A characterization of acyclic switching classes using forbidden subgraphs

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Abstract

We characterize the switching classes that do not contain an acyclic graph. The characterization is by means of a set of forbidden graphs. We prove that in addition to switches of the cycles C_n for $n \geq 7$ there are only finitely many such graphs. In fact, there are no such graphs with more than 9 vertices. We give a representative of each of the 24 classes.

1 Introduction

For a finite undirected graph $G = (V, E)$ and a set $\sigma \subseteq V$, the switch of G by σ is defined as the graph $G^\sigma = (V, E')$, which is obtained from G by removing all edges between σ and its complement $\overline{\sigma}$ and adding as edges all nonedges between σ and $\overline{\sigma}$. The switching class $[G]$ determined by G consists of all switches G^σ for subsets $\sigma \subseteq V$.

A switching class is an equivalence class of graphs under switching, see the survey papers by Seidel [7] and Seidel and Taylor [8]. Generalizations of this approach can be found in Gross and Tucker [4], Ehrenfeucht and Rozenberg [3], and Zaslavsky [10].

In this paper we solve a problem raised by Acharya [1] and by Zaslavsky in his dynamic survey in 1999 [11], which asks for a characterization of those graphs that have an acyclic switch. Forbidden graphs for perfect graphs in switching classes were treated by Hertz [6].

We show that apart from the simple cycles C_n for $n \geq 7$, there are only finitely many critically cyclic graphs (with respect to switching), that is, graphs G which have no acyclic switches G^σ, but all of whose induced proper subgraphs do have an acyclic switch.

In fact, we shall prove that a critically cyclic graph $G \notin [C_n]$ has order at most 9. These graphs are partitioned into 24 switching classes, and altogether there are 905 critically cyclic graphs of order at most 9 (up to isomorphism and excluding switches of the cycles C_n).[1]

In order to save the reader from long – and occassionally tedious – technical constructions for the small graphs, we rely on a computer program (in fact, two independent ones) for the cases of order at most 9. Therefore our purpose is to prove that if G is a critically cyclic graph of order $n \geq 10$, then $G \in [C_n]$. The proof of this result uses the characterization from [5] of the acyclic graphs G – henceforth called the special graphs – that have a non-trivial acyclic switch, see Section 3.

The paper is structured as follows: after some preliminaries we list the necessary details of the special graphs from [5]. Then we proceed with our actual results proving that critically cyclic graphs can have only a limited number of isolated vertices and as a consequence, a vertex in a critically cyclic graph has only a limited number of leaves adjacent to it. We prove that each critically cyclic switching class, except $[C_n]$ for $n \geq 8$, contains a (critically cyclic) graph, which is, except for two vertices, a special graph. By verifying that for each type of special graph a contradiction results – under the condition that the order of the graph is at least 10 – we finally prove our result. At the end of the paper we shall spend some time

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[1] Contact the first author for the ps-file containing all of them.

on the programs used and we consider also the question why not all of the critically cyclic switching classes are used in our proof.

2 Premilinaries

For a (finite) set V, let $|V|$ be the *cardinality* of V. We shall often identify a subset $A \subseteq V$ with its characteristic function $A : V \to \mathbf{Z}_2$, where $\mathbf{Z}_2 = \{0, 1\}$ is the cyclic group of order two. We use the convention that for $x \in V$, $A(x) = 1$ if and only if $x \in A$. The symmetric difference of two sets A and B is denoted $A + B$, and for the difference between A and B we write $A - B$. The restriction of a function $f : V \to W$ to a subset $A \subseteq V$ is denoted by $f|_A$.

The set $E(V) = \{\{x, y\} \mid x, y \in V,\ x \neq y\}$ denotes the set of all unordered pairs of distinct elements of V. A *graph* is a pair $G = (V, E)$ where V is the set of vertices and $E \subseteq E(V)$ the set of edges. We write xy or yx for the undirected edge $\{x, y\} \in E$; we call x and y *adjacent*. By convention we write $x \in G$ for $x \in V$. The graphs of this paper will be finite, undirected and simple, i.e., they contain no loops or multiple edges. The cardinalities $|V|$ and $|E|$ are called the *order*, respectively, *size* of G. Analogously to sets, a graph $G = (V, E)$ will be identified with the characteristic function $G : E(V) \to \mathbf{Z}_2$ of its set of edges so that $G(xy) = 1$ for $xy \in E$, and $G(xy) = 0$ for $xy \notin E$. Later we shall use both notations, $G = (V, E)$ and $G : E(V) \to \mathbf{Z}_2$, for graphs.

For a graph $G = (V, E)$ and $X \subseteq V$, let $G|_X$ denote the *subgraph* of G *induced* by X. Hence, $G|_X : E(X) \to \mathbf{Z}_2$. As shorthand we write $G - x$ for the graph $G|_{V-\{x\}}$ and, more generally, $G - I$ for $G|_{V-I}$. If for some $X \subseteq G$, $G|_X$ has edges between all pairs of distinct vertices, then we call it a *clique*. If $G|_X$ has no edges at all, then X is called *independent*.

For two graphs G and H on V we define $G + H$ to be the graph such that $(G + H)(xy) = G(xy) + H(xy)$ for all $xy \in E(V)$, where $+$ is addition modulo 2. We extend this operation to graphs on sets of vertices V and V' respectively, by first extending them to graphs on $V \cup V'$ and setting all new edges to 0.

The disjoint union of two graphs G and H on the other hand is denoted $G \cup H$. We use $k \cdot G$ as shorthand for the disjoint union of k copies of G.

Some graphs we shall encounter in the sequel are K_n, the clique on n vertices, and $K_{m,n}$ the complete bipartite graph on a two disjoint sets of m and n vertices respectively. The graph P_n denotes a path of n vertices and C_n denotes a cycle on n vertices.

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, the set $N_G(v) \subseteq V$ is the set of vertices adjacent to v in G. The *degree* of v is defined by $d_G(v) = |N_G(v)|$. An *isolated vertex* has degree zero, a *leaf* degree one. A vertex v is a *leaf at z* if v is a leaf adjacent to z.

An *acyclic* graph is a graph without cycles. A *tree* is a connected acyclic graph. If we fix the root of the tree, say r, then the *depth* of a vertex v in that tree is well-defined: it is the number of edges on the shortest path between v and x. Hence r has depth zero.

A *selector* for G is a subset $\sigma \subseteq G$, or alternatively a function $\sigma : G \to \mathbf{Z}_2$. A *switch* of a graph G by σ is the graph G^σ such that for all $xy \in E(V)$,

$$G^\sigma(xy) = \sigma(x) + G(xy) + \sigma(y) \ .$$

For a singleton selector $\sigma = \{x\}$ we shall write G^x instead of $G^{\{x\}}$ by convention.

It should be clear that this definition of switching is equivalent to the one given in the introduction. In Figure 1(7-3) a graph, the Chapel, is given and one of its switches is the graph in Figure 5(7-3'). As we shall continue to do in this paper, the selector is indicated by the black vertices.

The set $[G] = \{G^\sigma \mid \sigma \subseteq V\}$ is called the *switching class* of G. We reserve lower case σ, τ for selectors (subsets) used in switching.

A selector σ is *constant* on $X \subseteq V$ if $X \subseteq \sigma$, or $X \cap \sigma = \emptyset$. The name arises from the fact that $G|_X = G^\sigma|_X$. Note that always $G^\sigma = G^{V-\sigma}$.

This paper concerns itself with those graphs that do not have an acyclic switch. We call these graphs *forbidden*. Obviously, if a forbidden graph occurs in another graph, then the latter is also forbidden. For this reason we are interested in the graphs that are minimal in this respect: they do not have an acyclic switch, but all their induced subgraphs do have an acyclic switch. We call these graphs and the corresponding switching class *critically cyclic*. A switch of a critically cyclic graphs is also critically cyclic so the latter notion is well-defined.

