has a tabulation. We denote this tabulation by $\pi : X \otimes Y \to X$ and $\pi' : X \otimes Y \to Y$. We denote the category of pre-tabular unitary allegories, and structure preserving functors, by* **pTUA**.

We have the following lemma [FS91].

Lemma 2.3 $\mathbf{Map}(\mathbf{A})$ *is a cartesian category. The product of X and Y is given by the tabulation of $m_{X,Y}$.*

A *tabular unitary allegory* is a unitary allegory in which each relation has a tabulation. In [FS91] it is shown that every pre-tabular unitary allegory can be fully and faithfully embedded in a tabular unitary allegory. Furthermore, each tabular unitary allegory is isomorphic to the category $\mathbf{Rel}(\mathcal{C})$ of relations of a regular category $\mathcal{C}$ [FS91, JMP92]. Hence we can consider each pre-tabular unitary allegory as a subcategory of a $\mathbf{Rel}(\mathcal{C})$, with $\mathcal{C}$ regular.

Next we introduce the second categorical structure for axiomatizing relations, as proposed by Carboni and Walters [CW87].

Definition 2.4 *A category* **B** *is a* cartesian bicategory of relations *iff it is a locally ordered 2-category, equipped with a functorial tensor product $\otimes : \mathbf{B} \times \mathbf{B} \to \mathbf{B}$, which has an identity object I and natural isomorphisms*

$$\rho : X \to X \otimes I \qquad\qquad \gamma : X \otimes Y \to Y \otimes X$$

$$\alpha : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

satisfying the classical coherence conditions. Furthermore, for every object X in $\mathbf{B}$ there exists a comonoid structure

$$\Delta_X : X \to X \otimes X \qquad\qquad t_X : X \to I$$

satisfying the following axioms

1. *The arrows Δ_X and t_X satisfy the equations for X to be a cocommutative comonoid object [Mac71] (see figure 1).*
2. *Each morphism $R : X \to Y$ is a lax comonoid homomorphism. That is,*

$$\Delta_Y \cdot R \leq (R \otimes R) \cdot \Delta_X \quad \text{and} \quad t_Y \cdot R \leq t_X$$

3. *For each object X, Δ_X and t_X have a right adjoint Δ_X^* and t_X^*, respectively. This cocommutative comonoid structure is the* only *cocommutative comonoid structure on X with structure morphisms having right adjoints.*

Furthermore, every object X is discrete, *in the sense that*

$$\Delta_X \cdot \Delta_X^* = (\Delta_X^* \otimes 1) \cdot \alpha \cdot (1 \otimes \Delta_X)$$

We denote the category of cartesian bicategories of relations, and structure preserving functors, by **CRel**.

The remarkable thing about cartesian bicategories of relations is that local limits and the involution operator are definable [CW87]. To be precise, given two relations $R, S : X \to Y$, their intersection is given by

$$R \cap S = \Delta_Y^* \cdot (R \otimes S) \cdot \Delta_X$$

The involution of a relation $R : X \to Y$ is given by the composite

$$Y \xrightarrow{\ \rho\ } Y \otimes I \xrightarrow{1 \otimes \eta} Y \otimes X \otimes X \xrightarrow{1 \otimes R \otimes 1} Y \otimes Y \otimes X \xrightarrow{\epsilon \otimes 1} I \otimes X \xrightarrow{\rho^{-1}\gamma} X$$

where

$$\eta_X = I \xrightarrow{t_X^*} X \xrightarrow{\Delta_X} X \otimes X$$

$$\epsilon_X = X \otimes X \xrightarrow{\Delta_X^*} X \xrightarrow{t_X} I$$

Again we call a relation R having a right adjoint a map. One can prove that this right adjoint necessarily equals the involution. The following lemma is proved in [CW87].

