Structural Congruence in the pi-Calculus with Potential Replication

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Abstract

In the π-calculus with replication, potential structural congruence is introduced: a decidable notion of structural equivalence of processes close to the original congruence. It is sound and complete with respect to a semantic relation that maps processes to a submodel of the Multiset π-Calculus Mπ.

Introduction

Since its introduction in [8], the π-calculus has appeared in many forms — e.g., polyadic [6], asynchronous [5] — and has been subjected to some changes. For its elegance, the most striking one is the separation of behaviour and structure in the 'small π-calculus' of [7], a variant that excludes the operators of choice, matching, and recursion, but includes the theoretically appealing operation of replication. Here, inspired by the Chemical Abstract Machine of [1], rules of structural equivalence of processes are distilled from the transition system of [8], yielding both the essentials of process interaction in its new and simpler transition system, as well as a mathematically compact formulation of the statical bonds in an environment of agents. For many variants of the π-calculus, progress has been made in the understanding of behavioural equivalence of processes (see, e.g., [9, 10, 11], among numerous others), but much less is known about their anatomy (see [3, 4] though).

There are several reasons for the need of understanding the structure of processes. First of all, the stipulation that structurally equivalent processes should behave in the same manner (this is the STRUCT rule in the transition system of [7]) entails that to understand their behaviour fully, one must first know what processes look like. Secondly, since the choice of the laws that constitute structural congruence seems to be made slightly in favour of an elegant transition system, it is questionable whether a law that is initially put forward as a description of structure is not, in its essence, part of the dynamics. The law $!P \equiv !P \mid P$ for replicating a process contributes a typical example. One way to view replication is as a countably infinite parallel composition of copies of one process; this is established by 'recursively unfolding' this law. However, it was shown in [3] that this static view can be justified only after the addition of several other laws — e.g., closure properties of replication and distribution of replication over parallel composition. But this law also provides a different perspective on replication. In the style of the heating and cooling rules of the Chemical Abstract Machine of [1], its essence can be seen in the act of 'spinning off a process P from a deposit $!P$' (or absorbing P, when read from right to left). This second view is clearly a more dynamical one: $!P$ consists only *potentially* of an infinite number of copies of P. Hence, not until a proper model is provided for the structure of a process, one can be sure that the distillate is pure.

The Multiset π-Calculus Mπ of [2] was introduced originally as a concurrent model for π-calculus processes. In order to describe their 'true concurrent' behaviour, process agents are mapped to multisets of *molecules*. The general idea behind this semantics is that molecules can float freely in the 'multiset soup' — disregarding any spatial ordering between them — and that their interaction is local, i.e., independent of their surroundings. A natural question (answered positively by the *extended structural congruence* of [3]) that emerged from this approach is whether there could exist a set of laws that lay down the anatomy of processes, which is sound and complete with respect to the semantic mapping of [2]. In this paper we give a semantic mapping in such a way that the *original* list of structural laws of [7] is sound and complete with respect to it, that is, *almost* the original list. We compromise a bit in the treatment of replication. Although we do not support the view from the extended structural congruence of replication as an infinite parallel composition of a process anymore — we rather view replication as a potentially infinite supply of a process from which a finite but unbound number of copies can be drawn — we still want a replication to have the same capabilities of 'spinning off' a process as in the extended structural congruence. For instance, in the extended congruence, the replication of a parallel composition is identical to the replication of its components, as the law $!(P \mid Q) \equiv !P \mid !Q$ suggests. So, when we apply this law from left to right, the release of a copy of the component process P is prepared and it can be actuated by an application of the original law $!P \equiv !P \mid P$ for replication. Thus in the extended congruence, we can derive $!(P \mid Q) \equiv !(P \mid Q) \mid P$, or, to put it into words: a component process P can be 'spun off' from the compound replication $!(P \mid Q)$. This capability we also want in our new model and it is exactly the compromise mentioned before: we will exchange the (less strict) law $!(P \mid Q) \equiv !(P \mid Q) \mid P$ for the original one $!P \equiv !P \mid P$. The relation induced by the set of laws of [7] after this substitution we call *potential structural congruence*; it is the subject of this paper.

Let us illustrate the difference between the two laws by an example. Consider the following silly game, played by Bob and Gary. Both possess a bag of marbles: Bob's bag is filled with blue and red marbles only; the colours of the marbles Gary owns are green and red. We assume that the game does not come to an end just because one of the two players is out of marbles: each bag contains a sufficient number of both its colours. In between Bob and Gary there is a plate filled with a finite number of blue, green, and red marbles. The goal of the game is simple: in their urge for collecting still more marbles than they already possess, both Bob and Gary take from the plate as much marbles of their own colour as they can get hold of, one by one. There is one restriction, though: only if Bob holds a blue marble in his left hand *and* a red marble in his right hand, he may put them in his bag (if he wants to place marbles back onto the plate, it must be done in this pairwise fashion as well). A similar restriction applies to Gary. Now suppose that in the course of the game — in fact, very near the end — the plate contains one marble of each colour. Of course, if Bob is the quickest of the two he grabs his blue and red, and the game ends with one green marble remaining (which Gary cannot take). Frustrated by the unsatisfactory situation of one remaining marble that neither can take, Bob may altruistically decide to give Gary a second chance and place a pair of blue and red marbles back onto the plate. This act however will not temper Bob's (nor Gary's) annoyance, since evidently the game always ends either with a green, or with a blue marble remaining. Evidently, it is the pairing restriction that is the cause of this residue.

The above game is an informal specification of the structural equivalence class of the process term $R = \,!(b \mid r) \mid !(g \mid r) \mid b \mid g \mid r$, where the b, g, and r represent a blue, a green, and a red marble, respectively (thus, it does not specify the behaviour of R). By an application of the original law, R is structurally equivalent to both $R_1 = \,!(b \mid r) \mid !(g \mid r) \mid g$ (indicating that Bob has put a blue and a red marble into his bag) and $R_2 = \,!(b|r) \mid !(g|r) \mid b$ (where Gary has put a green and a red marble into his). Observe that the simpler $R_3 = \,!(b|r) \mid !(g|r)$ (where the plate does not contain any marbles) has a different structure, in the sense that it cannot be derived from R by a similar application. Only if we drop the rather artificial pairing restriction and allow Bob and Gary to take any marble of their colour that is left on the plate, R_3 can be derived by the new law for replication from both R_1 and R_2, as is expressed in the application of the rules $!(g \mid r) \mid g \equiv \,!(g \mid r)$ and $!(b \mid r) \mid b \equiv \,!(b \mid r)$, respectively. So the replication of the atomic processes b, g, and r, instead of the compound processes $(b \mid r)$ and $(g \mid r)$, allows Bob and Gary to finish the game with an empty plate.

As we mentioned earlier, one of our goals is to give a multiset semantics in the style of [2, 3] such that the laws of [7] together with the new one are sound and complete. As they should both describe the behaviour of a process, the *solutions* (i.e., the multiset soups of molecules) yielded by the semantic mapping of [2] and the one we propose here should be closely related. Thus, our new view of replication as a solution is much the same as the one of [2, 3], viz., it is a countably infinite union of clusters of molecules (these clusters are the *connected components* of [3] that represent independent processes) but with an additional entity called a *nucleus* to which every other connected component in the solution is drawn. This 'nucleus bond' is a weak one (as compared to the 'strong force' responsible for the formation of connected components) in the sense that if a solution is 'poured out over two vessels', only finitely many connected components can be separated from their nucleus. For instance, by applying the new law for replication to the process term $R_4 = !(b \mid r)$, we achieve that the solution representing R_4 is 'separated' into two solutions, viz., one that represents $!(b|r)$ (which need not be identical to the original solution), and one that represents b. Thus, although a solution representing a replication can *potentially* produce an infinite number of connected components (as, for instance, the law $!P \equiv !P \mid !P$ of the extended congruence suggests), the nucleus bond forbids just that.

Clearly, if we had two maps of cities, but we could not decide whether or not they belonged to the same city, those maps would probably be of little assistence. Thus, we should also focus on the decidability of the potential structural congruence, as was done for extended structural congruence (the second main result of [3]). The latter result directs us to a natural method of proof: using a computable transformation of process terms to show that two processes are congruent in the potential fashion if and only if their images are congruent in the extended style. So we reduce the decidability of potential structural congruence to the decidability of extended structural congruence. A pleasant by-product of this reduction is that it does not change the behaviour of a process; only its structure. Thus, the semantic mapping we propose is just the functional composition of the above-mentioned transformation of process terms with the semantic mapping of [2, 3].

To obtain an intuitive vision of the organization of this paper, the reader should bear in mind that the results in every section (except Section 8) are meant to serve the result of Section 7, which is the gravitational center of the paper. Here we show the completeness and decidability of potential structural congruence. Its inductive proof consists of five lemmas (one for each π-calculus operator) which rely heavily on the technical concepts introduced in Sections 4 and 5, where the foundation is laid for the proper treatment of restriction and replication, respectively. Section 1 recalls the basic concepts of the Multiset π-Calculus. The aforementioned transformation of processes is introduced in Section 2 together with the semantic mapping it induces. For the latter we show the soundness of potential structural congruence in Section 3 and we show the behavioural invariance of the transformation in Section 8. The normal form of processes introduced in Section 6 shows that a replication, a restriction, and a guarded process term are akin: all three correspond to a *cell* — a solution with one distinct connected component, its nucleus — which is the main technical concept of Section 5. More generally, it is shown in Section 6 that every process term corresponds to a finite union of cells.

This paper presents insight into the anatomy of π-calculus processes. We believe it will lead to a better understanding of their behaviour.

1 Preliminaries

We recall a few notions from [2, 3, 4] that we will use extensively, and we add a few new ones. The material in this paper relies heavily on the concepts and the results of [2], and in particular of [3]; the reader who is unfamiliar with [2, 3] is advised to read Sections 1–6 of [2] first.

In this paper we let $\mathbb{N}_+ = \{1, 2, \ldots\}$ be the set of positive natural numbers, and $\mathbb{N} = \mathbb{N}_+ \cup \{0\}$ the set of natural numbers (cf. the notation used in [4]; in [2, 3], these sets were denoted by $\mathbf{N}$ and $\mathbf{N} \cup \{0\}$, respectively). The set of π-calculus names is denoted by $\mathbf{N}$ (instead of N in [2, 3]). Recall that $\#I$ is the cardinality of a set I; if I is countably infinite, then $\#I = \omega$ (where ω stands for $\aleph_0$). For a mapping f and a set A, $f \restriction A$ denotes the restriction of f to A.

The set of π-calculus process terms we use is produced by the following syntactical description:

$$P ::= \mathbf{0},\ P \mid P,\ g.P,\ (\nu x)P,\ \ !P$$

where $\mathbf{0}$ denotes the inactive process, $P \mid Q$ is the parallel composition of the processes P and Q, $(\nu x)P$ restricts the use of $x \in \mathbf{N}$ to the scope P, and $!P$ is the replication of P. The remaining $g.P$ denotes a guarded process, where g can appear in two forms: as an input guard $x(y)$, and as an output guard $\overline{x}z$, with $x, y, z \in \mathbf{N}$. There are two ways to bind a name y in a process term: by input guarding $x(y)$ and by restriction (νy). The set $\mathrm{fn}(P)$ of *free names* of a process term P consists of those names that occur unbound in P. For names $y, z \in \mathbf{N}$, we denote by $P[z/y]$ the process term obtained from P by replacing every free occurrence of y in P by z, possibly renaming bound occurrences of z in P to avoid name collisions. We refer to [7] for the operational semantics of the π-calculus.

Recall from [2] that a *multiset* S is a countable set D_S together with a mapping $\phi_S : D_S \to \mathbb{N}_+ \cup \{\omega\}$ that defines the multiplicity of the elements of D_S in S. The *multiset union* is defined in the obvious way, adding the multiplicity of each element: for a countable set I, we let $S = \bigcup_{i \in I} S_i$ be the multiset defined by $D_S = \bigcup_{i \in I} D_{S_i}$ and $\phi_S(d) = \sum_{i \in I} \phi_{S_i}(d)$, where summation is extended to ω, as usual. Note that we consider only countable unions. For a set D, S is a *multiset over* D if $D_S \subseteq D$. If S is a multiset over D and $f : D \to E$ is a mapping, then the *multiset image* $f(S)$ of S under f is defined by $D_{f(S)} = f(D_S)$ and $\phi_{f(S)}(e) = \sum_{f(d)=e} \phi_S(d)$. Note that $f(\bigcup_{i \in I} S_i) = \bigcup_{i \in I} f(S_i)$ and that for $g : E \to F$, $(g \circ f)(S) = g(f(S))$. We refer to Section 3 of [3] for more basic properties of multisets. From [4] we recall that a multiset S is *contained* in a multiset T, denoted $S \subseteq T$, if there exists a multiset U such that $S \cup U = T$.