Let G be a critically cyclic graph. By definition, for all $x \in V$, there is a switch G^σ such that $G^\sigma - x$ is acyclic. As a consequence, all cycles in G^σ go through x and there is at least one such cycle. Note that this also holds for $G^{\sigma+\{x\}}$. Note that it does not hold that in every critically cyclic graph G there is a vertex x so that $G - x$ is acyclic; the graph $K_{3,3} \cup 3 \cdot K_1$ of Figure 3(9-2) is a counterexample.

Example 2.1
Let G be the graph of Figure 5(7-3'). We want to prove that it is a critically cyclic graph. For this we must show that it has no acyclic switches and removing any of the vertices allows for an acyclic switch. For the latter it is sufficient to observe that the vertices $2, \cdots, 6$ are all on the only cycle of G, and $G^{\{2,5\}} - 7$ and $G^{\{3,6\}} - 1$ are acyclic.

To prove that G has no acyclic switch observe that G has seven edges and an acyclic graph can have at most six. We shall now prove that applying any selector will not decrease the number of edges, and thereby we have proved that there is no acyclic switch.

First of all, the degree of every vertex in G is at most $3 = (n-1)/2$. Hence applying a singleton selector cannot decrease the number of edges.

For doubleton selectors, $\sigma = \{x_1, x_2\}$, we can do the same. The number of edges that changes is $|\sigma| \cdot (7 - |\sigma|) = 10$. We must make sure then that every selector makes at most five edges disappear. The only possible way, knowing that the maximum degree is three, is to take $\sigma = \{2, 6\}$, but in that case only four edges are removed, because one edge occurs in $G|_\sigma$.

For selectors of size 3, finally, twelve edges will change. Hence we must look for selectors that create less than six edges (or, in other words, make more than six edges disappear). For this, the selector must contain a vertex of degree three, say $\{2\}$. If we would also have $6 \in \sigma$, then the number of edges to be removed is four and there are no other vertices of degree three. Adding two vertices of degree two to σ results always in a selector having at most six edges going to its complement, because always either the two of them are adjacent, or one of them is adjacent to vertex 2. $\diamond$

Note that C_n for $n \leq 6$ have an acyclic switch: take an independent set of cardinality $\lfloor n/2 \rfloor$. However, the following was already proved by Acharya [1].

Lemma 2.2
The cycles C_n for $n \geq 7$ are critically cyclic.
Proof:
First of all, removing any vertex gives us an acyclic graph P_{n-1} and hence we have to prove that all switches of $C_n, n \geq 7$, have a cycle.

Let $\{x_1, \dots, x_n\}$ be the vertices of C_n. We first treat the selectors that select the same value, say 1, in two adjacent vertices, say x_1 and x_2. We need only consider nonconstant selectors and without loss of generality we may assume that $\sigma(x_n) = 0$. Now $\sigma(x_3) = 0$, because otherwise G^σ has a triangle $\{x_n, x_2, x_3\}$. Then $\sigma(x_4) = 1$, because otherwise $\{x_1, x_2, x_4\}$ is a triangle. The same holds for x_5 and now we have a triangle, $\{x_n, x_4, x_5\}$ in G^σ, since $n \geq 7$ implies that x_5 is not a neighbour of x_n. This takes care of all C_n, where $n \geq 7$ is odd.

The only case left is the selector σ that selects only the odd numbered vertices of C_n. It is easy to verify that C_8^σ is isomorphic to itself, and if $n \geq 10$ and even, then $\{x_1, x_2, x_4, x_5, x_7, x_8\}$ induces a C_6 in C_n^σ. $\square$

We now state the result of our computer search for the critically cyclic graphs.

Theorem 2.3
There are 27 switching classes of critically acyclic graphs of order $n \leq 9$. The representatives of these are given in the Figures 1, 2 and 3.

The main theorem proved in this paper is the following.

Theorem 2.4
The switching classes $[C_n]$ are the only critically cyclic switching classes of order $n \geq 10$.

In our proofs we shall refer to the graphs from Figure 1, 2, 3 and 5. The black vertices in the latter figure indicate how these graphs can be switched into the corresponding graph from the former three figures. We shall use Theorem 2.3 to the extent that they are in fact critically cyclic graphs. We shall not use that these are in fact *all* of them of order at most 9.

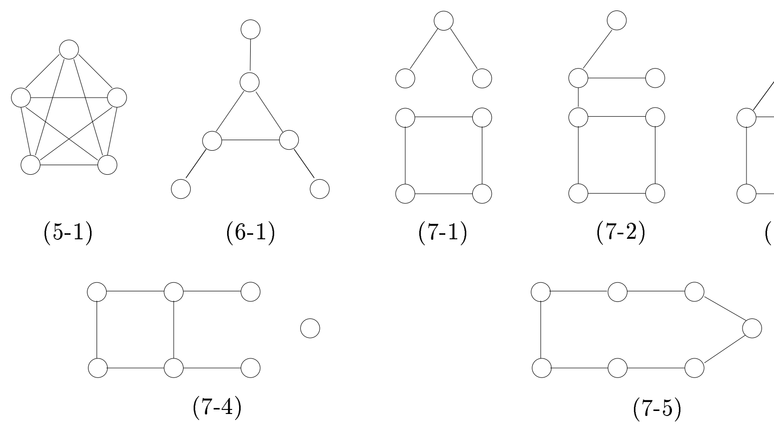

Figure 1: The critically cyclic graphs on five, six and seven vertices

3 The special graphs

We define the special graphs of [5] (see Figure 4). We shall use these graphs extensively in our proofs. The graphs have in common that they have nonconstant switchings into an acyclic graph.

The graph in Figure 4(1s) is denoted by $S_{k,m,\ell}$. It is a graph $K_{1,k+m}$ where k of the $k+m$ leaves are substituted by an edge, and to which ℓ isolated vertices have been added. We let, see also Figure 4(1s),

(S1) z be the centre of S,

(S2) $H = \{(z, y_i, x_i) \mid i = 1, 2, \ldots, k\}$ be the extended star of S, where $S(zy_i) = 1 = S(y_i x_i)$ for all i,

(S3) $I = \{u_1, u_2, \ldots, u_\ell\}$ be the set of isolated vertices of S, and

(S4) $M = \{v_1, v_2, \ldots, v_m\}$ be the set of leaves adjacent to z in S.

Note that by the selector $\{z\}$, the graph $S_{k,m,\ell}$ is switched into $S_{k,\ell,m}$.

The types (2s)-(8s) of graphs are denoted $S(k, m)$, where k and m indicate the number of leaves of the (black) vertices z_1 and z_2. Because of the symmetry in k and m in each of these graphs we may assume that $k \geq m$.

A graph of type (2s) is simply the disjoint union $K_{1,k} \cup K_{1,m}$. Adding an isolated vertex to a graph of this type gives a graph of type (3s).

We denote by $P_t(m, k)$ the tree that is obtained from the path P_t of t vertices when the leaves are substituted by $K_{1,m}$ and $K_{1,k}$, see Figure 4(4s) for $P_3(k, m)$ (adding an isolated vertex gives a graph of type (5s)), Figure 4(6s) for $P_2(k, m)$ and Figure 4(8s) for $P_4(k, m)$. Further, $K_{1,3}(k, m)$ denotes the tree, where two of the leaves of $K_{1,3}$ are substituted by the stars $K_{1,k}$ and $K_{1,m}$, see Figure 4(7s).