Lemma 2.5 Map(B) *is cartesian category. For two objects X and Y, the product is given by $X \otimes Y$, with projections $\pi = \rho^{-1} \cdot (1 \otimes t)$ and $\pi' = \rho^{-1} \cdot \gamma \cdot (t \otimes 1)$. Furthermore, for two relations R and S, $R \otimes S = (\pi^\circ \cdot R \cdot \pi) \cap (\pi'^\circ \cdot S \cdot \pi')$.*

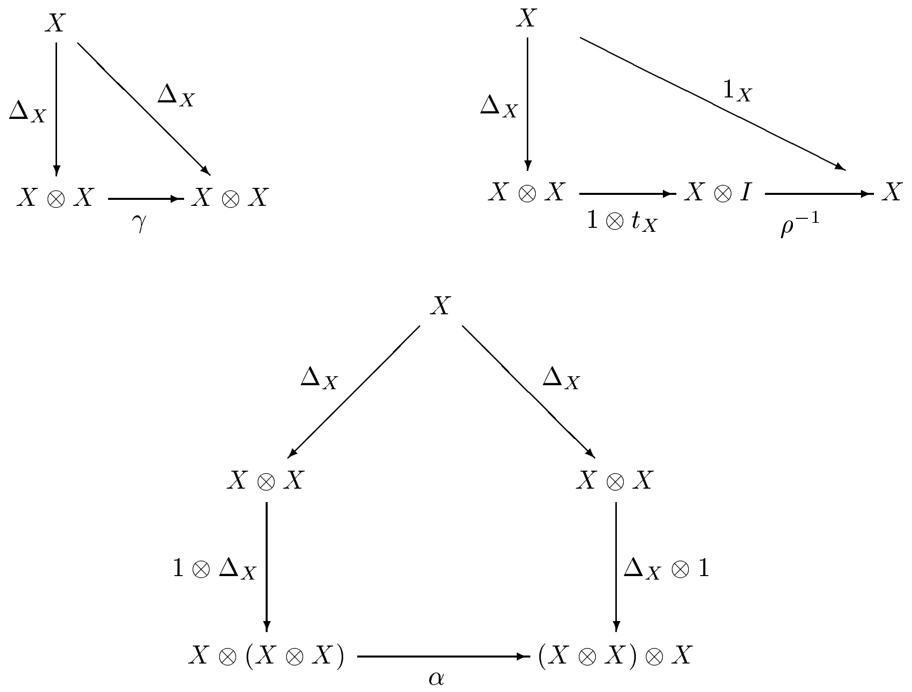


Figure 1: Cocommutative comonoid object X

3 The isomorphism

It has been observed earlier [Car93] that the two categories **pTUA** and **CRel** are equivalent. In this section we prove that they are in fact isomorphic.

First we describe the functor $\mathcal{A} : \mathbf{CRel} \to \mathbf{pTUA}$. Let $\mathbf{B}$ be a cartesian bicategory of relations. $\mathcal{A}(\mathbf{B})$ is given by the following data.

- The objects and morphisms of $\mathcal{A}(\mathbf{B})$ are the objects and morphisms of $\mathbf{B}$. The order on the homsets in $\mathcal{A}(\mathbf{B})$ is the order in $\mathbf{B}$.
- The involution is given by the (definable) involution of $\mathbf{B}$.
- The unit is given by I.
- The tabulation of $m_{X,Y}$ is given by $\pi : X \otimes Y \to X$ and $\pi' : X \otimes Y \to Y$.

Lemma 3.1 $\mathcal{A}(\mathbf{B})$ *is a pre-tabular unitary allegory.*

Proof The proof follows from a number of theorems in [CW87]. By Theorem 1.6, the homsets of $\mathbf{B}$ have finite products. By Theorem 2.4, there exists an anti-involution such that $(R \cap S)^\circ = R^\circ \cap S^\circ$. The modular law follows from Remark 2.9(ii). The axioms for the unit follow from Theorem 1.6(ii). Finally, by Theorem 2.8(i), the projections form a tabulation of m. $\square$

Next we describe the functor $\mathcal{C} : \mathbf{pTUA} \to \mathbf{CRel}$. Let $\mathbf{A}$ be a pre-tabular unitary allegory. $\mathcal{C}(\mathbf{A})$ is given by the following data.