We recall from [2] that strings of the form $x(-)$ or $\overline{x}z$, where $x, z \in \mathbf{N} \cup \mathbb{N}_+ \cup \mathrm{New}$, are called *schematic guards* (where New is an uncountably infinite set of *new names*, disjoint with $\mathbf{N}$ and $\mathbb{N}$); a *guard* is a string of the form $x(y)$ or $\overline{x}z$, where $x, z \in \mathbf{N} \cup \mathrm{New}$ and $y \in \mathbf{N}$. Observe that only *guards over* $\mathbf{N}$, i.e., guards with names from $\mathbf{N}$, occur in a π-calculus process term. A *solution* is a multiset over the set of molecules Mol, where a *molecule* is a pair $g.S$ with S a solution, and g a schematic guard; we refer to Section 4 of [2] for a formal definition of Mol and the set of solutions Sol. For a guard $x(y)$, we denote by $x(y).S$ the molecule $x(-).\mathrm{inc}(S)[1/y]$, where $\mathrm{inc}(S)$ increases all natural numbers occurring in S, and $[1/y]$ substitutes the number 1 for all occurrences of y. For a solution S (or a molecule, or a guard), the set $\mathrm{fn}(S)$ contains all names from $\mathbf{N} \cup \mathrm{New}$ that occur in S; the set of new names in S is $\mathrm{new}(S) = \mathrm{fn}(S) \cap \mathrm{New}$. A solution S is a *copy* of a solution S' if there exists a bijection $f : \mathrm{new}(S) \to \mathrm{new}(S')$ such that $f(S) = S'$ (where $f(S)$ is the result of replacing every occurrence of a new name n with $f(n)$; see also Section 4 of [2] and Section 2 of [3]).

The semantic relation $\Rightarrow$ of [2] that maps each π-calculus process term to a solution (modulo copies) is defined as the smallest relation that satisfies the following requirements:

(S0) $\mathbf{0} \Rightarrow \varnothing$

(S1) if $P_1 \Rightarrow S_1$ and $P_2 \Rightarrow S_2$, then $P_1 \mid P_2 \Rightarrow S_1 \cup S_2$,
provided $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$

(S2) if $P \Rightarrow S$ and $x \in \mathbf{N}$, then $(\nu x)P \Rightarrow_{\mathbf{F}} S[n/x]$,
provided $n \in \mathrm{New} - \mathrm{new}(S)$

(S3) if $P \Rightarrow S$ and g is a guard over $\mathbf{N}$, then $g.P \Rightarrow \{g.S\}$

(S4) if $P \Rightarrow S_i$ for all $i \in \mathbb{N}$, then $!\,P \Rightarrow \bigcup_{i \in \mathbb{N}} S_i$,
provided $\mathrm{new}(S_i) \cap \mathrm{new}(S_j) = \varnothing$ for all $i \neq j$.

By Lemma 3 of [2], if $P \Rightarrow S$, then $\mathrm{fn}(P) = \mathrm{fn}(S) \cap \mathbf{N}$. Process terms are *multiset congruent*, denoted as $P \equiv_m Q$, if $\{S \mid P \Rightarrow S\} = \{S \mid Q \Rightarrow S\}$.

Recall from Section 4 of [3] that a solution V is *connected* if there do not exist nonempty solutions S and T with $\mathrm{new}(S) \cap \mathrm{new}(T) = \varnothing$ such that $V = S \cup T$. Connectedness of a solution is preserved under substitutions $[n/x]$, where $x \in \mathbf{N}$ and $n \in \mathrm{New}$: if V is connected, then $V[n/x]$ is connected (see the proof of Lemma 17 of [3]). By Lemma 7 of [3], every copy of a connected solution is connected. As described in Lemma 10 and Lemma 11 of [3], every solution S can be represented as a union $S = \bigcup_{i \in I} V_i$ of nonempty connected solutions V_i with mutually disjoint $\mathrm{new}(V_i)$ in essentially one way (i.e., modulo a renaming of the index set I); the V_i are called the *connected components* of S when we have this representation in mind. For a nonempty connected solution V, the *multiplicity of* V *in* S is $\mathrm{mult}(V, S) = \#\{i \in I \mid V_i \text{ is a copy of } V\}$; it yields the number (with its value in $\mathbb{N} \cup \{\omega\}$) of copies of a connected component in S. Since the 'copy-of' relation is an equivalence relation, it induces a partition of I into equivalence classes, where $i \in I$ and $i' \in I$ are equivalent if and only if V_i is a copy of $V_{i'}$, i.e., we view I as the union of mutually disjoint sets I_k, $k \in K$, where V_i is a copy of $V_{i'}$ if and only if i and i' are member of the same I_k. So $S = \bigcup_{k \in K} \bigcup_{i \in I_k} V_i$, where the I_k are essentially unique. For a nonempty connected solution V, we let $[V]$ be the equivalence class under the copy-of relation that contains V. Note that $[V]$ only consists of nonempty connected solutions (see Lemma 7 of [3]). Recall from [4] that S is *strongly contained* in a solution T, denoted $S \subseteq^{\mathrm{n}} T$, if $S \cup U = T$ for some solution U (i.e., S is contained in T) and $\mathrm{new}(S) \cap \mathrm{new}(U) = \varnothing$. Hence if $T = \bigcup_{j \in J} W_j$, where the W_j are the connected components of T, this is equivalent to the existence of a partition J_S, J_U of J such that $S = \bigcup_{j \in J_S} W_j$ and $U = \bigcup_{j \in J_U} W_j$ (by Lemma 9 of [3]; cf. Lemma 3.3 of [4]).

Below we collect three basic properties of 'copy-of' and 'mult'; they will form the building blocks of many of the proofs in subsequent sections. Note that the first property relates the two notions. Let S, S_i and T, T_i, $i \in I$, be solutions such that the $\mathrm{new}(S_i)$ are mutually disjoint and the $\mathrm{new}(T_i)$ are mutually disjoint.

(Pa) S is a copy of T if and only if for all nonempty connected V, $\mathrm{mult}(V, S) = \mathrm{mult}(V, T)$

(Pb) If S_i is a copy of T_i for every $i \in I$, then $\bigcup_{i \in I} S_i$ is a copy of $\bigcup_{i \in I} T_i$

(Pc) For every nonempty connected V, $\mathrm{mult}(V, \bigcup_{i \in I} S_i) = \sum_{i \in I} \mathrm{mult}(V, S_i)$

Property (Pa) is implicitly shown in the proof of Lemma 3.14 of [4]; properties (Pb) and (Pc) are Lemmas 3 and 12 of [3], respectively. Observe that (Pb) follows from (Pa) and (Pc).

2 Potential Structural Congruence

We introduce potential structural congruence, a relation that captures the structural equivalence of π-calculus process terms in a way that differs only slightly from the standard structural congruence relation of [7]. Unlike the latter, it does not only identify process terms $!P$ and $!P \mid P$, but more generally it identifies $!P$ and $!P \mid Q$, where $P = Q \mid Q'$. Thus, $!P$ can replicate the *components* Q *of* P without changing structure. Unlike the (extended) structural congruence of [2, 3], it does not identify $!P$ and $!Q \mid !Q'$. Thus, in potential structural congruence, replication is treated in a way that differs essentially from that in the extended version, where replication is modelled as a countably infinite parallel composition.

In Fig. 1, a number of possible laws of structural congruence are listed. The list is the same as the one in [3], except for the last law (3.6), which is added here. In [7], structural congruence $\equiv$ (denoted by $\equiv^{\mathbf{std}}$ in Fig. 1 and

$\equiv$

$\equiv^{\mathbf{ptl}}$

$\equiv^{\mathbf{std}}$

(α)	$P \equiv Q$ whenever P and Q are α-convertible,
(1.1)	$P \mid \mathbf{0} \equiv P$,
(1.2)	$P \mid Q \equiv Q \mid P$,
(1.3)	$P \mid (Q \mid R) \equiv (P \mid Q) \mid R$,
(2.1)	$(\nu x)(\nu y)P \equiv (\nu y)(\nu x)P$,
(2.2)	$(\nu x)P \equiv P$, provided $x \notin \mathrm{fn}(P)$,
(2.3)	$(\nu x)(P \mid Q) \equiv P \mid (\nu x)Q$, provided $x \notin \mathrm{fn}(P)$,
(2.4)	$(\nu x)g.P \equiv g.(\nu x)P$, provided x does not occur in g,
(3.1)	$!\,P \equiv\, !\,P \mid P$,
(3.2)	$!\,(P \mid Q) \equiv\, !\,P \mid\, !\,Q$,
(3.3)	$!\,!\,P \equiv\, !\,P$,
(3.4)	$!\,\mathbf{0} \equiv \mathbf{0}$,
(3.5)	$!\,P \mid\, !\,P \equiv\, !\,P$, and
(3.6)	$!\,(P \mid Q) \equiv\, !\,(P \mid Q) \mid P$.

Figure 1: The laws of structural congruence

by $\equiv_M$ in [3]) is defined as the smallest congruence satisfying the laws (α), (1.1), (1.2), (1.3), (2.1), (2.2), (2.3), and (3.1). In [2], *(extended) structural congruence* (inconveniently also denoted $\equiv$) is defined by adding the structural laws (3.2), (3.3), (3.4), and (2.4). We define a third relation, *potential structural congruence*, denoted $\equiv^{\mathbf{ptl}}$, as the smallest congruence satisfying the laws (α), (1.1), (1.2), (1.3), (2.1), (2.2), (2.3), and (3.6). In order to keep uniformity with extended structural congruence (which we denote by $\equiv$), the structural congruence of [7] is referred to as *standard structural congruence* and is denoted by $\equiv^{\mathbf{std}}$.

Note that, as shown in [7], law (2.2) is equivalent to the law $(\nu x)\mathbf{0} \equiv \mathbf{0}$ (using laws (1.1) and (2.3)). Law (3.5) is included for its usefulness in the version of extended structural congruence of [3]; its dependency on laws (3.1), (3.2), and (3.3) is shown there. Structural law (3.1) is a consequence of (3.6) (and structural law (1.1)): $!\,P \equiv\, !\,(P|\mathbf{0}) \equiv\, !\,(P|\mathbf{0})|P \equiv\, !\,P|P$, so (3.1) is more restrictive than (3.6). On the other hand, in the presence of the law (3.2) (and the laws (1.2) and (1.3)), the laws (3.1) and (3.6) are equivalent: if we assume (3.1), then $!(P \mid Q) \equiv\ !P \mid !Q \equiv\ !P \mid !Q \mid P \equiv\ !(P \mid Q) \mid P$. Note that potential structural congruence differs from standard structural congruence only in the exchange of structural law (3.6) for (3.1). Thus, in terms of inclusion, it is in between standard and extended structural congruence.

The main difference between standard structural congruence (or potential structural congruence, for that matter) and extended structural congruence is found in the treatment of replication. The combination of the laws (3.1)–(3.4) express the *infinite* nature of replication. As was observed in the Introduction of [3], the laws (3.2), (3.3), and (3.4) are basically cardinality laws that justify the perception of replication as a countably infinite parallel composition of copies of a process. Structural law (3.1) *by itself* does not justify this perception; it rather expresses that from a deposit $!P$ one can extract a *finite but unbounded* number of copies of P, but conceals the nature of the deposit itself. Similarly, structural law (3.6) expresses that one can extract a finite number of *components* of P from the deposit. Thus, for the laws (3.1) and (3.6), this number is only *potentially infinite*.

Another difference between extended and standard congruence that is closely related, albeit more technical, and in which standard and potential congruence coïncide, is the preservation of the *nesting depth* $\mathrm{ndr}(P)$ *of replications* in a process term P, where $\mathrm{ndr}(P)$ is defined as expected: $\mathrm{ndr}(\mathbf{0}) = 0$, $\mathrm{ndr}(P \mid Q) = \max(\mathrm{ndr}(P), \mathrm{ndr}(Q))$, $\mathrm{ndr}((\nu x)P) = \mathrm{ndr}(g.P) = \mathrm{ndr}(P)$, and $\mathrm{ndr}(!P) = \mathrm{ndr}(P) + 1$. Because of the structural laws (3.3) and (3.4), extended structural congruence does not preserve the nesting depth of replications in a process term. By a simple proof on the definition of $\equiv^{\mathbf{ptl}}$ it can be shown that potential (and hence standard) structural congruence does preserve it.