The acyclic graphs P_7, T_7, P_6 and $P_4 \cup P_2$ are listed in Figure 4(9s), (10s), (11s) and (12s) respectively. Their role is strictly limited in this paper, because of their low order.

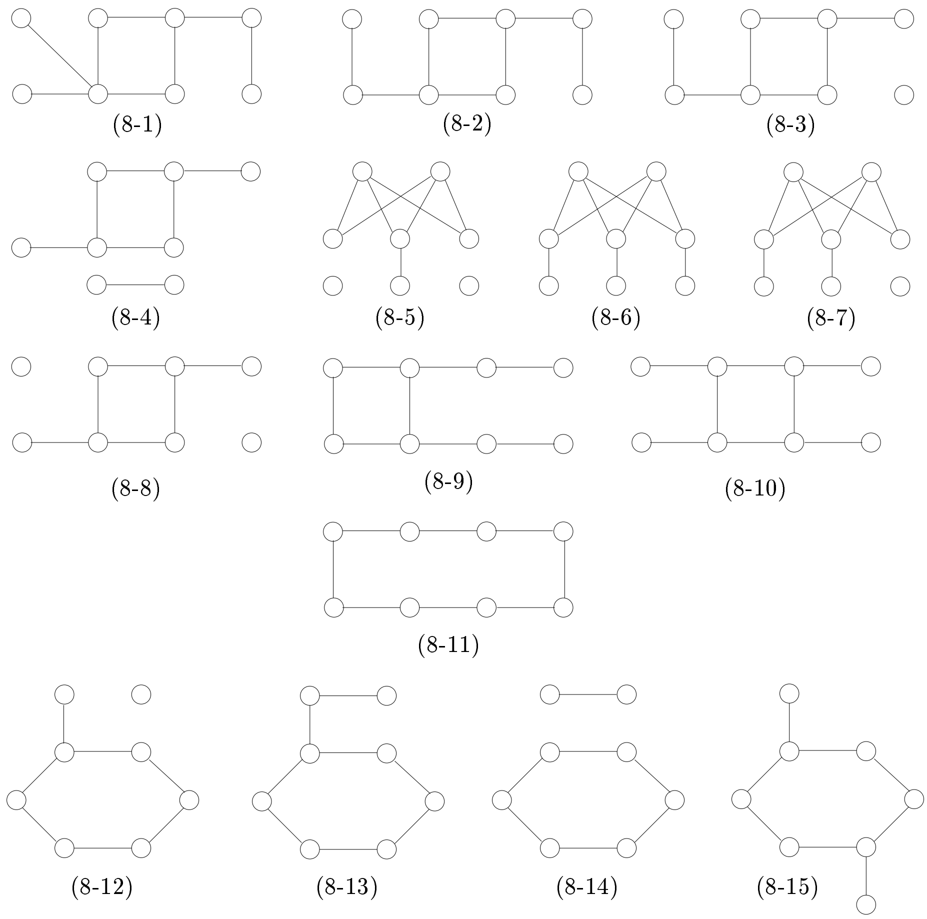

Figure 2: Critically cyclic graphs on eight vertices

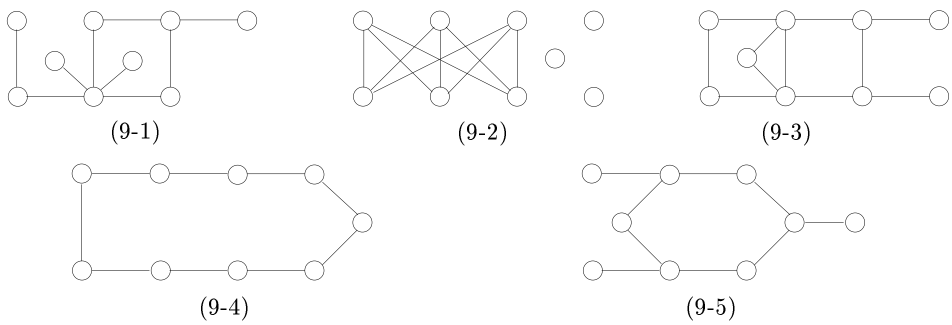

Figure 3: Critically cyclic graphs on nine vertices

Notice that P_6 equals $P_4(1,1)$ of the type (8s), but we wish to treat this small instance independently.

In [5] we proved

Theorem 3.1

i. Every switching class contains at most one tree up to isomorphism. The trees that have a nonconstant switch into a tree are fully characterized by (6s)-(10s), and (1s) for $m, \ell = 0$.

ii. Every switching class contains at most three acyclic graphs up to isomorphism. The acyclic graphs that have a nonconstant acyclic switch are fully characterized by (1s)-(12s) (the switches are indicated by the black vertices).

The graphs of all except a few of the types, switch into an isomorphic copy of themselves if we apply the selector indicated by the blackened vertices, the *centres* of the special graphs. There are five exceptions: a graph $S_{k,m,\ell}$ of type (1s) switches into $S_{k,\ell,m}$ and these are only isomorphic if $m = \ell$, and a graph of type (3s) switches into a graph of type (4s) (and vice versa). Finally, the graphs (11s) and (12s) switch into each other.

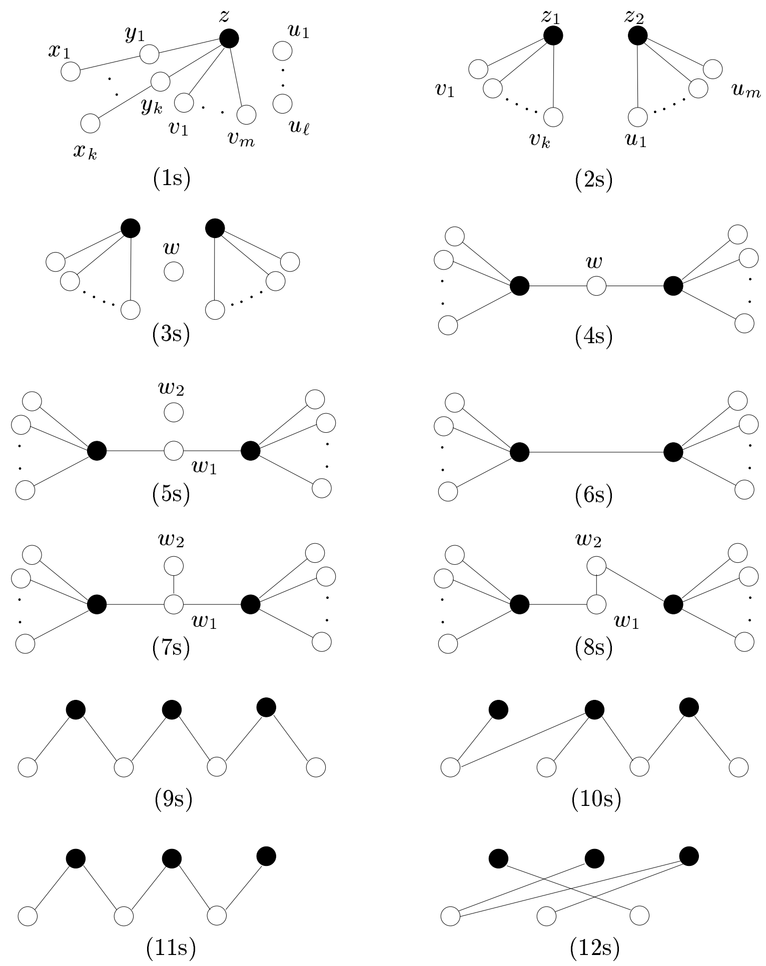

Figure 4: The special graphs (1s)-(12s)

In the following we shall often want to use the fact that a certain special graph has a unique switch. For instance, the graph $S_{1,2,0}$ is of type (1s), but also (4s), (6s) and (7s). These give rise to a number of "extra" selectors that map $S_{1,2,0}$ into an acyclic graph. In this case the extra selectors are $\{x_1, z\}$, $\{y_1, z\}$, and $\{y_1, v_1\}$ respectively.