- The objects and morphisms of $\mathcal{C}(\mathbf{A})$ are the objects and morphisms of $\mathbf{A}$. The order on the homsets in $\mathcal{C}(\mathbf{A})$ is the order in $\mathbf{A}$.
- The tensor product of two objects A and B is given by the domain of the tabulation of $m_{A,B}$.
- The tensor product of two arrows $R : A \to C$ and $S : B \to D$ is given by
$$R \otimes S = \pi^{\circ}_{C,D} \cdot R \cdot \pi_{A,B} \;\cap\; {\pi'_{C,D}}^{\circ} \cdot S \cdot \pi'_{A,B}$$
- The identity object of the tensor product is given by the unit.
- The required natural isomorphisms are given by
$$\rho = \langle 1, t\rangle = \pi^{\circ} \qquad \gamma = \langle \pi', \pi\rangle \qquad \alpha = \langle\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle$$
where $\langle\cdot,\cdot\rangle$ is the pairing operator of $\mathbf{Map}(\mathbf{A})$.
- $\Delta = \langle 1, 1\rangle$ and t is already present in $\mathbf{A}$.

Lemma 3.2 $\mathcal{C}(\mathbf{A})$ *is a cartesian bicategory of relations.*

Proof We work in the internal language of the regular category associated with $\mathbf{A}$. Then to prove that $\otimes$ is functorial, we need to show that $(T \times U) \cdot (R \times S) = (T \cdot R) \otimes (U \cdot S)$. This amounts to showing that in the internal language, for any relations R, S, T, U,

$$\exists c, d. T(c, e) \wedge U(d, f) \wedge R(a, c) \wedge S(b, d) \dashv\vdash (\exists c. T(c, e) \wedge R(a, c)) \wedge (\exists d. U(d, f) \wedge S(b, d))$$

This equivalence holds in any regular category, by Frobenius Reciprocity (see [MR77]). It follows immediately from the definition that $\otimes$ preserves the order on hom sets. Hence $\otimes$ is a homomorphism of bicategories.

The arrows ρ, γ and α are maps, and that they are isomorphisms satisfying the classical coherence conditions follows from the fact that they are so in $\mathbf{Map}(\mathbf{A})$. Using the internal language again, it is easy to show that they are natural.

For each object X, the maps Δ_X and t_X satisfy the axioms for X to be a cocommutative comonoid structure, since they do so in $\mathbf{Map}(\mathbf{A})$.

Let $R : X \to Y$ be an arrow in $\mathbf{A}$. Then $t_Y \cdot R \leq t_X$, since it holds in $\mathbf{A}$. Furthermore,

$$\begin{aligned}
\Delta \cdot R &= (\pi^{\circ} \cap \pi'^{\circ}) \cdot R \\
&\leq \pi^{\circ} \cdot R \;\cap\; \pi'^{\circ} \cdot R \\
&= \pi^{\circ} \cdot R \cdot \pi \cdot \Delta \;\cap\; \pi'^{\circ} \cdot R \cdot \pi' \cdot \Delta \\
&= (R \otimes R) \cdot \Delta
\end{aligned}$$

Hence R is a lax comonoid homomorphism.

Suppose $\tilde{\Delta}$ and $\tilde{t}$ is another cocommutative comonoid structure. Then both t and $\tilde{t}$ are maps to the terminal object in $\mathbf{Map}(\mathbf{A})$ and hence they are equal.

From the counit axiom (second diagram in figure 1), it follows that $\pi \cdot \tilde{\Delta} = 1$ and $\pi' \cdot \tilde{\Delta} = 1$. Hence $\tilde{\Delta} = \langle 1, 1 \rangle = \Delta$, since $\mathbf{Map}(\mathbf{A})$ is cartesian.

It is straightforward to show that Discreteness holds, using the internal language. $\square$

Lemma 3.3 $\mathcal{C}(\mathcal{A}(\mathbf{B})) = \mathbf{B}$.

Proof Obviously, $\mathcal{C}(\mathcal{A}(\mathbf{B}))$ and $\mathbf{B}$ are the same bicategory. So we only need to check that the extra structure of $\mathcal{C}(\mathcal{A}(\mathbf{B}))$ and $\mathbf{B}$ coincide. For example, suppose Δ is the diagonal in $\mathbf{B}$ and Δ' is the diagonal in $\mathcal{C}(\mathcal{A}(\mathbf{B}))$. Since $\mathbf{Map}(\mathbf{B})$ and $\mathbf{Map}(\mathcal{A}(\mathbf{B}))$ are the same cartesian categories, $\Delta = \langle 1, 1 \rangle = \Delta'$. The other cases are proved similarly. $\square$