One peculiar difference between standard and potential structural congruence should be noted: in general, standard structural congruence forbids P as a spin-off of $!(P \mid Q)$. Yet in the vicinity of $!Q$ the following all of a sudden becomes possible: $!(P \mid Q) \mid !Q \equiv^{\mathbf{std}} !(P \mid Q) \mid (P \mid Q) \mid !Q \equiv^{\mathbf{std}} !(P \mid Q) \mid P \mid !Q$, by applying structural law (3.1) twice.

The main goal of this paper is twofold. First of all we want to understand the structure of processes as expressed in the potential structural congruence; we want to give a *model* that describes this structure, in the same way as was done in [2, 3] for $\equiv$. In particular, the new model should be suited to determine the behaviour of processes. Secondly, we want to show that potential structural congruence is decidable (as $\equiv$ was shown to be decidable in [3]).

The strategy we use is simple: we define a mapping $\mathbf{F}$ on process terms and show that $\mathbf{F}$ reduces $\equiv^{\mathbf{ptl}}$ to $\equiv$, i.e., we prove that for all process terms P and Q, $P \equiv^{\mathbf{ptl}} Q$ if and only if $\mathbf{F}(P) \equiv \mathbf{F}(Q)$. By the decidability of $\equiv$ (Theorem 34 of [3]), the decidability of $\equiv^{\mathbf{ptl}}$ follows, because the reduction $\mathbf{F}$ is effective. Since $\equiv$ is sound and complete with respect to $\Rightarrow$ (by Theorem 33 of [3]), we obtain the model that describes $\equiv^{\mathbf{ptl}}$ by proposing $\Rightarrow_{\mathbf{F}} = \Rightarrow \circ \mathbf{F}$ as the new semantic mapping.

Definition 2.1 The mapping $\mathbf{F}$ on process terms is defined inductively by

$\mathbf{F}(\mathbf{0}) = \mathbf{0}$,
$\mathbf{F}(P \mid Q) = \mathbf{F}(P) \mid \mathbf{F}(Q)$,
$\mathbf{F}((\nu x)P) = (\nu x)\mathbf{F}(P)$,
$\mathbf{F}(g.P) = g.(\mathbf{F}(P) \mid (\nu v)(\nu w)(\overline{v}v.\overline{w}w.\mathbf{F}(P) \mid \overline{v}v.\overline{w}w.\mathbf{0}))$,
provided $v \neq w$ and $v, w \notin \mathrm{fn}(P)$,
$\mathbf{F}(\,!\,P) = \,!\,\mathbf{F}(P) \mid (\nu w)\overline{w}w.\overline{w}w.\mathbf{F}(P)$,
provided $w \notin \mathrm{fn}(P)$.

Thus, $\mathbf{F}$ respects parallel composition, restriction, and the inactive process $\mathbf{0}$. Note that by the choice of v and w in the definition of $\mathbf{F}(g.P)$ and $\mathbf{F}(\,!\,P)$, $\mathbf{F}$ is technically a relation, such that for every $Q, Q' \in \mathbf{F}(P)$, $Q \equiv_\alpha Q'$, where $\equiv_\alpha$ denotes α-conversion of process terms; by Theorem B of [2], Q and Q' have the same semantics under $\Rightarrow$.

Intuitively, the reduction of $\equiv^{\mathbf{ptl}}$ to $\equiv$ can be split into two parts: the first concerns the objective of "invalidating" the structural laws (3.2), (3.3), and (3.4), which all involve replication; the second concerns the "invalidation" of structural law (2.4), which involves guarding and restriction. Now the additional $(\nu w)\overline{w}w.\overline{w}w.\mathbf{F}(P)$ in parallel composition with $!\,\mathbf{F}(P)$ in the definition of $\mathbf{F}(\,!\,P)$ intuitively stands for an inactive agent (or 'deposit') that records the replication of P, or rather, the replication of $\mathbf{F}(P)$. In this way, every single replication inside a process term is registered in a separate and distinctive agent, and thus, at least intuitively, this method "invalidates" structural laws (3.2), (3.3), and (3.4), simply because the agents record different replications on either side of these laws, as opposed to structural law (3.6), where identical replications are registered. The role of the guarded triplet $g.(\mathbf{F}(P) \mid (\nu v)(\nu w)(\overline{v}v.\overline{w}w.\mathbf{F}(P) \mid \overline{v}v.\overline{w}w.\mathbf{0}))$ in the definition of $\mathbf{F}(g.P)$ is the invalidation of structural law (2.4). Intuitively, this construction acts as a 'detection mechanism' for the location of a restriction (νx) that binds subprocesses of P. Assume, for instance, that P contains the free name x and g is a guard in which x does not occur. Then in $\mathbf{F}((\nu x)g.P) = (\nu x)g.(\mathbf{F}(P) \mid (\nu v)(\nu w)(\overline{v}v.\overline{w}w.\mathbf{F}(P) \mid \overline{v}v.\overline{w}w.\mathbf{0}))$, the free x occurring in both subterms $\mathbf{F}(P)$ becomes bound by the *same* restriction (νx) (where, for now, we assume that x occurs free in $\mathbf{F}(P)$ as well), whereas in $\mathbf{F}(g.(\nu x)P) = g.((\nu x)\mathbf{F}(P) \mid (\nu v)(\nu w)(\overline{v}v.\overline{w}w.(\nu x)\mathbf{F}(P) \mid \overline{v}v.\overline{w}w.\mathbf{0}))$, two *separate* restrictions are responsible for the binding of x. Thus, this method detects on which side of g the restriction (νx) occurs. The reason why $\mathbf{F}(g.P)$ has two new agents will be explained in Section 4. We need the requirement $v \neq w$ to separate the new agents in $\mathbf{F}(\,!\,P)$ from those in $\mathbf{F}(g.P)$, which guarantees that there is no 'interference' between the two methods.

We want to stress the fact that in the definition of $\mathbf{F}(\,!\,P)$ and $\mathbf{F}(g.P)$, the agents that are added cannot perform any communication action because of the restrictions (νw) and $(\nu v)(\nu w)$, respectively, and since their outer guards are only output guards. Thus, $\mathbf{F}$ does not change the parallel behaviour of a process. This will be partially formalized in Section 8.

Below we state some easy to prove properties of $\mathbf{F}$, which show that $\mathbf{F}$ behaves well on the free names of a process term; in the above paragraph we already used the first informally.

Lemma 2.2 *Let $y, z \in \mathbf{N}$. For every process term P,*

(1) $\mathrm{fn}(\mathbf{F}(P)) = \mathrm{fn}(P)$, *and*

(2) $\mathbf{F}(P[z/y]) = \mathbf{F}(P)[z/y]$.

Proof Straightforward by induction on the structure of P. $\square$

The semantic relation we propose is just the composition of $\mathbf{F}$ with the semantic relation $\Rightarrow$ of [2]. As explained earlier in this section, the goal of this paper is to show that the composed relation serves as a model for potential structural congruence.

Definition 2.3 For a process term P and a solution S, we define $P \Rightarrow_{\mathbf{F}} S$ if and only if $\mathbf{F}(P) \Rightarrow S$. Thus, by the definition of $\Rightarrow$, $\Rightarrow_{\mathbf{F}}$ is the smallest relation that satisfies the following requirements:

(SF0) $\mathbf{0} \Rightarrow_{\mathbf{F}} \varnothing$

(SF1) if $P_1 \Rightarrow_{\mathbf{F}} S_1$ and $P_2 \Rightarrow_{\mathbf{F}} S_2$, then $P_1 \mid P_2 \Rightarrow_{\mathbf{F}} S_1 \cup S_2$, provided $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$

(SF2) if $P \Rightarrow_{\mathbf{F}} S$ and $x \in \mathbf{N}$, then $(\nu x)P \Rightarrow_{\mathbf{F}} S[n/x]$, provided $n \in \mathrm{New} - \mathrm{new}(S)$

(SF3) if $P \Rightarrow_{\mathbf{F}} S_0, S_1$ and g is a guard over $\mathbf{N}$, then $g.P \Rightarrow_{\mathbf{F}} \{g.T\}$ where $T = S_0 \cup \{\overline{m}m.\{\overline{m'}m'.S_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$, provided $\mathrm{new}(S_0) \cap \mathrm{new}(S_1) = \varnothing$, and $m, m' \in \mathrm{New} - \mathrm{new}(S_0 \cup S_1)$ with $m \neq m'$

(SF4) if $P \Rightarrow_{\mathbf{F}} S, S_i$ for all $i \in \mathbb{N}$, then $!P \Rightarrow_{\mathbf{F}} (\bigcup_{i \in \mathbb{N}} S_i) \cup \{\overline{m}m.\{\overline{m}m.S\}\}$, provided $\mathrm{new}(S_i) \cap \mathrm{new}(S_j) = \varnothing$ for all $i, j \in \mathbb{N}$ with $i \neq j$, $\mathrm{new}(S) \cap \mathrm{new}(S_i) = \varnothing$ for all $i \in \mathbb{N}$, and $m \in \mathrm{New} - \mathrm{new}(S)$ with $m \notin \mathrm{new}(S_i)$ for all $i \in \mathbb{N}$.

Process terms P and Q are *multiset congruent by* $\mathbf{F}$, denoted $P \equiv_m^{\mathbf{F}} Q$, if $\{S \mid P \Rightarrow_{\mathbf{F}} S\} = \{S \mid Q \Rightarrow_{\mathbf{F}} S\}$.

Observe that if $P \Rightarrow_{\mathbf{F}} S$ then $\mathrm{fn}(P) = \mathrm{fn}(S) \cap \mathbf{N}$, by Lemma 3 of [2] and Lemma 2.2(1). Also note that $P \equiv_m^{\mathbf{F}} Q$ is equivalent with $\mathbf{F}(P) \equiv_m \mathbf{F}(Q)$, and hence with $\mathbf{F}(P) \equiv \mathbf{F}(Q)$, by Theorem 33 of [3]. Thus, the main objective of this paper is to show that $\equiv^{\mathbf{ptl}}$ and $\equiv_m^{\mathbf{F}}$ are the same relation, i.e., that potential structural congruence is sound and complete with respect to $\Rightarrow_{\mathbf{F}}$.

We need the next two lemmas in the following sections; the first is the analogon of Lemma 5 of [2] and states that the set $\{S \mid P \Rightarrow_{\mathbf{F}} S\}$ consists only of copies of one another, the second is the analogon of Lemma 6 of [2] and shows that the semantic relation $\Rightarrow_{\mathbf{F}}$ is compositional with respect to substitution, i.e., $\{S \mid P[z/y] \Rightarrow_{\mathbf{F}} S\} = \{S[z/y] \mid P \Rightarrow_{\mathbf{F}} S\}$.

Lemma 2.4 *If $P \Rightarrow_{\mathbf{F}} S$, then $P \Rightarrow_{\mathbf{F}} S'$ if and only if S' is a copy of S.*

Proof Immediate by Lemma 5 of [2] and by the definition of $\Rightarrow_{\mathbf{F}}$. $\square$

Lemma 2.5 *Let $y, z \in \mathbf{N}$.*

(1) If $P \Rightarrow_{\mathbf{F}} S$, then $P[z/y] \Rightarrow_{\mathbf{F}} S[z/y]$.

(2) If $P[z/y] \Rightarrow_{\mathbf{F}} S'$, then there exists S such that $P \Rightarrow_{\mathbf{F}} S$ and $S' = S[z/y]$.

Proof Immediate by Lemma 6 of [2] and Lemma 2.2(2). $\square$

3 The Soundness of Potential Structural Congruence

We show the first (and easy) half of our main objective in the following lemma and corollary; note that the implication $P \equiv^{\mathbf{ptl}} Q \Longrightarrow \mathbf{F}(P) \equiv \mathbf{F}(Q)$ suffices for our purposes. Nevertheless, we show the stronger statement below.

Lemma 3.1 *For all process terms P and Q, if $P \equiv^{\mathbf{ptl}} Q$ then $\mathbf{F}(P) \equiv^{\mathbf{ptl}} \mathbf{F}(Q)$.*

Proof Let $\mathcal{R}$ be the relation on process terms defined by $P\,\mathcal{R}\,Q$ if $\mathbf{F}(P) \equiv^{\mathbf{ptl}} \mathbf{F}(Q)$. We show that $\mathcal{R}$ is a congruence relation that satisfies the laws of $\equiv^{\mathbf{ptl}}$. Since $\equiv^{\mathbf{ptl}}$ is the smallest congruence satisfying these laws, clearly $P \equiv^{\mathbf{ptl}} Q$ implies $P\,\mathcal{R}\,Q$.