We want to avoid situations such as these in our proofs and as it will turn out, it will not bother us. However, to be precise, we shall list the condition on each of the types, that guarantees that the acyclic switch is unique. There is one rather tricky thing that has to be taken into account: although $S_{1,0,2}$ is only of type (1s), switching at the centre gives $S_{1,2,0}$ of which we have seen that the nonconstant acyclic switch is not unique. This shows that the condition cannot simply be found by checking that a certain graph is of one single type only: this has to hold for all acyclic switches in the switching class. Note that this problem only occurs for the types (1s), (3s), (4s), (11s) and (12s), because they can switch to non-isomorphic acyclic graphs.

Lemma 3.2
A special graph $S_{k,m,\ell}$ has a unique nonconstant switch into an acyclic graph if $k \geq 3$, or $k = 2$ and $m + \ell \geq 2$, or $k \leq 1$ and $m, \ell \geq 3 - k$.
Proof:
The graphs $3 \cdot P_2$ and $2 \cdot P_2 \cup 2 \cdot K_1$ have no nonconstant switch to an acyclic graph. This follows from Theorem 3.1 and the fact that they are not special.

Let σ be a nonconstant selector containing z. Suppose σ is not the switch $\{z\}$. If $k \geq 2$, then we have $k = 2$ and $|I \cup M| \leq 1$ by the previous paragraph.

Let $k = 1$ and $m, \ell \geq 2$. Now, S has three components and hence the only possibility for overlap is with (3s). But $k = 1$ and $m \geq 2$ exclude the possibility that the nontrivial component of $S_{k,m,\ell}$ is a star.

For $k = 0$ and $m, \ell \geq 3$, S is exclusively of type (1s), because no other type of special graph has more than three components. □

The types (9s) and (10s) are obviously unique. The graph (11s) is also of type (8s), and hence has two nonconstant acyclic switches. The same holds for the graph (12s). For the other types (2s)-(8s) we now list the conditions.

Lemma 3.3
Under the following conditions do the special graphs $S(k, m)$, $k \geq m$ have a unique nonconstant switch to an acyclic graph.

- (2s) needs $k, m \geq 2$,
- (3s)-(5s), (7s), (8s) need $k \geq 2, m \geq 1$,
- (6s) needs $k, m \geq 3$.

Proof:
Let $S = S(k, m)$ be a special graph and let $\{z_1, z_2\} \subseteq \sigma$ with σ nonconstant. We prove that $\sigma = \{z_1, z_2\}$ if S^σ is to be an acyclic graph.

First of all, $K_{1,2} \cup K_{1,2}$ has one nonconstant acyclic switch (either select the leaves, or select the two inner vertices). For all types, except (6s), it now follows that $k, m \geq 2$ implies the existence of a unique nonconstant switch to an acyclic graph.

For (2s) this is all we can do, because $K_{1,2} + K_{1,1}$ has two switches: the choice of z_2 in $K_{1,1}$ is arbitrary. In the cases (3s)-(5s), (7s) and (8s) we do have a unique switch for $k \geq 2, m \geq 1$, because the vertex z_2 can only be chosen in one way: it is the vertex in $K_{1,1}$ that is not a leaf in S.

In the case of (6s) we get $k, m \geq 2$, because of overlap with (1s). Because (6s) for $k \geq m, m = 2$ overlaps with (7s) we arrive at the condition $k, m \geq 3$. □

Note, that there are cases that do overlap, but in which case the switches happen to be equivalent: (5s)($k = 0 = m$) and (2s)($k = 2, m = 0$) are the same graph, but the corresponding switches are complements.

4 Isolated vertices

In this section we give constraints for the isolated vertices in critically cyclic graphs. In particular, we prove our main tool for the final proof: if G is critically cyclic such that $G - x$ is acyclic for a vertex x, then $G - x$ has no isolated vertices.

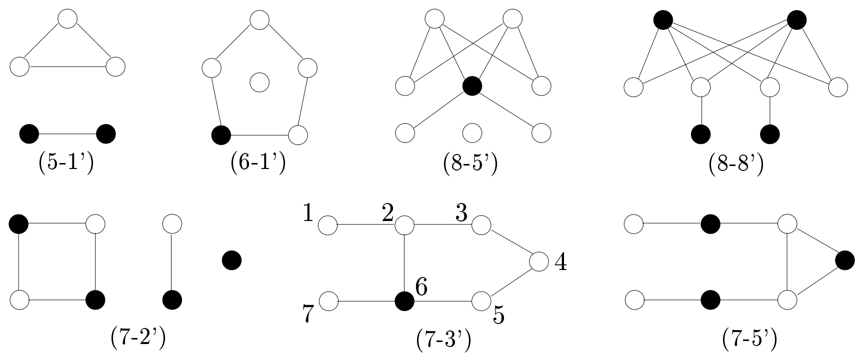

Figure 5: Switches of known critically cyclic graphs that are used in the proofs

Lemma 4.1
Let G be a critically cyclic graph. Then G has at most two isolated vertices or $G = K_{3,3} \cup 3 \cdot K_1$ ((9-2) in Figure 3).
Proof:
Let $I = \{x_1, x_2, \ldots, x_m\}$ be the set of isolated vertices of G, and assume that $m \geq 3$. Now $G - x_1$ is not acyclic, and it has an acyclic switch $(G - x_1)^\tau$. Hence τ is not constant on $G - I$, say $\tau(v_0) = 0$ and $\tau(v_1) = 1$ for some $v_0, v_1 \notin I$.

If two vertices of $I - \{x_1\}$ have the same value for τ, say $\tau(x_2) = i = \tau(x_3)$, then v_{1-i} is the unique vertex of $V - I$ with $\tau(v_{1-i}) = 1 - i$. Indeed, if it were $\tau(v) = 1 - i$ for another $v \in V - I$, then (x_2, v_{1-i}, x_3, v) would form a cycle in $(G - x_1)^\tau$. Moreover, in this case, there exists a vertex of I, say x_4, such that $\tau(x_4) = 1 - i$, for, otherwise, extending τ by setting $\tau(x_1) = i$ would result x_1 to be a leaf of G^τ contradicting the fact that all cycles of G^τ go through x_1. However, now (x_2, v_1, x_3, x_4) forms a cycle in $(G - x_1)^\tau$, which is a contradiction. In particular, $m \leq 3$ to avoid triangles with x_2 or x_3. The switching class of the discrete graph $\overline{K}_n$ consists of the complete bipartite graphs of order n, see [7], and therefore $m = 3$, and $\tau(x_2) \neq \tau(x_3)$. Since the graph $(G - x_1)^\tau$ is acyclic and $G^\tau(x_2 x_3) = 1$, it follows that $V - I$ is independent in $(G - x_1)^\tau$. Therefore $G = K_{r,s} \cup 3 \cdot K_1$ for some $r, s \geq 2$. Since $K_{3,3} \cup 3 \cdot K_1$ is a critically cyclic graph, and each $K_{2,s} \cup 3 \cdot K_1$, for $s \geq 4$, has an acyclic switch (by switching one of the vertices in the part of size 2 of $K_{2,s}$), the claim follows. □

Lemma 4.2
Let G be critically cyclic of order $n \geq 10$. Then no vertex $z \in V$ is adjacent to more than two leaves of G.
Proof:
If a set L of leaves of G is adjacent to a vertex z, then by switching at z, $\sigma = \{z\}$, the vertices of L become isolated in G^σ. □

Lemma 4.3
Let G be a critically cyclic graph of order $n \geq 10$. Then G has at most one isolated vertex.
Proof:
Suppose that G has exactly two isolated vertices, $I = \{x_1, x_2\}$. Let $(G - x_1)^\tau$ be acyclic, where we assume that $\tau(x_2) = 0$ without restriction. The set τ is independent in G and in $(G - x_1)^\tau$, for, otherwise, there would be a triangle (containing x_2) in $(G - x_1)^\tau$. In fact, τ contains at most one vertex from each connected component of $(G - I)^\tau$. Notice that these connected components are trees, because $(G - x_1)^\tau$ is acyclic.