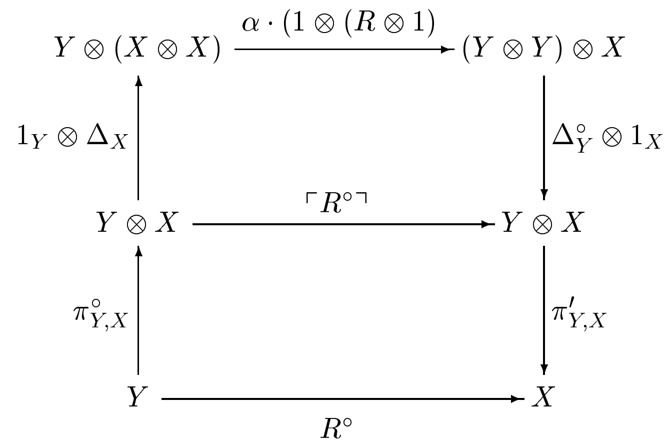
Lemma 3.4 $\mathcal{A}(\mathcal{C}(\mathbf{A})) = \mathbf{A}$.

Proof Again, it is easy to see that $\mathcal{A}(\mathcal{C}(\mathbf{A}))$ and $\mathbf{A}$ are the same bicategory. Let (π, π') be the tabulation of the maximal relation $m_{X,Y}$ in $\mathbf{A}$, and $(\tilde{\pi}, \tilde{\pi}')$ the tabulation of $m_{X,Y}$ in $\mathcal{A}(\mathcal{C}(\mathbf{A}))$. Now, reasoning in $\mathbf{Map}(\mathbf{A})$,

$$\tilde{\pi}_{X,Y} = \rho_X^{-1} \cdot (1_X \otimes t_Y) = \pi_{X,U} \cdot (1_X \otimes t_Y) = \pi_{X,U} \cdot \langle \pi_{X,Y}, t_Y \cdot \pi'_{X,Y} \rangle = \pi_{X,Y}$$

Hence, for all objects X and Y, $m_{X,Y}$ has the same tabulation in $\mathbf{A}$ and $\mathcal{A}(\mathcal{C}(\mathbf{A}))$.

Next we have to show that the involution in $\mathbf{A}$ and in $\mathcal{A}(\mathcal{C}(\mathbf{A}))$ coincide. Consider the following diagram.



where $\ulcorner S \urcorner = 1 \cap \pi'^{\circ} \cdot S \cdot \pi$. It is easy to see that $(1 \times \Delta) \cdot \pi^{\circ} = (1 \otimes \eta) \cdot \rho$. Hence the uppermost path from Y to X is the converse of R according to Carboni and Walters. For any relation S, $S = \pi' \cdot \ulcorner S \urcorner \cdot \pi^{\circ}$:

$$\pi' \cdot (1 \cap \pi'^{\circ} \cdot S \cdot \pi) \cdot \pi^{\circ} \leq \pi' \cdot \pi'^{\circ} \cdot S \cdot \pi \cdot \pi^{\circ} \leq S$$

$$S = m \cap S = \pi' \cdot \pi^{\circ} \cap S \leq \pi' \cdot \ulcorner S \urcorner \cdot \pi^{\circ}$$

where the last inequality follows from the modular law. This shows that the lower diagram commutes.

Next, since $\ulcorner R^\circ \urcorner \leq 1$, we have that $\ulcorner R^\circ \urcorner^\circ = \ulcorner R^\circ \urcorner$ [FS91]. Hence $\ulcorner R^\circ \urcorner = 1 \cap \pi^\circ \cdot R \cdot \pi'$. From this it easily follows that the upper diagram commutes. □

We can extend the operation $\mathcal{A}$ to a functor $\mathcal{A} : \mathbf{CRel} \to \mathbf{pTUA}$, by stipulating that $\mathcal{A}(F) = F$. It is easy to see that $\mathcal{A}(F)$ is a functor preserving the structure of pre-tabular unitary allegories, hence $\mathcal{A}$ is a functor. Analogously, we extend the operation $\mathcal{C}$ to a functor $\mathcal{C} : \mathbf{pTUA} \to \mathbf{CRel}$.

We now arrive at the main theorem of this paper.

Theorem 3.5 $\mathbf{pTUA} \cong \mathbf{CRel}$.

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