It is easily shown that $\mathcal{R}$ is a congruence; in fact, this follows immediately from $\equiv^{\mathbf{ptl}}$ being a congruence. For instance, to show transitivity of $\mathcal{R}$, we have that $\mathbf{F}(P) \equiv^{\mathbf{ptl}} \mathbf{F}(R) \equiv^{\mathbf{ptl}} \mathbf{F}(Q)$ implies $\mathbf{F}(P) \equiv^{\mathbf{ptl}} \mathbf{F}(Q)$ by transitivity of $\equiv^{\mathbf{ptl}}$, and $\mathbf{F}(P) \equiv^{\mathbf{ptl}} \mathbf{F}(Q)$ implies $\mathbf{F}(\,!\,P) = \,!\,\mathbf{F}(P) \mid (\nu w)\overline{w}w.\overline{w}w.\mathbf{F}(P) \equiv^{\mathbf{ptl}} \,!\,\mathbf{F}(Q) \mid (\nu w)\overline{w}w.\overline{w}w.\mathbf{F}(Q) = \mathbf{F}(\,!\,Q)$, since by Lemma 2.2(1), $\mathrm{fn}(P) = \mathrm{fn}(\mathbf{F}(P)) = \mathrm{fn}(\mathbf{F}(Q)) = \mathrm{fn}(Q)$, which shows that $\mathcal{R}$ is compatible with replication.

We show that $\mathcal{R}$ satisfies (α), in particular that $P \equiv_\alpha Q$ implies $\mathbf{F}(P) \equiv_\alpha \mathbf{F}(Q)$. Since $\mathcal{R}$ is a congruence, it follows from the properties of $\equiv_\alpha$ that it suffices to prove the special cases (a) $\mathbf{F}((\nu z)P[z/y]) \equiv_\alpha \mathbf{F}((\nu y)P)$, and (b) $\mathbf{F}(x(z).P[z/y]) \equiv_\alpha \mathbf{F}(x(y).P)$, for $x, y, z \in \mathbf{N}$ with $z \notin \mathrm{fn}(P)$ and $z \neq y$.

(a)

$$\begin{aligned}
\mathbf{F}((\nu z)P[z/y]) = (\nu z)\mathbf{F}(P[z/y]) \quad &= \quad (\nu z)\mathbf{F}(P)[z/y] \\
&\equiv_\alpha \quad (\nu y)\mathbf{F}(P) \\
&= \quad \mathbf{F}((\nu y)P),
\end{aligned}$$

by Lemma 2.2(2) and α-conversion, respectively.

(b) Similarly, for $v, w \neq y, z$,

$$\begin{array}{rcl}
\mathbf{F}(x(z).P[z/y]) & = & x(z).(\mathbf{F}(P[z/y]) \mid \\
& & \quad (\nu v)(\nu w)(\overline{v}v.\overline{w}w.\mathbf{F}(P[z/y]) \mid \overline{v}v.\overline{w}w.\mathbf{0})) \\
& \equiv_\alpha & x(y).(\mathbf{F}(P) \mid (\nu v)(\nu w)(\overline{v}v.\overline{w}w.\mathbf{F}(P) \mid \overline{v}v.\overline{w}w.\mathbf{0})) \\
& = & \mathbf{F}(x(y).P).
\end{array}$$

Next, observe that since $\mathbf{F}$ is compatible with parallel composition and restriction, and since $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, it is easily shown that $\mathcal{R}$ satisfies structural laws (1.1), (1.2), (1.3), and (2.1); e.g., $\mathbf{F}(P \mid (Q \mid R)) = \mathbf{F}(P) \mid \mathbf{F}(Q \mid R) = \mathbf{F}(P) \mid (\mathbf{F}(Q) \mid \mathbf{F}(R)) \equiv^{\mathbf{ptl}} (\mathbf{F}(P) \mid \mathbf{F}(Q)) \mid \mathbf{F}(R) = \mathbf{F}(P \mid Q) \mid \mathbf{F}(R) = \mathbf{F}((P \mid Q) \mid R)$, which shows the case for (1.3). By Lemma 2.2(1), (2.2) and (2.3) are also obvious. Finally, we have

$$\begin{array}{rcl}
\mathbf{F}(\,!\,(P \mid Q)) & = & !\,(\mathbf{F}(P) \mid \mathbf{F}(Q)) \mid (\nu w)\overline{w}w.\overline{w}w.\mathbf{F}(P \mid Q) \\
& \equiv^{\mathbf{ptl}} & (\,!\,(\mathbf{F}(P) \mid \mathbf{F}(Q)) \mid \mathbf{F}(P)) \mid (\nu w)\overline{w}w.\overline{w}w.\mathbf{F}(P \mid Q) \\
& \equiv^{\mathbf{ptl}} & (\,!\,\mathbf{F}(P \mid Q) \mid (\nu w)\overline{w}w.\overline{w}w.\mathbf{F}(P \mid Q)) \mid \mathbf{F}(P) \\
& = & \mathbf{F}(\,!\,(P \mid Q)) \mid \mathbf{F}(P) \\
& = & \mathbf{F}(\,!\,(P \mid Q) \mid P),
\end{array}$$

which shows the case for (3.6). □

Corollary 3.2 *For all process terms P and Q, if $P \equiv^{\mathbf{ptl}} Q$, then $P \equiv^{\mathbf{F}}_m Q$.*

Proof By Lemma 3.1, $P \equiv^{\mathbf{ptl}} Q$ implies $\mathbf{F}(P) \equiv^{\mathbf{ptl}} \mathbf{F}(Q)$, and thus $\mathbf{F}(P) \equiv \mathbf{F}(Q)$. By Theorem B of [2], we have $\mathbf{F}(P) \equiv_m \mathbf{F}(Q)$. Consequently, $P \equiv^{\mathbf{F}}_m Q$. □

4 Berries and Cherries

Due to the particular form of $\mathbf{F}(\,!\,P)$ and $\mathbf{F}(g.P)$, the molecules that appear in a solution that is the semantics (by $\Rightarrow_{\mathbf{F}}$) of a process term have two different overall shapes, viz., $\overline{m}m.\{\overline{m}m.S\}$, and $g.(S_0 \cup \{\overline{m}m.\{\overline{m'}m'.S_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\})$, according to (SF4) and (SF3), respectively. We will need the following properties of such molecules.

Definition 4.1 A *berry* is a molecule $\overline{m}m.\{\overline{m}m.S\}$ with $m \in \mathrm{New}$. A *cherry* is a molecule $g.T$ with $T = S_0 \cup \{\overline{m}m.\{\overline{m'}m'.S_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$, where g is a schematic guard, $m, m' \in \mathrm{New}$ with $m \neq m'$, S_0 and S_1 are solutions with $\mathrm{fn}(S_0) \cap \mathbf{N} = \mathrm{fn}(S_1) \cap \mathbf{N}$, and all molecules of S_0 of the form $g_1.\{g_2.T'\}$ are berries.

The denomination of the two may become more apparent after the inspection of Fig. 2. Clearly, berries and cherries can be distinguished only by their

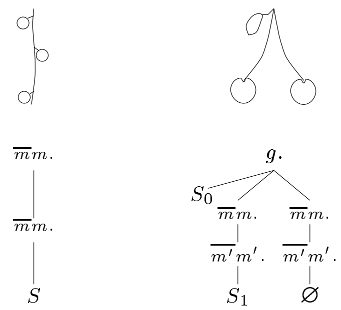

Figure 2: A berry and a cherry

form: if $g.T$ is a berry, then T is a singleton solution, whereas if $g.T$ is a cherry, then T consists of at least two molecules. Because of this easy identification two inactive agents were needed in the definition of $\mathbf{F}(g.P)$. The structural difference between berries and cherries also guarantees that after a substitution $[z/y]$ (where $y, z \in \mathbf{N} \cup \mathbb{N}_+ \cup \text{New}$), a berry does not change into a cherry (although it might not *remain* a berry), and vice versa. By the last requirement in Definition 4.1 the representation of T in a cherry $g.T$ is unique, i.e., if also $T = U_0 \cup \{\overline{n}n.\{\overline{n'}n'.U_1\}, \overline{n}n.\{\overline{n'}n'.\varnothing\}\}$ with all the requirements in Definition 4.1, then $U_0 = S_0$, $U_1 = S_1$, $n = m$, and $n' = m'$, since $\overline{n}n.\{\overline{n'}n'.U_1\}$ and $\overline{n}n.\{\overline{n'}n'.\varnothing\}$ are evidently not berries.

In Section 7, where we prove the completeness of potential structural congruence, we want to give a characterization of the sets $\{P \mid P \equiv_m^{\mathbf{F}} Q\}$ for arbitrary process terms Q. The case in which Q is a restriction $(\nu x)Q'$ (see Lemma 7.4) is rather cumbersome, in the sense that it can 'clash' with the restrictions introduced in $\mathbf{F}(P)$ (that evidently are of a different kind) when P is a guarded process term $g.P'$ or a replication $!P'$. Consider for instance the latter. Let U be a solution for which both $!P' \Rightarrow_{\mathbf{F}} U$ and $(\nu x)Q' \Rightarrow_{\mathbf{F}} U$ hold. By inspection of (SF4) and (SF2), we derive that $U = \bigcup_{i \in \mathbb{N}} S_i \cup \{\overline{m}m.\{\overline{m}m.S'\}\} = T'[n/x]$, where $P' \Rightarrow_{\mathbf{F}} S_i, S'$ and $Q' \Rightarrow_{\mathbf{F}} T'$ (and assuming the S_i, S', and T' satisfy the conditions in (SF4) and (SF2)). Suppose that in the course of our investigation we considered the case $m = n$. It is easily verified that then $T' = \bigcup_{i \in \mathbb{N}} S_i \cup \{\overline{x}x.\{\overline{x}x.S'\}\}$. But T' would not make sense in this case since the molecule $\overline{x}x.\{\overline{x}x.S'\}$ cannot appear in any solution that lies in the range of $\Rightarrow_{\mathbf{F}}$. To exclude these cases *a priori*, we introduce properties of solutions that make the range of $\Rightarrow_{\mathbf{F}}$ more explicit.

Definition 4.2 A solution S has the *first berry-cherry (BC) property* if it contains only berries and cherries. It has the *second berry-cherry (BC) property* if $x \in \text{fn}(S) \cap \mathbf{N}$ implies that $x \in \text{fn}(g.T)$ for a cherry $g.T$ of S.

Observe that in the above example, the first BC property of T' forbids $m = n$. In fact, the second BC property of T' also forbids that $m = n$ and, moreover, excludes that $n \in \mathrm{new}(S')$ because $\mathrm{new}(S') \cap \mathrm{new}(S_i) = \varnothing$ for all $i \in \mathbb{N}$. Below we show that every solution that is the semantics of a process term satisfies the BC properties.

Lemma 4.3 *For every process term P, if $P \Rightarrow_{\mathbf{F}} S$, then S has the first and the second BC property.*

Proof The proof is by induction on the structure of P; it is trivial for $P = \mathbf{0}$.

Let $P = P_1 \mid P_2$ where $P_i \Rightarrow_{\mathbf{F}} S_i$ with $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$. Then $P \Rightarrow_{\mathbf{F}} S_1 \cup S_2$. Since $D_{S_1 \cup S_2} = D_{S_1} \cup D_{S_2}$, both BC properties are immediate by induction.

Let $P = (\nu x)P'$ where $P' \Rightarrow_{\mathbf{F}} S$ and let $n \in \mathrm{New} - \mathrm{new}(S)$. Then $P \Rightarrow_{\mathbf{F}} S[n/x]$. Note that $D_{S[n/x]} = \{g[n/x].T[n/x] \mid g.T \in D_S\}$. By induction, S has both BC properties. Observe that, evidently, a molecule $g[n/x].T[n/x]$ is a berry if $g.T$ is a berry. Now assume that $g.T \in D_S$ is a cherry, and let $T = S_0 \cup \{\overline{m}m.\{\overline{m'}m'.S_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$. Then

$$g[n/x].T[n/x] = g[n/x].(S_0[n/x] \cup \{\overline{m}m.\{\overline{m'}m'.S_1[n/x]\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}).$$

Now $\mathrm{fn}(S_0[n/x]) \cap \mathbf{N} = (\mathrm{fn}(S_0) \cap \mathbf{N}) - \{x\} = (\mathrm{fn}(S_1) \cap \mathbf{N}) - \{x\} = \mathrm{fn}(S_1[n/x]) \cap \mathbf{N}$. Moreover, since $\{g_1'.\{g_2'.U'\} \in D_{S_0[n/x]}\} = \{g_1[n/x].\{g_2[n/x].U[n/x]\} \mid g_1.\{g_2.U\} \in D_{S_0}\}$, all molecules of the form $g_1'.\{g_2'.U'\}$ in $S_0[n/x]$ are berries by the above observation. Hence $g[n/x].T[n/x]$ is a cherry, and thus $S[n/x]$ has the first BC property. To show that $S[n/x]$ has the second BC property, let $y \in \mathrm{fn}(S[n/x]) \cap \mathbf{N}$. Obviously, $y \in \mathrm{fn}(S)$ and $y \neq x$. By the second BC property of S, there exists a cherry $g.T$ in S with $y \in \mathrm{fn}(g.T)$. By the above, $g[n/x].T[n/x]$ is a cherry in $S[n/x]$, and, since $y \neq x$, $y \in \mathrm{fn}(g[n/x].T[n/x])$.