Let $\tau = \{z_1, \ldots, z_r\}$, and set $\tau(x_1) = 0$. Then

$$G^\tau = (H + (T_1 \cup T_2 + \ldots \cup T_r)) \cup F,$$

where $H = K_{2,r}$ has the bipartition $(\{x_1, x_2\}, \{z_1, \ldots, z_r\})$, and the induced subgraphs T_i are disjoint trees with $H \cap T_i = \{z_i\}$; and F is an acyclic induced subgraph or it is empty. Since G^τ is not acyclic, we must have $r \geq 2$.

- By (7-1) and (7-2'), either F is discrete or it is a path P_2. In both cases, $|F| \leq 2$, by Lemma 4.1.
- By (8-6), there can be at most two nontrivial trees among $T_1, \dots, T_r$.
- Let T_i be nontrivial a tree. By (7-1) the depth of T_i from the root z_i is at most 3 and there are no vertices of degree more than 2 at depth higher than 1. The graph (7-2) excludes the possibility that a child of z_i has degree larger than two, and by (7-2') the tree cannot contain both an induced P_4 and an induced P_3. Hence each nontrivial tree T_i has the form
$$T_i = S_{k_i,s_i,0} \text{ or } P_4(s_i, 0),$$
where $S_{k_i,s_i,0}$ (for $k_i \geq 0$) is one of the special trees with z_i as its centre, and in $P_4(s_i, 0)$, z_i is the centre adjacent to the s_i leaves. By Lemma 4.2, $s_i \leq 2$.

We shall now consider the three cases for zero, one and two nontrivial T_i.

(0) If G^τ has no nontrivial components among $T_1, \dots, T_r$, then G^τ equals either $K_{2,r}$, $K_{2,r} \cup K_1$, $K_{2,r} \cup 2 \cdot K_1$ or $K_{2,r} \cup P_2$. All these have an acyclic switch; a contradiction.

(1) Suppose G^τ has exactly one nontrivial tree among $T_1, \dots, T_r$, say T_1.

Let $T_1 = P_4(s_1, 0)$.

- By (7-1), $r = 2$ (otherwise remove z_1).
- By (7-2'), $|F| = 0$ (otherwise remove the vertices of T_1 adjacent to z_1).

However, now $n \leq 9$ contradicts our assumption on n.

Let $T_1 = S_{k_1,s_1,0}$ with $k_1 > 0$, and let $r \geq 3$.

- By (7-2'), $|F| = 0$, $s_1 = 0$ and $k_1 = 1$ (otherwise remove z_1).

In this case T_1 is a path P_3, and G^τ has an acyclic switch for all $r \geq 3$ (switch all z_i's and the other end point of T_1); a contradiction.

Then the case for $r = 2$. In this case, by (7-2'), F cannot be a path P_2, and so it is discrete. Now G^τ has an acyclic switch (switch at z_1).

Finally, if $T_1 = S_{0,s_1,0}$, then $|T_1| \leq 3$ (by Lemma 4.2), and therefore $r \geq 4$, since $|F| \leq 2$.

- By (7-2'), F is discrete (otherwise remove z_1).
- By (8-5), $|F| \leq 1$, and by (8-5'), if $|F| = 1$, then $s_1 = 1$ (and in this case, T_1 is a path P_2).

The remaining cases, $s_1 = 1$ and $|F| = 1$, and $s_1 = 2$ and $F = \emptyset$, have acyclic switches for all r (switch with respect to x_1, x_2 and a leaf at z_1); a contradiction.

(2) Suppose that G^τ has exactly two nontrivial trees in $T_1, \dots, T_r$, say T_1 and T_2, and assume without loss of generality that $|T_1| \geq |T_2|$.

- By (8-8'), $r \leq 3$.
- By (8-4) and (8-7), $|F| \leq 1$.
- By (8-7), if $r = 3$, then $|F| = 0$.

Hence $r + |F| \leq 3$. Since $n \geq 10$, it follows that $|T_1| + |T_2| \geq 10 - r - |F| \geq 7$.

First we treat the trees of depth at most 1. In this case, $T_1 = S_{0,s_1,0}$. By Lemma 4.2 $s_1 \leq 2$, and hence $|T_1| \leq 3$. Therefore $|T_2| \geq 4$, which contradicts the assumption $|T_1| \geq |T_2|$.

Let t_1 be the depth of T_1 and suppose that $t_1 \geq 2$. Now by (8-1) and (8-2), $|T_2| = 2$, that is, T_2 is a path P_2, and consequently $|T_1| \geq 5$. If $T_1 = S_{k_1,s_1,0}$ ($k_1 > 0$), then $k_1 = 1$ otherwise we have (8-3) by removing a middle vertex from a P_3).

It follows that $|T_1| = t_1 + 1 + s_1 \geq 5$. Recall that $s_1 \leq 2$. However, the case $t_1 \geq 2$ and $s_1 = 2$ is excluded by (9-1), and the cases $t_1 = 3$ and $1 \leq s_1 \leq 2$ are excluded by (8-4) (remove the child of z_1 on the path of depth t_1). □

As in the above, we have

Lemma 4.4
Let G be a critically cyclic graph of order $n \geq 10$. Then no vertex $z \in V$ is adjacent to more than one leaf of G.

Lemma 4.5
Let G be a critically cyclic graph of order $n \geq 10$ and let $x \in G$.

i. $G - x$ can have at most two isolated vertices. Moreover, if $G - x$ has two isolated vertices, then x is adjacent to exactly one of these in G.

ii. If a vertex $z \neq x$ is adjacent to m leaves of $G - x$, then $m \leq 2$. Moreover, if $m = 2$, then x is adjacent to exactly one of these.

Proof:
For (i) we only need to observe that if $G - x$ has three isolated vertices, then in either G^x or G at least two of these are isolated and we can apply Lemma 4.3. The same holds if the number of isolated vertices is two, but x is not adjacent to exactly one of them in G.

For (ii), assume that there is a vertex $z \neq x$ adjacent to more than two leaves. The vertex x is adjacent to at most one of these in either G or G^x and the result then follows from Lemma 4.4. □

We say that a vertex $y \in V$ is *compatible* with x, if

- $G - x$ is acyclic,
- $G - y$ and $G^x - y$ are not acyclic.

Note that if y is compatible with x, then all cycles in G (and G^x) go through x, but not all of them go through y.

Lemma 4.6
Let G be critically cyclic graph such that $G - x$ is acyclic.

i. If y is compatible with x, then $G - \{x, y\}$ is a special graph.

ii. If G is of order $n \geq 8$, then there exists a vertex $y \in V$ that is compatible with x unless $G \in [C_n]$.

Proof:
Let $(G - y)^\tau$ be acyclic and set $S = G - \{x, y\}$. Because S and S^τ are both acyclic graphs it follows that either (a) S is special or (b) τ is constant on S.

In the case (b) all cycles go through x and y which contradicts the fact that $G - y$ is not acyclic. To see this, let there be a cycle that does not go through y. There are two selectors constant on S. The first of these is $\tau = S \cup \{x\}$. But then $(G - y)^\tau$ equals $G - y$ which is a contradiction, because the former is acyclic and the latter is not. If on the other hand $\tau = S$, then $(G - y)^\tau = (G^x - y)^{S \cup \{x\}} = G^x - y$ and again we have a contradiction.

For the second part, suppose $G \notin [C_n]$. Since G has no acyclic switches, there are cycles in G and G^x, and they all pass through x, because $G - x$ is acyclic. Moreover, since C_k is critically cyclic for $k \geq 7$, the induced cycles of G and G^x have length at most 6.

If G or G^x has an induced cycle C_5 or C_6, then let y be a vertex that is not on such a cycle. It is clear that $G - y$ and $G^x - y$ both contain cycles, and therefore each such y is compatible with x.

If G and G^x have both an induced cycle of length at most 4, then these two cycles have altogether at most 7 vertices (since they share the vertex x), and, by $n \geq 8$, there exists a vertex y that is not on these cycles. For each such vertex y, both $G - y$ and $G^x - y$ are not acyclic. This proves the claim. □

Lemma 4.7
Let G be critically cyclic of order $n \geq 10$ such that $G - x$ is acyclic. Then $G - x$ has no isolated vertices.
Proof:
Assume to the contrary of the claim that u is isolated in $G - x$. In this case u is either a leaf adjacent to x (or isolated) in G and isolated (or a leaf adjacent to x) in G^x. Hence $G - u$ and $G^x - u$ are not acyclic and by Lemma 4.6(i), $S = G - \{x, u\}$ is a special graph.

In this case, by Lemma 4.5(ii) and the fact that $n \geq 10$, S must be either of type (1s) or one of (5s), (7s), (8s) with $k = 2 = m$.

In the latter three cases S has a unique switch at the two centres $\tau = \{z_1, z_2\}$ by Lemma 3.3, and it is easy to see that $(G - u)^\tau$ is not acyclic, since x is adjacent to exactly one leaf adjacent to both centres in S and remains to be so in S^τ.

Consider then the case $S = S_{k,m,\ell}$. Without restriction we can assume that $\tau(z) = 1$. Extend τ to the whole domain by setting $\tau(u) = 0$.

We have $n = (2k+1) + m + \ell + 2 \geq 10$, and thus $k \geq \frac{1}{2}(7 - (m + \ell))$. By Lemma 4.5, $m \leq 2$ and $\ell \leq 1$. (Recall that u is an isolated vertex of $G - x$.) In particular, $k \geq 2$, and if $k = 2$, then $m = 2$, $\ell = 1$ and $n = 10$. In these cases, the special acyclic graph S has a unique switch S^ρ to another acyclic graph (by Lemma 3.2), where $\rho = \{z\}$. By the uniqueness of ρ, we have that $\rho(v) = \tau(v)$ for all $v \notin \{x, u\}$.

Now, the only vertices in G that can become adjacent to u in G^τ are x and z and because G^τ is not acyclic, these connections must exist: $G^\tau(ux) = 1 = G^\tau(uz)$ and they are the only edges of G^τ incident with u. Moreover, x is adjacent in G^τ to exactly one vertex $v \in H \cup I$, since G^τ contains a cycle but $G^\tau - u$ does not.

If $v = x_i$, say $v = x_1$. If $\ell \geq 1$, then $\{x, x_1, z, u, y_1, u_1, y_2\}$ induces an (7-4) in G^τ. Therefore $\ell = 0$. If $|M| \geq 1$, then $\{x, x_1, z, u, y_1, w, v_1\}$ induces an (7-4) in G^τ for $w = x_2$ or $w = y_2$ depending on the value $G^\tau(xv_1)$. Therefore also $m = 0$. Now $k \geq 4$, and G^τ contains an induced (7-4) obtained by removing x_2.

If $v = y_i$, say $v = y_1$, then $G^\tau|_{\{x, y_1, x_1, z, u\}}$ is an induced C_5, and hence G^τ has an induced (6-1) obtained by removing x_2.

If $v = u_i$, say $v = u_1$. To avoid (8-3) as being induced by $\{x, u_1, z, u, x_1, y_1, y_2, v_i\}$ (for any $v_i \in M$), we must have $G^\tau(xv_i) = 0$ (if $m > 0$). Now, however, $(G^\tau)^z$ is acyclic.

If $v = z$, then G^τ has an acyclic switch for $\{z\}$. This contradiction completes the proof of the lemma. □

5 The cases

In this section, let G be a critically cyclic graph of order $n = |V| \geq 10$, let $x \in V$ be a fixed vertex.

Since G is critically cyclic, there exists an acyclic switch $(G - x)^\sigma$ of the subgraph $G - x$. Since the switches of critically cyclic graphs are criticially cyclic, we can assume that σ is constant on V, and therefore that $G - x$ is acyclic already.

Assume that y is a vertex compatible with x, that is, $G - y$ and $G^x - y$ are both not acyclic. We know by Lemma 4.6(ii) that vertices such as x and y defined above exist if the switching class does not contain C_n. In the following we shall consider every type of special graph in turn and show that each case leads to a contradiction, thereby proving our main theorem that besides graphs in $[C_n]$ there no critically cyclic graphs of order $n \geq 10$.

By Lemma 4.6(i), $S = G - \{x, y\}$ is a special acyclic graph, and $(G - y)^\sigma$ is acyclic for a nonconstant selector σ. The special graph S cannot be of type (9s), (10s), (11s) or (12s), because the order of S should be at least 8 to ensure that $n \geq 10$.

Without restriction we can assume that $\sigma(x) = 0$. This follows from the symmetry in the definition of compatibility, i.e. the fact that both $G - y$ and $G^x - y$ are not acyclic. We extend σ to the whole domain by setting $\sigma(y) = 0$. Note that $(G - y)^\sigma = G^\sigma - y$.

In the following proofs a number of simple properties are often used, and we note them here: first of all, the vertex y is adjacent to at most one vertex of each component of S. If not, $G - x$ would not be acyclic. Also, there must be a cycle in G that does not contain y, because $G - y$ is not acyclic. This also holds for $G^x - y$.

We shall now formulate a few conditions that hold for the, still remaining, special graphs (1s)-(8s). Let $L_H(z)$ be the set of leaves adjacent to z in H, and let I_H denote the set of isolated vertices in H.

Lemma 5.1
Given the definitions above, we have that

i. $I_S \subseteq N_G(y)$.

ii. for all $z \in S$, $|L_S(z)| \leq 3$. Moreover, $|L_S(z)| = 3$ implies $|N_G(x) \cap L_S(z)| \geq 1$ and $|N_G(y) \cap L_S(z)| = 1$.

Proof:
Claim (i) follows from Lemma 4.7.

We have $|N_G(y) \cap L_S(z)| \leq 1$, since $G - x$ is acyclic. If $|L_S(z)| \geq 3$, then, by Lemma 4.5(ii), $|L_S(z) - N_G(y)| \leq 2$, and x is adjacent to at most one vertex of $L_S(z) - N_G(y)$. Hence, in this case, we must have $|L_S(z)| = 3$ and in that case x and y are each adjacent to at least one vertex. In the case of y it is exactly one vertex. □

Note how the previous Lemma restricts the values of k and m for the types (2s)-(8s) and m for (1s). On the other hand $n \geq 10$ gives a lower bound on these values for most types.

5.1 The case (1s)

We shall now consider first the most difficult case, $S_{k,m,\ell}$. Suppose that $S = S_{k,m,\ell}$, and adopt the notations of (S1)-(S4) for it. Without restriction we may assume that $\sigma(z) = 1$.