We show the case $P = g.P'$ for an input guard $g = x(y)$ only. Let $P' \Rightarrow_{\mathbf{F}} S_0, S_1$ with $\mathrm{new}(S_0) \cap \mathrm{new}(S_1) = \varnothing$. Then $P \Rightarrow_{\mathbf{F}} \{x(y).T\}$, where $T = S_0 \cup \{\overline{m}m.\{\overline{m'}m'.S_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$. We show that $x(y).T$ is a cherry. First recall that $x(y).T$ abbreviates $x(-).\mathrm{inc}(T)[1/y]$, where $\mathrm{inc}(T)[1/y] = \mathrm{inc}(S_0)[1/y] \cup \{\overline{m}m.\{\overline{m'}m'.\mathrm{inc}(S_1)[1/y]\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$. Now

$$\begin{aligned} \mathrm{fn}(\mathrm{inc}(S_0)[1/y]) \cap \mathbf{N} &= (\mathrm{fn}(S_0) \cap \mathbf{N}) - \{y\} \\ &= (\mathrm{fn}(S_1) \cap \mathbf{N}) - \{y\} \\ &= \mathrm{fn}(\mathrm{inc}(S_1)[1/y]) \cap \mathbf{N}, \end{aligned}$$

because $\mathrm{fn}(S_0) \cap \mathbf{N}$ and $\mathrm{fn}(S_1) \cap \mathbf{N}$ are both equal to $\mathrm{fn}(P')$, cf. the remark after Definition 2.3. Moreover, since by induction S_0 has the first BC property, all molecules of S_0 of the form $g_1.\{g_2.U\}$ are berries, and hence all molecules of $\mathrm{inc}(S_0)[1/y]$ of the form $g_1'.\{g_2'.U'\}$ are also berries, because $\{g_1'.\{g_2'.U'\} \in D_{\mathrm{inc}(S_0)[1/y]}\} = \{g_1.\{g_2.\mathrm{inc}(U)[1/y]\} \mid g_1.\{g_2.U\} \in D_{S_0}\}$. Consequently, $x(y).T$ is a cherry, so immediately $\{x(y).T\}$ has the first and second BC property.

Finally, let $P = \,!P'$ where $P' \Rightarrow_{\mathbf{F}} S, S_i$, $i \in \mathbb{N}$, such that the $\mathrm{new}(S_i)$ are mutually disjoint and disjoint with $\mathrm{new}(S)$, and let $m \in \mathrm{New}$ with $m \notin \mathrm{new}(S)$ and $m \notin \mathrm{new}(S_i)$. Then $P \Rightarrow_{\mathbf{F}} S' = (\bigcup_{i\in\mathbb{N}} S_i) \cup \{\overline{m}m.\{\overline{m}m.S\}\}$. Since $\overline{m}m.\{\overline{m}m.S\}$ is a berry, both BC properties are immediate by induction; the second holds because $x \in \mathrm{fn}(S') \cap \mathbf{N}$ implies $x \in \mathrm{fn}(S_i) \cap \mathbf{N}$, since $\mathrm{fn}(S) \cap \mathbf{N} = \mathrm{fn}(P') = \mathrm{fn}(S_i) \cap \mathbf{N}$. □

The BC properties are rather flexible in the sense that they are closed under, for instance, arbitrary multiset union (as is used in the above proof), and copying, both of which are trivial to show. The first BC property is also closed under multiset containment $\subseteq$; the second fails, which can be seen by taking $S = \{g.T_{\mathrm{c}}, g.T_{\mathrm{b}}\}$, where $g.T_{\mathrm{c}}$ is a cherry and $g.T_{\mathrm{b}}$ is a berry with $x \in \mathrm{fn}(T_{\mathrm{c}}) \cap \mathrm{fn}(T_{\mathrm{b}}) \cap \mathbf{N}$. Then S has the second BC property, but clearly $\{g.T_{\mathrm{b}}\} \subseteq S$ has not. The BC properties are also closed under all substitutions $[n/x]$ with $n \in \mathrm{New}$ and $x \in \mathbf{N}$; this follows implicitly from the case $P = (\nu x)P'$ in the proof of the above lemma. Evidently, they are not closed under inverse substitutions $[x/n]$. However, we do have the following remarkable result: the BC properties of a solution S are guaranteed if they hold for *two* (different) substitutions $S[m/y]$ and $S[n/x]$. This will be used in the proof of Lemma 7.4.

Lemma 4.4 *Let S be a solution such that both $S[m/y]$ and $S[n/x]$ have the first and the second BC property, where $x, y \in \mathbf{N}$ with $x \neq y$ and $m, n \in \mathrm{New} - \mathrm{new}(S)$. Then S has the first and the second BC property.*

Proof To show that S has the first BC property, we first observe that $D_S = \{(g.T)[x/n] \mid g.T \in D_{S[n/x]}\} = \{(g.T)[y/m] \mid g.T \in D_{S[m/y]}\}$. Now let $g.T \in D_{S[n/x]}$ be a cherry with $T = S_0 \cup \{\overline{m_1}m_1.\{\overline{m_2}m_2.S_1\}, \overline{m_1}m_1.\{\overline{m_2}m_2.\varnothing\}\}$. We show that $(g.T)[x/n] \in D_S$ is a cherry. Note that by the first BC property of $S[m/y]$, $(g.T)[x/n][m/y] \in D_{S[m/y]}$ is a cherry. Now assume that $m_1 = n$. Then

$$T[x/n][m/y] = S_0[x/n][m/y] \cup \{\overline{x}x.\{\overline{m_2'}m_2'.S_1[x/n][m/y]\}, \overline{x}x.\{\overline{m_2'}m_2'.\varnothing\}\},$$

with $m_2' = m_2[x/n]$. Since neither $\overline{x}x.\{\overline{m_2'}m_2'.S_1[x/n][m/y]\}$ nor $\overline{x}x.\{\overline{m_2'}m_2'.\varnothing\}$ is a berry, this contradicts that $(g.T)[x/n][m/y]$ is a cherry. Hence $m_1 \neq n$, and similarly it can be shown that $m_2 \neq n$. Thus,

$$T[x/n] = S_0[x/n] \cup \{\overline{m_1}m_1.\{\overline{m_2}m_2.S_1[x/n]\}, \overline{m_1}m_1.\{\overline{m_2}m_2.\varnothing\}\}.$$

Suppose $z \in \mathrm{fn}(S_0[x/n])$ with $z \in \mathbf{N}$. If $z \neq x$, then, since $z \in \mathrm{fn}(S_0)$ and hence $z \in \mathrm{fn}(S_1)$ because $g.T$ is a cherry, $z \in \mathrm{fn}(S_1[x/n])$ is immediate. If $z = x$, then $z \in \mathrm{fn}(S_0[x/n][m/y])$ since $x \neq y$. Thus $z \in \mathrm{fn}(S_1[x/n][m/y])$ since $(g.T)[x/n][m/y]$ is a cherry, and so $z \in \mathrm{fn}(S_1[x/n])$. This shows that $\mathrm{fn}(S_0[x/n]) \cap \mathbf{N} \subseteq \mathrm{fn}(S_1[x/n]) \cap \mathbf{N}$, and by symmetry we have $\mathrm{fn}(S_0[x/n]) \cap \mathbf{N} = \mathrm{fn}(S_1[x/n]) \cap \mathbf{N}$. Finally, let $\mathsf{m} = h[x/n].\{h'[x/n].S'[x/n]\}$ be a molecule in $S_0[x/n]$. Then $h.\{h'.S'\} \in D_{S_0}$ is a berry; let $h = h' = \overline{n_1}n_1$. Again we have $n_1 \neq n$ since otherwise $\overline{x}x.\{\overline{x}x.S'[x/n][m/y]\}$ is a molecule in $S_0[x/n][m/y]$

that is not a berry. Hence $h'[x/n] = h[x/n] = \overline{n_1}n_1$, and thus m is a berry. By the same argument it can be shown that $(g.T)[x/n] \in D_S$ is a berry whenever $g.T \in D_{S[n/x]}$ is a berry. This shows that S has the first BC property.

To show that S has the second BC property, let $z \in \mathrm{fn}(S)$. If $z \neq x$, then $z \in \mathrm{fn}(S[n/x])$ and so, by the second BC property of $S[n/x]$, there exists a cherry $g.T \in D_{S[n/x]}$ with $z \in \mathrm{fn}(g.T)$. Thus $z \in \mathrm{fn}((g.T)[x/n])$ and, by the argument in the previous paragraph, $(g.T)[x/n] \in D_S$ is a cherry. The case that $z = x$ (and hence $z \neq y$) follows from a symmetric argument. $\square$

5 Cells and Colonies

To simplify our notation in this section, let for an arbitrary guard g and $m, m' \in$ New with $m \neq m'$, $\mathsf{C}(g, m, m')$ be the cherry

$$g.\{\overline{m}m.\{\overline{m'}m'.\varnothing\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\},$$

and let for an arbitrary solution S and $m \in$ New, $\mathsf{B}(m, S)$ be the berry

$$\overline{m}m.\{\overline{m}m.S\}.$$

Note that $g.\mathbf{0} \Rightarrow_{\mathbf{F}} \{\mathsf{C}(g, m, m')\}$ for every guard g over $\mathbf{N}$ and $m, m' \in$ New with $m \neq m'$.

We have now come to the heart of the paper, where we will give a 'model' for structural law (3.6), the one remaining law of structural congruence that involves replication (see Fig. 1). Since the other laws that involve replication, (3.2)–(3.5), are no longer valid in $\equiv^{\mathbf{ptl}}$, we can deduce from a close inspection of (3.6) only that any model which gives meaning to the structure of processes (as expressed in the laws of $\equiv^{\mathbf{ptl}}$) should treat a replicated process as an atomic entity, similar to the connected components of a solution. This is because, contrary to the aforementioned laws of replication, in (3.6) the replication $!(P \mid Q)$ is the same on either side. The entity we propose is a *cell*, a solution with an infinite structure that is distinguished by a special connected component: its *nucleus*. In much the same way as the process term P on the right side of structural law (3.6) can be seen as the *offspring* of $!(P \mid Q)$, the *division* of a cell produces an offspring solution that consists of a finite number of daughter cells. The key technical construction we use is a partial order on connected solutions that relates (the nucleus of) a cell to its offspring.

Suppose we are interested in the following question: for given process terms Q_1 and Q_2, which process terms P can be found such that $P \equiv^{\mathbf{F}}_m Q_1 \mid Q_2$ (Again, we will be interested in this type of question in Section 7, in particular Lemma 7.3.) In Lemma 28 of [3], similar problems were investigated involving $\equiv_m$ instead of $\equiv^{\mathbf{F}}_m$, so let us first consider solutions S with $P \Rightarrow S$.

As an example, let $P = \,!((\nu z)R_1 \mid R_2)$, where $R_1 = \overline{x}z.\mathbf{0}$ and $R_2 = x(y).\mathbf{0}$.