Lemma 5.2 We have

i. $k = 2$,

ii. $1 \leq \ell, m \leq 2$ and $m + \ell \geq 3$,

iii. $M \subseteq N_G(x)$,

iv. if $\ell = 2$, then $|N_G(x) \cap I| = 1$,

v. if $m = 2$, then $|N_G(y) \cap M| = 1$.

vi. $|N_G(x) \cap (H \cup I) - \{z\}| \leq 1$,

Proof:
By Lemma 4.3, $|N_G(x) \cap I| \leq 1$ for, otherwise, switch with $\{x, y\}$ to obtain two isolated vertices. By Lemma 4.5(ii) we have both $\ell \leq 2$ and Claim (iv).

If $k = 0$, then $m + \ell \geq 7$ contradicting the bounds $m \leq 3$ from Lemma 5.1(ii) and $\ell \leq 2$.

If $k = 1$, then $m + \ell \geq 5$, since $n \geq 10$. In this case, $\ell = 2$ and $m = 3$. If $k = 2$, then $m + \ell \geq 3$. Therefore by Lemma 3.2, in all cases $k \geq 1$, S^z is the unique acyclic switch of S. It follows that $\sigma|_S = \{z\}$, and therefore $M \subseteq N_G(x)$, for, otherwise the acyclic graph $G^\sigma - y$ would have an isolated vertex contradicting Lemma 4.7 (remember that we have $\sigma(x) = 0 = \sigma(y)$). Lemma 4.5(ii) then implies $m \leq 2$, and as a consequence $k \geq 2$, because as was shown above, if $k = 1$, then we must have $m = 3$. The Claim (v) follows from Lemma 4.5(ii).

Claim (vi) follows from the fact that $G^\sigma|_{H \cup I}$ is connected and $G^\sigma - y$ is acyclic.

Suppose then that $k \geq 3$. By Claim (vi) it follows that there are at least two pairs $x_i y_i$ such that $G(xx_i) = 0 = G(xy_i)$, say for $i = 1, 2$. Let the selectors τ_i be such that $(G - x_i)^{\tau_i}$ are acyclic, where we may choose $\tau_i(z) = 1$. The special graph $S - x_i$, which is $S_{k-1,m+1,\ell}$, has a unique acyclic switch $(S - x_i)^z$, since $n \geq 10$ and $k \geq 3$ by Lemma 3.2 (note that $\ell = 0$ implies $m \geq 4$, because $n \geq 10$).

It is then clear that $\tau_i = \sigma$ when we set $\tau_i(x_i) = 0$. By Lemma 4.7, the vertex y_i is not isolated in $G^{\tau_i} - x_i$, and therefore $G^{\tau_i}(yy_i) = G(yy_i) = 1$ for i=1,2 (since $G(xy_i) = 0 = G^{\tau_i}(xy_i)$) and we have a cycle in $G - x$. This contradiction proves case (i). □

Notice that, in this case, the above Lemma implies that $n \leq 11$.

We finish the case $S = S_{k,m,\ell}$.

Assume $G(xu_1) = 1$. Then $G(xz) = 1$, since otherwise (x, u_1, z) would be a triangle in $G^\sigma - y$. Also, $G(xx_i) = 0 = G(xy_i)$ for $i = 1, 2$, because $G^\sigma - y$ is acyclic.

We have $G(xy) = 0$, for, otherwise (x, y, u_1) is a triangle in G, and to avoid (5-1) with the edges $G(x_i y_i) = 1$, we would have to have that y is adjacent to two vertices in $H - \{z\}$ giving a cycle to $G - x$. Note that now all edges involving x are known.

Now (x, z, v_1) is a triangle in G, and to avoid (7-5'), necessarily (1) $G(yz) = 1$ or (2) $G(yv_1) = 1$, and y is adjacent to no other vertices of $H \cup M$.

(1) If $G(yz) = 1$, then $|M| = 1$, because otherwise y must be adjacent to either one of the v_i (Lemma 5.2(v)), but then (y, z, v_i) is a cycle of $G - x$. Lemma 5.2(ii) implies $|I| = 2$ and $\{x, u_1, y, z, y_1, x_2, u_2\}$ induces a (7-4). We have $G(xu_2) = 0$ by Lemma 5.2(iv).

(2) If $G(yv_1) = 1$, then $\{u_1, y, x, v_1, z, y_1, x_2\}$ induces a (7-3).

Therefore $G(xu_1) = 0$, and consequently $I = \{u_1\}$ by Lemma 5.2(iv).

By Lemma 5.2, $m = 2$, and we have $G(xv_1) = 1 = G(xv_2)$, $G(yv_1) = 1$, $G(yv_2) = 0$, and $G(yu_1) = 1$, $G(xu_1) = 0$.

In this case $G(yw) = 0$ for all $w \in S - \{u_1, v_1\}$, since $G - x$ is acyclic.

To avoid a cycle in $G^\sigma - y$, $G(xx_i) = 0 = G(xy_i)$ for $i = 1$ or 2, say $i = 1$. There are two cases here.

(1) $G(xz) = 0$. Now $G(xy) = 1$, since otherwise $\{x, v_1, v_2, z, y, x_1, y_1, u_1\}$ induces an (8-9) in G.

(2) $G(xz) = 1$. To avoid $\{x, z, v_1, y_1, x_1, y, u_1\}$ inducing a (7-5') we must have $G(xy) = 1$.

In both cases, $G(xy) = 1$. But $\{x, y, v_1, x_1, y_1\}$ induces a (5-1'). This contradiction proves the present case.

5.2 The other cases

Let $S = S(k, m)$ where we assume that $k \geq m$. Let z_1 and z_2 be the two centres of S, and $L = \{v_1, \ldots, v_k\}$ and $M = \{u_1, \ldots, u_m\}$ be the sets of leaves of S adjacent to z_1 and z_2, respectively.

Lemma 5.3

i. If S is of type (3s)-(8s), then $|L_S(z_i)| \leq 2$ for $i = 1$ or 2.

ii. If S^σ is the unique acyclic switch of S and $z \in S$, such that $S \neq S^\sigma$ and $|L_S(z)| = 3$, then x and y are each adjacent to exactly one, but different leaf at z.

Proof:
For Claim (i), assume both z_1 and z_2 have three leaves adjacent to them in S. By Lemma 5.1(ii), y is adjacent to one leaf at z_1 and one at z_2 giving a cycle in $G - x$ for the types (4s)-(8s). For (3s) we can apply the same reasoning, but taking y instead of x: $G^\sigma - y$ has a cycle. Note that we need that σ is the unique nonconstant selector mapping S into an acyclic graph. However, we have $k = 3 = m$ and by Lemma 3.3 the result follows.

To avoid a cycle in $G^\sigma - y$, x is adjacent to at most one of the leaves. Now, case (ii) follows from Lemma 4.5(ii) and Lemma 5.1(ii). □

Note that by Lemma 5.1(ii), Lemma 5.2(i) and (ii), Lemma 5.3(i) it already follows that there are no critically cyclic graphs of order at least 12 unless they are in $[C_n]$ for $n \geq 12$.

5.3 The cases (2s)-(4s)

By the fact that $n \geq 10$ and Lemma 5.1(ii), we have $k = 3$ and $2 \leq m \leq 3$. In all these cases the unique nonconstant switch mapping S into an acyclic graph is $\sigma = \{z_1, z_2\}$ by Lemma 3.3. Recall that we still have $\sigma(x) = 0 = \sigma(y)$.

By Lemma 5.3(ii), x is adjacent to one of the v_i, say v_1, and y is adjacent to an other v_i, say v_3. To avoid a cycle in $G^\sigma - y$, x must be adjacent to z_2, and y is not adjacent to any of the other v_i or z_1.

We now go over the cases one by one.

(2s) $S = S(k,m) = K_{1,k} \cup K_{1,m}$. Because $n \geq 10$ and the bounds on k and m, we know that $k = 3 = m$. By Lemma 5.3, x is adjacent to a leaf u_i, say u_1 and y to a leaf u_i different from u_1, say u_3. Because of the unicity of σ, x must be adjacent to z_1 to ensure that $G^\sigma - y$ is acyclic.

The only remaining unknown is $G(xy)$. If $G(xy) = 0$, then we have (5-1) $\{x, v_1, z, u_3, y\}$, and if $G(xy) = 1$, then we have (7-4) $\{u_1, x, y, v_3, z_1, v_2, u_2\}$.

(3s) $S = S(k,m) = K_{1,k} \cup K_{1,m} \cup K_1$. Because of the uniqueness of σ, S is mapped into a tree of type (4s). To avoid cycles in G^σ, necessarily $G(xz_1) = 1$, $G(xw) = 0$ (for the isolated vertex w of G) and $G(xu_i) = 0$ for all $u_i \in M$. By the above, $G(xz_2) = 1$ and $G(xv_2) = 0 = G(xv_3)$.

By Lemma 4.5(ii), $m = 2$ and y is adjacent to one of the u_i, say u_2.

The only unknown is the edge xy. If $G(xy) = 0$, then we have (7-5') $\{v_1, x, z_2, u_1, y, v_3, z_1\}$, and if $G(xy) = 1$, then we have (7-4) $\{v_1, x, y, u_2, z_2, v_2, u_1\}$.

(4s) Now $S = S(k,m) = P_3(k,m)$. Because S is connected, y is not adjacent to any other vertex of S (except v_3). Hence, $m = 2$ and x is adjacent to one of the u_i, say u_1 (Lemma 4.5(ii)). To prevent cycles in $G^\sigma - y$, x must be adjacent to z_1. If $G(xy) = 0$, then we have (5-1) $\{x, u_1, z_2, v_3, y\}$ and if $G(xy) = 1$, then we have (7-4) in G^x $\{v_1, z_1, v_2, v_3, x, u_1, y\}$.

5.4 The cases (5s)-(8s)

(6s) $S(k,m) = P_2(k,m)$. In this case, $n \geq 10$ implies $k = 3 = m$, but then $G - x$ is not acyclic.

In the remaining cases (5s), (7s) and (8s), let w_1 be the neighbour of z_1 of degree 2 and let w_2 be the single unnamed vertex ($d_S(w_2)$ equals 0, 1 or 2 depending on the case), see Figure 4(5s), (7s) and (8s).

By Lemma 5.1(ii) and $n \geq 10$, $2 \leq k \leq 3$, $m \geq 1$, and $k + m \geq 4$. In all these cases the unique nonconstant switch mapping S into an acyclic graph is $\sigma = \{z_1, z_2\}$ by Lemma 3.3.

We can assume that x is adjacent to a vertex in L, say $G(xv_1) = 1$. This follows from Lemma 5.3(ii) if $k = 3$. On the other hand, if $k = 2$, then necessarily $m = 2$, since $n \geq 10$, and in this case $N_G(y) \cap L = \emptyset$ or $N_G(y) \cap M = \emptyset$ in order to avoid a cycle in $G - x$. By Lemma 4.5(ii), $N_G(x) \cap M \neq \emptyset$ or $N_G(x) \cap L \neq \emptyset$, respectively. Since now $k = m$ $(= 2)$, the assumption is validated.

Claim 1: $G(xz_1) = 1 = G(xz_2)$, and $G(xu) = 0$ for all $u \notin \{v_1, z_1, z_2, w_2, y\}$. Moreover, $G(xw_2) = 0$ if $d_S(w_2) \neq 0$ (that is, excepting the case (5s)).

Proof:

Recall that $\sigma(x) = 0$, and, indeed, $\sigma = \{z_1, z_2\}$. The claim follows, since $G^\sigma - y$ is acyclic.

Claim 2: $G(yv) = 1$ holds for exactly one vertex $v \in S - \{w_2\}$, and either (i) $v \in L$, say $G(yv_3) = 1$, in which case $k = 3$ and $m = 1$, (ii) $v \in M$, say $G(yu_2) = 1$, in which case $k = 2$, $m = 2$. Moreover, $G(yw_2) = 1$ holds only in the case (5s).

Proof:

The first statement follows from the fact that $G - x$ is acyclic. Now if y is not adjacent to a vertex of M, then $|M| = 1$ by Lemma 4.5(ii) and the fact that $G(xu) = 0$ for all $u \in M$. It follows that $k = 3$, and, consequently, y is adjacent to a vertex of L. On the other hand, if $G(yu) = 1$ for a $u \in M$, then $G(yv) = 0$ for all $v \in L$ to avoid a cycle in $G - x$, and in this case, $k = 2$ by Lemma 4.5. That $G(yw_2) = 1$ in the case (5s) follows from Lemma 5.1(i). In the other two cases, $G(yw_2) = 1$ would result in a cycle in $G - x$.

These two claims together determine G with the exception of the value for $G(xy)$.

The cases are all excluded:

(5s) x is not adjacent to w_1 and neither is y. Hence in $G^\sigma - y$ the vertex w_1 is isolated contradicting Lemma 4.7.

(7s) In both cases, $G(xy) = 1$ to avoid (7-4) as being the subgraph induced by the vertices $\{x, z_1, w_1, z_2, v_2, w_2, y\}$. Now G contains a switch of (7-4) if $k = 3$ and $m = 1$ (this is $G^{z_1} - \{v_1, v_3, u_2\}$), and G contains (7-5') if $k = 2 = m$ (this is $G - \{u_1, v_2, z_2\}$).

(8s) In both cases, $G(xy) = 1$ to avoid (6-1) as being the subgraph induced by the vertices $\{x, z_1, w_1, w_2, z_2, y\}$. Now $\{x, z_1, w_1, w_2, z_2, y, u_1\}$ induces (7-3').

This proves Theorem 2.4.

6 Concluding remarks

Finding the critically cyclic graphs was done as follows: a program was written in `C` that listed for a number n of vertices a representative of each switching class that did not contain any acyclic switches. In a later phase, when we were looking for critically cyclic graphs on n vertices, we only had to make sure that all critically cyclic graphs of lower order could not anymore occur in these graphs. The program was run in this way for up to 12 vertices. We used here the files from [9] which list generators for the switching classes up to isomorphism and up to complementation for up to 10 vertices.

A computer program in the functional language Scheme verified that the critically cyclic graphs found were in fact critically cyclic. Also, the authors verified this by hand.

In our proofs, not all of the critically cyclic graphs were used. The graphs that were not used are (8-10)-(8-15) and (9-3)-(9-5). Lemma 4.6 excludes the cycles C_8 and C_9. For the other graphs, except (8-12), the reason is that if they are induced subgraphs of any graph of order at least 10, then this graph also contains one of the cyclic graphs from Figure 1, 2 and 3 or it contains (8-12). The graph (8-12) does not occur in our proofs, because it is overruled by Lemmas 4.6 and 4.7, that is, if G is a forbidden graph of order 10 that does not have 2 isolated vertices and such that $G - x$ is acyclic and $G - \{x, y\}$ is special, then G contains an induced critically acyclic graph that was used in the proofs.

As an aside we note that our program found that the graphs (8-9) and (8-12) have a similar property: adding two vertices to either of these graphs in any way, always results in a graph that contains a switch of one of the other critically cyclic graphs.

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