The reader can easily verify by inspection of (S0)–(S4) that

$$S = (\bigcup_{i\in\mathbb{N}} V_i) \cup (\bigcup_{j\in\mathbb{N}} U_j)$$

where $V_i = \{\overline{x}n_i.\varnothing\}$, $U_j = \{x(y).\varnothing\}$, and the $n_i \in \text{New}$ are all distinct. Thus, in fact, the V_i and the U_j constitute the connected components of S. Now if we require $P \equiv_m Q_1 \mid Q_2$, i.e., $S = T_1 \cup T_2$ with $Q_k \Rightarrow T_k$, $k \in \{1,2\}$, and $\text{new}(T_1) \cap \text{new}(T_2) = \varnothing$, then T_1 and T_2 must form a partition (multiset-wise) of the connected components of S, i.e., we can deduce that $T_k = (\bigcup_{i\in I_k} V_i) \cup (\bigcup_{j\in J_k} U_j)$, such that both I_1, I_2 and J_1, J_2 partition $\mathbb{N}$. Depending on whether I_k and J_k are finite or infinite, T_1 and T_2 (and hence Q_1 and Q_2) can appear in various forms. For instance, if I_1 is finite and both J_1 and J_2 are infinite, then

$$Q_1 \equiv_m Q_1' = \overbrace{(\nu z)R_1 \mid \ldots \mid (\nu z)R_1}^{\#I_1} \mid !\,R_2$$

and

$$Q_2 \equiv_m Q_2' = \;!\,(\nu z)R_1 \mid \;!\,R_2.$$

Not surprisingly, $P \equiv Q_1' \mid Q_2'$, which can be shown by applying structural laws (3.1), (3.2), and (3.5) for this particular instance of P. Our example shows that, in general, the problem of finding P such that $P \equiv_m Q_1 \mid Q_2$ is now reduced to the problem of finding Q_k' such that $Q_k' \equiv_m Q_k$. This is exactly the inductive argument that was used in the proof of the completeness of extended structural congruence in [3].

Now let us return to our original question: find P such that $P \equiv_m^{\mathbf{F}} Q_1 \mid Q_2$. Again, by inspection of (SF0)–(SF4), the reader can check that $P \Rightarrow_{\mathbf{F}} S$ such that

$$S = (\bigcup_{i\in\mathbb{N}} V_i) \cup (\bigcup_{j\in\mathbb{N}} U_j) \cup W,$$

where $V_i = \{\mathsf{C}(\overline{x}n_i, o_i, o_i')\}$, $U_j = \{\mathsf{C}(x(y), p, p')\}$, and

$$W = \{\mathsf{B}(m, \{\mathsf{C}(\overline{x}n, o, o'), \mathsf{C}(x(y), q, q')\})\},$$

where the $m, n, n_i, o_i, o_i', o, o', p, p', q, q' \in \text{New}$ are all distinct. Observe that now the V_i, the U_j, and W form the connected components of S. Thus, as before, if we set $S = T_1 \cup T_2$ with $Q_k \Rightarrow_{\mathbf{F}} T_k$, $k \in \{1,2\}$, then clearly $T_k = (\bigcup_{i\in I_k} V_i) \cup (\bigcup_{j\in J_k} U_j) \cup W_k$, where $W_k = W$ if and only if $W_{(k \bmod 2)+1} = \varnothing$. But now our choice of I_k and J_k is limited; not every combination yields solutions T_k that make sense, i.e., are the semantics (by $\Rightarrow_{\mathbf{F}}$) of process terms Q_k. Observe that, in fact, the mere presence or the absence of W in T_k is the cause of these constraints: by inspection of (SF0)–(SF4), it is straightforward that if $W_k = \varnothing$, then both I_k and J_k are finite, and if $W_k = W$, then both I_k and J_k are infinite. To put it differently, if W is a connected component of T_k, then (and only then) W *forces* the V_i and U_j to be of multiplicity ω in T_k. So W is the only connected component of T_k with finite multiplicity (viz., one); the other connected components all have multiplicity ω in T_k. In fact, this also holds for the multiplicities in S: we call W the *nucleus* of the *cell* S. Observe that W 'contains a hidden copy of' V_i and of U_j: it is guarded by the ms in the berry W consists of. It is this 'forcing of hidden copies' we want to capture in the next two definitions.

To conclude our example, $Q_1 \equiv^{\mathbf{F}}_m Q'_1 = P$ and

$$Q_2 \equiv^{\mathbf{F}}_m Q'_2 = \overbrace{(\nu z)R_1 \mid \ldots \mid (\nu z)R_1}^{\#I_2} \mid \overbrace{R_2 \mid \ldots \mid R_2}^{\#J_2},$$

or conversely. Thus, the division of S into T_1 and T_2 produces a finite number (viz., $\#I_2 + \#J_2$ in the above case) of daughter cells. Observe that in either case $P \equiv^{\mathbf{ptl}} Q'_1 \mid Q'_2$, using structural law (3.6), and that, again, we reduced finding $P \equiv^{\mathbf{F}}_m Q_1 \mid Q_2$ to finding $Q'_k \equiv^{\mathbf{F}}_m Q_k$. This inductive argument is used in Section 7 in the proof of completeness of potential structural congruence.

Definition 5.1 Let V and W be nonempty connected solutions. Then V *is directly hidden in* W, denoted $V \prec W$, if, for some $m \in \mathrm{New}$ and solutions S and T, $W = \{\overline{m}m.\{\overline{m}m.S\}\} \cup T$ and $V' \subseteq^{\mathrm{n}} S$ for some copy V' of V with $\mathrm{new}(V') \cap \mathrm{new}(T) = \varnothing$. We let $< \, = \, \prec^+$; if $V < W$, then we say that V *is hidden in* W.

Recall from the Preliminaries (or from [4]) that $V' \subseteq^{\mathrm{n}} S$ if $V' \cup U' = S$ with $\mathrm{new}(V') \cap \mathrm{new}(U') = \varnothing$. Since V' is nonempty and connected, this just means that it is a connected component of S and hence $\mathrm{mult}(V, S) \geq 1$.

Definition 5.1 is of a rather technical nature due to the presence of T. However, we want to stress that every result in this section (except the last) relies solely on the fact that $<$ is a partial order on nonempty connected solutions that is closed under taking copies. Therefore, we postpone the illustration of the technical details until the moment we need them, viz., in Lemma 5.10.

Lemma 5.2 *The relation $<$ is transitive, irreflexive, and is preserved under taking copies.*

Proof Clearly, $<$ is transitive by Definition 5.1. We first show that $<$ is preserved under taking copies, i.e., that $V' < W'$ whenever $V < W$, where V' is a copy of V and W' is a copy of W. Observe that it suffices to show that $V \prec W$ implies $V' \prec W'$. Let $W = \{\overline{m}m.\{\overline{m}m.S\}\} \cup T$, where $V_1 \subseteq^{\mathrm{n}} S$ for some copy V_1 of V with $\mathrm{new}(V_1) \cap \mathrm{new}(T) = \varnothing$. Then, for some injection $f : \mathrm{new}(W) \to \mathrm{New}$, $W' = \{\overline{f(m)}f(m).\{\overline{f(m)}f(m).f(S)\}\} \cup f(T)$. By Lemma 3.2(2) of [4], $f(V_1) \subseteq^{\mathrm{n}} f(S)$. Moreover, $\mathrm{new}(f(V_1)) \cap \mathrm{new}(f(T)) = f(\mathrm{new}(V_1)) \cap f(\mathrm{new}(T)) = \varnothing$, since f is injective. Clearly, $f(V_1)$ is a copy of V'. This proves that $<$ is preserved under copying. Finally, we show that there exists no infinite sequence

$W_0 \succ W_1 \succ \ldots$. Assume the contrary. Since $\prec$ is preserved under copying, the W_i, $i \geq 0$, can be chosen such that $W_i = \{\overline{m_i}m_i.\{\overline{m_i}m_i.S_i\}\} \cup T_i$ and $W_{i+1} \subseteq^{\mathrm{n}} S_i$, for solutions S_i and T_i, and with $m_i \in$ New. But then, when we view the molecule $\mathsf{m} = \overline{m_0}m_0.\{\overline{m_0}m_0.S_0\}$ as a rooted directed tree (cf. [2]), there exists an infinite directed path starting at the root of m of which the nodes are labeled $\overline{m_0}m_0$, $\overline{m_0}m_0$, $\overline{m_1}m_1$, $\overline{m_1}m_1$, $\overline{m_2}m_2$, ... respectively. However, this contradicts the definition of Sol and Mol of [2]. It follows that $<$ is irreflexive. $\square$

Thus, the partial order $<$ is rather on equivalence classes $[V]$ than on the connected solutions V. Hence, trivially, $V < W$ only if V is not a copy of W.

Next, as explained in the example, we are interested in *cells*, which are solutions of which all but one of its connected components have multiplicity ω. More generally, we are interested in finite unions of such solutions, which we will call *colonies*. As will be shown in the next section, every solution that is the semantics by $\Rightarrow_{\mathbf{F}}$ of a process term is such a colony.

Definition 5.3 Let W be a nonempty connected solution. A solution X with $\mathrm{new}(W) \cap \mathrm{new}(X) = \varnothing$ and $\mathrm{mult}(V, X) \in \{0, \omega\}$ for all nonempty connected V is an *initial segment of* W if for all nonempty connected V, $\mathrm{mult}(V, X) = \omega$ if and only if $V < W$. For any pair (W, X) of such solutions, the union $C = W \cup X$ is a *cell*; the *nucleus of* C is $\mathrm{nuc}(C) = W$. A solution S is a *colony* if $S = \bigcup_{p=1}^{s} C_p$, $s \geq 0$, where the C_p are cells such that $\mathrm{new}(C_p) \cap \mathrm{new}(C_{p'}) = \varnothing$ for all $p, p' \in \{1, \ldots, s\}$ with $p \neq p'$.

Note that in the previous example, $(\bigcup_{i \in \mathbb{N}} V_i) \cup (\bigcup_{j \in \mathbb{N}} V)$ is an initial segment of W, and thus S is a cell with nucleus W.

Obviously, by the irreflexivity of $<$ (Lemma 5.2), the nucleus $\mathrm{nuc}(C)$ of a cell C is well defined, since it is the only connected component of multiplicity one. We show that for every nonempty connected solution there exists a cell of which it is the nucleus. In fact, let W be a nonempty connected solution and consider the set $\{[V] \mid V < W\}$. Since by Definition 5.1 the set $\{[V] \mid V \prec W\}$ is countable (because solutions are countable multisets), $\{[V] \mid V < W\}$ is a countable set. Let V_i, $i \in I$, be representatives of these equivalence classes. Let furthermore for all $i \in I$ and $j \in \mathbb{N}$, $f_{i,j} : \mathrm{new}(V_i) \to \mathrm{New} - \mathrm{new}(W)$ be an injection such that the $\mathrm{new}(f_{i,j}(V_i))$ are mutually disjoint. Observe that the $f_{i,j}$ exist because $\mathrm{New} - \mathrm{new}(W)$ is uncountably infinite and $\mathrm{new}(V_i)$ is countable for every $i \in I$ (as is observed just below Lemma 1 of [2], $\mathrm{new}(S)$ is countable for every solution S). Then $f_{i,j}(V_i)$ is a copy of V_i for every $i \in I$ and $j \in \mathbb{N}$. Thus $X = \bigcup_{i \in I} \bigcup_{j \in \mathbb{N}} f_{i,j}(V_i)$ is an initial segment of W and so $W \cup X$ is a cell with nucleus W.

Using property (Pa) of Section 1, the following facts can easily be proved. If X and Y are initial segments of a nonempty connected solution W, then X is a copy of Y. By the distribution of multiset mappings over multiset union and the preservation of $<$ under the 'copy-of' relation (Lemma 5.2), every copy of a cell is a cell and every copy of a colony is a colony.

For a cell $C = \mathrm{nuc}(C) \cup X$, if a connected component V of X is itself the nucleus $\mathrm{nuc}(D)$ of a cell D, i.e., $\mathrm{nuc}(D) < \mathrm{nuc}(C)$, then, as the following lemma shows in a general setting, (a copy of) the cell D is already part of C.

Lemma 5.4 *Let C and D_i, $i \in I$, be cells such that the* $\mathrm{new}(D_i)$ *are mutually disjoint,* $\mathrm{new}(C) \cap \mathrm{new}(D_i) = \varnothing$, *and* $\mathrm{nuc}(D_i) < \mathrm{nuc}(C)$ *for all $i \in I$. Then $C \cup \bigcup_{i \in I} D_i$ is a copy of C.*

Proof Using (Pa), we show that $\mathrm{mult}(V, C \cup \bigcup_{i \in I} D_i) = \mathrm{mult}(V, C)$ for all nonempty connected V. By (Pc) this holds if $\mathrm{mult}(V, D_i) = 0$ for all $i \in I$. Now let $\mathrm{mult}(V, D_i) \geq 1$ for some $i \in I$. Since D_i is a cell, either $V < \mathrm{nuc}(D_i)$ or V is a copy of $\mathrm{nuc}(D_i)$. Hence, since $<$ is transitive and is preserved under taking copies, $V < \mathrm{nuc}(C)$. Consequently, since C is a cell, $\mathrm{mult}(V, C) = \omega$ and so $\mathrm{mult}(V, C \cup \bigcup_{i \in I} D_i) = \mathrm{mult}(V, C)$, by (Pc). $\square$

Note that we could have applied Lemma 14 of [3], but that in fact Lemma 14 of [3] follows directly from (Pa) and (Pc).

Clearly, a finite union of new-disjoint colonies is a colony. We use the following property of a colony S often in subsequent proofs and will refer to it as *the colony property*:

> *for all nonempty connected V,* $\mathrm{mult}(V, S) = \omega$ *if and only if there exists a connected component W of S with $V < W$;*

this is an easy consequence of (Pc) and the transitivity of $<$ (see Lemma 5.2). Using the irreflexivity of $<$ it is also easy to show that a colony S is either empty, or it has at least one connected component V with finite multiplicity, i.e., $1 \leq \mathrm{mult}(V, S) < \omega$. For every such V, if $S = \bigcup_{p=1}^{s} C_p$, the union of new-disjoint cells C_p, then V is a copy of the nucleus of a particular cell C_p. The reverse need not hold, since it is possible that $\mathrm{nuc}(C_p)$ has multiplicity ω in one of the other cells, and hence in S; however the following lemma — which, in fact, we will not need in the sequel but we will state anyway — shows that there exists at least one representation of the colony S as a finite union of new-disjoint cells C in which all the nuclei have finite multiplicity. Moreover, this representation of S is unique (up to taking copies of the cells C).

Lemma 5.5 *For every colony S there exist new-disjoint cells $C_1, \ldots, C_s$, $s \geq 0$, such that $S = \bigcup_{p=1}^{s} C_p$ and for all nonempty connected V, V is a copy of* $\mathrm{nuc}(C_p)$ *for some $p \in \{1, \ldots, s\}$ if and only if* $1 \leq \mathrm{mult}(V, S) < \omega$.

Proof The if-part is obvious. Let $S = \bigcup_{q=1}^{t} D_q$, where the D_q are new-disjoint cells. The proof of the only-if part is by induction on t. It is trivial for $t = 0$. Assume $t \geq 1$ and let V be a copy of $\mathrm{nuc}(D_q)$ for some $q \in \{1, \ldots, t\}$ such that $\mathrm{mult}(V, S) = \omega$. Then, by (Pc), there exists $q' \in \{1, \ldots, t\}$ such that $\mathrm{mult}(V, D_{q'}) = \omega$. Hence $\mathrm{nuc}(D_q) < \mathrm{nuc}(D_{q'})$. By Lemma 5.4, $D_q \cup D_{q'}$ is a cell. The proof now follows by induction. $\square$

The representation of the colony S in Lemma 5.5 can be viewed as a minimal one, in the sense that S cannot be written as the union of a smaller number of cells. However, even if we remove from a colony any infinite number of copies of one cell, we still end up with a copy of the colony.

Lemma 5.6 *Let C_i, $i \in I$, be cells with mutually disjoint* $\mathrm{new}(C_i)$ *and let $S = \bigcup_{i\in I} C_i$ be a colony. If, for some cell C, the set $K = \{i \in I \mid C_i \text{ is a copy of } C\}$ is infinite, then $\bigcup_{i\in I-K} C_i$ is a copy of S.*

Proof Since, by (Pa), the nuclei of the C_i, $i \in K$, are copies of $\mathrm{nuc}(C)$ we have $\mathrm{mult}(\mathrm{nuc}(C), S) = \omega$. Hence, by the colony property of S, there exists a connected component W of S with $\mathrm{nuc}(C) < W$. Clearly, W is not a connected component of C_i with $i \in K$, because if it were, $W < \mathrm{nuc}(C)$, which is impossible by the irreflexivity of $<$. Thus W is a connected component of C_j for some $j \in I - K$, so $W < \mathrm{nuc}(C_j)$ or $W = \mathrm{nuc}(C_j)$ since C_j is a cell. In each case, $\mathrm{nuc}(C_i) < \mathrm{nuc}(C_j)$ for all $i \in K$. By Lemma 5.4, C_j is a copy of $C_j \cup \bigcup_{i\in K} C_i$, and so $\bigcup_{i\in I-K} C_i$ is a copy of S, by (Pb). $\square$

The designation of the term cell to the union of a nonempty connected solution with its initial segment is due to the following *fundamental cell-division property*:

> *If a cell is split into two new-disjoint colonies, then the one containing the nucleus is always a copy of the cell.*

We view the second colony as the 'offspring' of the original cell.

Lemma 5.7 *Let S_1, S_2 be colonies such that* $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$. *Let W be a nonempty connected solution and X an initial segment of W. If $W \cup X = S_1 \cup S_2$, then there exists a copy X' of X such that either $S_1 = W \cup X'$ and $X = X' \cup S_2$, or conversely, $S_2 = W \cup X'$ and $X = X' \cup S_1$.*

Proof Observe that since W is a nonempty connected solution, by Lemmas 6 and 8 of [3], there exists X' such that either $S_1 = W \cup X'$ and $X = X' \cup S_2$, or the converse: $S_2 = W \cup X'$ and $X = S_1 \cup X'$. By symmetry, it suffices to prove only the first case. We show that $\mathrm{mult}(V, X) = \mathrm{mult}(V, X')$ for all nonempty connected V, since then, by (Pa), X is a copy of X'. If $\mathrm{mult}(V, X) = 0$, then obviously $\mathrm{mult}(V, X') = 0$, by (Pc). If $\mathrm{mult}(V, X) \neq 0$, then $\mathrm{mult}(V, X) = \omega$ since X is an initial segment. Hence $V < W$ and so, by the colony property of S_1, $\mathrm{mult}(V, S_1) = \omega$. Consequently, $\mathrm{mult}(V, X') = \omega$, by (Pc). $\square$

We will need the analogon of Lemma 6 of [3], which characterizes the division of multisets, for an arbitrary number of colonies, i.e., if S_i, $i \in I$, and T_j, $j \in J$, are colonies with mutually disjoint $\mathrm{new}(S_i)$ and mutually disjoint $\mathrm{new}(T_j)$, and if $\bigcup_{i\in I} S_i = \bigcup_{j\in J} T_j$, then there exist colonies $U_{i,j}$, such that $S_i = \bigcup_{j\in J} U_{i,j}$ and $T_j = \bigcup_{i\in I} U_{i,j}$. Unfortunately, this does not even hold for the smallest nontrivial case where $\#I = \#J = 2$ and the S_i and T_j are single cells. To see this, let $P = \,!(\nu z)R_1$, where $R_1 = \overline{x}z.\mathbf{0}$ is the process term taken from the example earlier in this section. The reader easily verifies that $P \,|\, P \Rightarrow_{\mathbf{F}} S_1 \cup S_2$, where $S_i = \mathrm{nuc}(S_i) \cup X_i$, $i \in \{1,2\}$, with

$$\mathrm{nuc}(S_i) = \{\mathsf{B}(m_i, \{\mathsf{C}(\overline{x}n_i, o_i, o_i')\})\},$$

$$X_i = \bigcup_{k \in K_i} \{\mathsf{C}(\overline{x}n_{k,i}, o_{k,i}, o_{k,i}')\},$$

K_1 and K_2 are infinite and disjoint sets, and the $m_i, n_i, o_i, o_i', n_{k,i}, o_{k,i}, o_{k,i}' \in$ New are all distinct. Observe that the S_i are both cells. Now let $T_j = \mathrm{nuc}(S_j) \cup X_{(j \bmod 2)+1}$. Clearly, the S_i and T_j are all cells that are copies of one another. Since the connected components of $S_1 \cup S_2$ are all distinct, the division depicted

	T_1	T_2
S_1	$\mathrm{nuc}(S_1)$	X_1
S_2	X_2	$\mathrm{nuc}(S_2)$

	T_1'	T_2'
S_1	S_1	$\varnothing$
S_2	$\varnothing$	S_2

Figure 3: Quartet (S_1, S_2, T_1, T_2) is not divisible into colonies

on the left of Fig. 3 is unique, but clearly the solutions inside the small squares are not colonies. However, in the division depicted on the right of Fig. 3 they are, and, although not equal, T_j is a copy of T_j'. So the general result we are aiming for rather claims the existence of colonies $U_{i,j}$ such that S_i *is a copy of* $\bigcup_{j \in J} U_{i,j}$ and T_j *is a copy of* $\bigcup_{i \in I} U_{i,j}$ (cf. Lemma 15 of [3]). The situation in Fig. 3 depicts precisely the construction we will use in the proof: we locate the nuclei of all the cells the S_i and the T_j consist of by determining the 'small square' to which they belong (since nuclei are connected components, a unique column i and row j can be assigned to them). Then the $U_{i,j}$ are constructed by adding the proper initial segments to the 'squares' that contain nuclei, otherwise leaving them empty. So the $U_{i,j}$ are determined only by the nuclei of the cells of the S_i and the T_j. As the following lemma shows, this construction indeed yields copies of the S_i and the T_j.

Lemma 5.8 *Let $S = \bigcup_{p=1}^{s} C_p$ be a colony of cells C_p. Let also $S = \bigcup_{k \in K} W_k$, where the W_k are the connected components of S. Let $K' \subseteq K$ be such that there exists an injection $\psi : \{1, \ldots, s\} \to K'$ with $\mathrm{nuc}(C_p) = W_{\psi(p)}$ for all p, $1 \le p \le s$. Let X_k, $k \in K'$, be an initial segment of W_k such that the* $\mathrm{new}(X_k)$ *are mutually disjoint and* $\mathrm{new}(X_k) \cap \mathrm{new}(S) = \varnothing$. *Then $\bigcup_{k \in K'} (W_k \cup X_k)$ is a copy of S.*

Proof Let K_p, $1 \leq p \leq s$, be mutually disjoint sets with $\bigcup_{p=1}^{s} K_p = K$ such that $C_p = \bigcup_{k \in K_p} W_k$ (see Lemma 9 of [3]). Clearly, since C_p is a cell, $W_k < \mathrm{nuc}(C_p) = W_{\psi(p)}$ for all $k \in K_p - \{\psi(p)\}$; in particular, $W_k < W_{\psi(p)}$ for all $k \in (K_p \cap K') - \{\psi(p)\}$. Hence by Lemma 5.4, for every $p \in \{1, \ldots, s\}$, $\bigcup_{k \in K_p \cap K'} (W_k \cup X_k)$ is a copy of $W_{\psi(p)} \cup X_{\psi(p)}$, and the latter is a copy of C_p. Consequently, by (Pb), $\bigcup_{k \in K'} (W_k \cup X_k) = \bigcup_{p=1}^{s} \bigcup_{k \in K_p \cap K'} (W_k \cup X_k)$ is a copy of $\bigcup_{p=1}^{s} C_p = S$. □

We now prove the analogon of Lemma 6 of [3] for colonies in a slightly generalized version.

Lemma 5.9 *Let S_i, $i \in I$, and T_j, $j \in J$, be solutions with mutually disjoint* $\mathrm{new}(S_i)$ *and mutually disjoint* $\mathrm{new}(T_j)$. *If $\bigcup_{i \in I} S_i = \bigcup_{j \in J} T_j$, then there exist colonies $U_{i,j}$, such that if S_i is a colony, then S_i is a copy of $\bigcup_{j \in J} U_{i,j}$, and if T_j is a colony, then T_j is a copy of $\bigcup_{i \in I} U_{i,j}$.*

Proof Let $I' = \{i \in I \mid S_i \text{ is a colony}\}$ and $J' = \{j \in J \mid T_j \text{ is a colony}\}$. For $i \in I'$, let $S_i = \bigcup_{p=1}^{s_i} C_{p,i}$, where the $C_{p,i}$ are new-disjoint cells, and similarly $T_j = \bigcup_{q=1}^{t_j} D_{q,j}$, for $j \in J'$. Let $C_{p,i} = \mathrm{nuc}(C_{p,i}) \cup X_{p,i}$, where $X_{p,i}$ is an initial segment of $\mathrm{nuc}(C_{p,i})$, and similarly for $D_{q,j} = \mathrm{nuc}(D_{q,j}) \cup Y_{q,j}$. By Lemma 6 of [3], there exist $U'_{i,j}$ such that $S_i = \bigcup_{j \in J} U'_{i,j}$ and $T_j = \bigcup_{i \in I} U'_{i,j}$. Hence $\bigcup_{j \in J} U'_{i,j} = \bigcup_{p=1}^{s_i} \mathrm{nuc}(C_{p,i}) \cup \bigcup_{p=1}^{s_i} X_{p,i}$ for all $i \in I'$. Let $U'_{i,j} = \bigcup_{k \in K_{i,j}} W_k$, where the W_k, $k \in K_{i,j}$, are the connected components of $U'_{i,j}$ and the $K_{i,j}$ are mutually disjoint. By Lemma 10 of [3], there exist injections $\psi_i : \{1, \ldots, s_i\} \to \bigcup_{j \in J} K_{i,j}$, $i \in I'$, such that $\mathrm{nuc}(C_{p,i}) = W_{\psi_i(p)}$. Similarly, there exist injections $\phi_j : \{1, \ldots, t_j\} \to \bigcup_{i \in I} K_{i,j}$, $j \in J'$, such that $\mathrm{nuc}(D_{q,j}) = W_{\phi_j(q)}$. Now let $L_{i,j} = \{k \in K_{i,j} \mid \exists p : k = \psi_i(p) \text{ or } \exists q : k = \phi_j(q)\}$. Observe that $L_{i,j}$ is finite for every $i \in I$ and $j \in J$ (and $L_{i,j} = \varnothing$ for every pair (i,j) with $i \in I - I'$ and $j \in J - J'$). Let, for every $k \in L_{i,j}$, Z_k be an initial segment of W_k such that the $\mathrm{new}(Z_k)$ are mutually disjoint and $\mathrm{new}(Z_k) \cap \mathrm{new}(\bigcup_{i' \in I} S_{i'}) = \varnothing$. Define $U_{i,j} = \bigcup_{k \in L_{i,j}} (W_k \cup Z_k)$. Clearly $U_{i,j}$ is a colony for all $i \in I$ and $j \in J$. Moreover, since for $i \in I'$, ψ_i is an injection $\{1, \ldots, s_i\} \to \bigcup_{j \in J} L_{i,j}$ and $\bigcup_{j \in J} L_{i,j} \subseteq \bigcup_{j \in J} K_{i,j}$, by Lemma 5.8, $\bigcup_{j \in J} U_{i,j}$ is a copy of S_i, and similarly for T_j, $j \in J'$. □

Observe that Lemma 5.9 is valid even when $\bigcup_{i \in I} S_i$ is a copy of $\bigcup_{j \in J} T_j$; let $\bigcup_{i \in I} S_i = f(\bigcup_{j \in J} T_j) = \bigcup_{j \in J} f(T_j)$ for some injection $f : \mathrm{New} \to \mathrm{New}$. By Lemma 5.9 there exist colonies $U_{i,j}$, $i \in I$, $j \in J$, such that $\bigcup_{j \in J} U_{i,j}$ is a copy of S_i if S_i is a colony, and $\bigcup_{i \in I} U_{i,j}$ is a copy of $f(T_j)$ if $f(T_j)$ is a colony. Hence $\bigcup_{i \in I} U_{i,j}$ is a copy of T_j if T_j is a colony.

We have now reached the point where we need to explain the technical details of Definition 5.1. In the next section we show that every solution that is the semantics of a process term (by $\Rightarrow_{\mathbf{F}}$) is a colony. More specifically, we show that for process terms in a normal form, every replication, every guarded process term, and every restriction corresponds to a cell (see Lemma 6.3).

If we inspect the semantic rules (SF0)–(SF4), we see that whenever a solution with connected components of ω-multiplicity is introduced (viz. in (SF4)), it contains a connected component of the form $\{\mathsf{B}(m,S)\}$ that hides a copy (viz. in S) of every other connected component in that solution, and vice versa. This means that the solution corresponding to a replicated process term is a cell with nucleus $\{\mathsf{B}(m,S)\}$. The construction of (SF3) yields a cell as well: $\{g.T\}$ with $T = S_0 \cup \{\overline{m}m.\{\overline{m'}m'.S_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$ is a cell with an empty initial segment, since it does not contain a berry. By the remaining (SF2) — which 'glues together' connected components that contain the free name x — a solution $\{\mathsf{B}(m,S)\}$ containing a berry can become encapsulated in a larger connected solution $\{\mathsf{B}(m,S[n/x])\} \cup T'[n/x]$, where T' collects the 'sticky' components containing x; note that this is the case whenever $x \in \mathrm{fn}(S)$. For this reason, the solution T was needed in the environment of the solution $\{\overline{m}m.\{\overline{m}m.S\}\}$ in Definition 5.1. To explain the condition $\mathrm{new}(V') \cap \mathrm{new}(T) = \varnothing$ in Definition 5.1, consider the process terms $P' = \,!R_1$ and $P = (\nu z)P'$, where $R_1 = \overline{x}z.\mathbf{0}$, the process term from the beginning of this section. We have

$$P' \Rightarrow_{\mathbf{F}} C = \{\mathsf{B}(m, \{\mathsf{C}(\overline{x}z, o, o')\})\} \cup \bigcup_{i \in \mathbb{N}} \{\mathsf{C}(\overline{x}z, o_i, o'_i)\},$$

and so

$$P \Rightarrow_{\mathbf{F}} C[n/z] = \{\mathsf{B}(m, \{\mathsf{C}(\overline{x}n, o, o')\})\} \cup \bigcup_{i \in \mathbb{N}} \{\mathsf{C}(\overline{x}n, o_i, o'_i)\}$$

with all the names from New distinct. Clearly, both C and $C[n/z]$ are cells (with $\mathrm{nuc}(C) = \{\mathsf{B}(m, \{\mathsf{C}(\overline{x}z, o, o')\})\}$ and $\mathrm{nuc}(C[n/z]) = C[n/z]$, i.e., $C[n/z]$ has an empty initial segment). However, if the condition were dropped, the latter would no longer be a cell (nor a colony), since Definition 5.1 would then require $C[n/z]$ to have $\bigcup_{i \in \mathbb{N}} \{\mathsf{C}(\overline{x}n_i, p_i, p'_i)\}$ as an initial segment (with distinct new names).

We need one more result for the restriction case in the proof of Lemma 6.3 in the next section: a substitution $[n/x]$ applied to a nonempty colony S of cells C_p that each contain x yields a cell, provided that S has the first BC property (recall from Lemma 4.3 that every solution that is the semantics of a process term has the two BC properties). Without the first BC property this would fail, since it would allow us to create unwanted berries of the form $\mathsf{B}(n,T)$, i.e., the new n in the berry would be the result of an application of (SF2), rather than of (SF4). Consider, e.g., the solution $S = \{\overline{x}x.\{\overline{x}x.S_1\}\}$ where S_1 is an arbitrary nonempty solution. Note that S is a cell (with empty initial segment) for which the first BC property fails, since $\overline{x}x.\{\overline{x}x.S_1\}$ is a berry nor a cherry. Then $S[n/x] = \{\overline{n}n.\{\overline{n}n.S_1[n/x]\}\}$ cannot be a cell (nor a colony), since it requires the connected components of $S_1[n/x]$ to be of multiplicity ω.

Lemma 5.10 *Let* $S = \bigcup_{p=1}^{s} C_p$, $s \geq 1$, *be a colony of cells* C_p *such that* $x \in \mathrm{fn}(C_p) \cap \mathbf{N}$. *Let* $n \in \mathrm{New}$ *with* $n \notin \mathrm{new}(S)$. *If* S *has the first BC property, then* $S[n/x]$ *is a cell with* $n \in \mathrm{new}(\mathrm{nuc}(S[n/x]))$.

Proof We first show that $V < W$ implies $\mathrm{fn}(V) \cap \mathbf{N} \subseteq \mathrm{fn}(W) \cap \mathbf{N}$. By the definition of $<$ it suffices to show that $\mathrm{fn}(V) \cap \mathbf{N} \subseteq \mathrm{fn}(W) \cap \mathbf{N}$ if $V \prec W$; however, this is trivial, since $\mathrm{fn}(V') \cap \mathbf{N} = \mathrm{fn}(V) \cap \mathbf{N}$ for every copy V' of V.

Now, let $C_p = W_p \cup X_p$, where $W_p = \mathrm{nuc}(C_p)$ and X_p is an initial segment of W_p. By the statement above, $x \in \mathrm{fn}(W_p)$ for every $p \in \{1, \ldots, s\}$. Let $X_p = \bigcup_{k \in K_p} V_k$, where the V_k, $k \in K_p$, are the connected components of X_p and the K_p are mutually disjoint. Let, furthermore, $K_p^x = \{k \in K_p \mid x \in \mathrm{fn}(V_k)\}$, and let $W = \bigcup_{p=1}^{s} (W_p \cup \bigcup_{k \in K_p^x} V_k)$ and $X = \bigcup_{p=1}^{s} \bigcup_{k \in K_p - K_p^x} V_k$. Clearly, $S[n/x] = W[n/x] \cup X$ and $\mathrm{new}(W[n/x]) \cap \mathrm{new}(X) = \varnothing$. Also note that $W[n/x]$ is connected by Lemma 16 of [3].

In the remainder of the proof we show that X is an initial segment of $W[n/x]$. Observe that it follows that $n \in \mathrm{new}(\mathrm{nuc}(S[n/x]))$, since then $\mathrm{nuc}(S[n/x]) = W[n/x]$. Let V be a nonempty connected solution. Obviously, $\mathrm{mult}(V, X) = 0$ if $x \in \mathrm{fn}(V)$. If $x \notin \mathrm{fn}(V)$, then $\mathrm{mult}(V, \bigcup_{k \in K_p - K_p^x} V_k) = \mathrm{mult}(V, X_p)$. Hence by (Pc), $\mathrm{mult}(V, X) \in \{0, \omega\}$. Also, by the definition of X, $\mathrm{mult}(V, X) \neq 0$ if and only if $x \notin \mathrm{fn}(V)$ and, for some $p' \in \{1, \ldots, s\}$, $V < W_{p'}$, since $X_{p'}$ is an initial segment of $W_{p'}$. Consequently, to complete the proof, it suffices to show that $V < W[n/x]$ if and only if $x \notin \mathrm{fn}(V)$ and $V < W_{p'}$ for some $p' \in \{1, \ldots, s\}$.

To show the only-if part, note that it suffices to show that $V \prec W[n/x]$ implies $x \notin \mathrm{fn}(V)$ and $V < W_{p'}$. Assume $V \prec W[n/x]$. Then $x \notin \mathrm{fn}(V)$ is immediate by the first statement of the proof. By Definition 5.1, $W[n/x] = \{\overline{m}m.\{\overline{m}m.Z\}\} \cup T$, where $V' \subseteq^{\mathrm{n}} Z$ for some copy V' of V with $\mathrm{new}(V') \cap \mathrm{new}(T) = \varnothing$. Observe that $m \neq n$ by the first BC property of S. We show that $n \notin \mathrm{new}(V')$. Suppose to the contrary that $n \in \mathrm{new}(V')$; then $n \notin \mathrm{new}(T)$. Hence $W = \{\overline{m}m.\{\overline{m}m.Z[x/n]\}\} \cup T$, because $n \notin \mathrm{new}(S)$. Since $n \notin \mathrm{new}(T)$, $W = W_{p'}$ for some $p' \in \{1, \ldots, s\}$ and so W is a connected component of S. Clearly $V'[x/n] \subseteq^{\mathrm{n}} Z[x/n]$ and $\mathrm{new}(V'[x/n]) \cap \mathrm{new}(T) = \varnothing$. Now let Y be an arbitrary connected component of $V'[x/n]$. Then $Y \prec W$ with $x \in \mathrm{fn}(Y)$. By the colony property of S and by (Pc), $\mathrm{mult}(Y, S) = \omega = \mathrm{mult}(Y, X)$. But this is impossible, since by definition $x \notin \mathrm{fn}(X)$. Hence $n \notin \mathrm{new}(V')$.

To conclude the only-if part, by Lemmas 6 and 5 of [3], there exists $p' \in \{1, \ldots, s\}$ and a solution $T_{p'}$, such that

$$(W_{p'} \cup \bigcup_{k \in K_{p'}^x} V_k)[n/x] = \{\overline{m}m.\{\overline{m}m.Z\}\} \cup T_{p'}$$

and $T_{p'} \subseteq T$. Again by Lemmas 6 and 5 of [3], there exists a solution $T'_{p'} \subseteq T_{p'}$ such that either $W_{p'}[n/x] = \{\overline{m}m.\{\overline{m}m.Z\}\} \cup T'_{p'}$, or there exists $k' \in K_{p'}^x$ such that $V_{k'}[n/x] = \{\overline{m}m.\{\overline{m}m.Z\}\} \cup T'_{p'}$. Note that $\mathrm{new}(V') \cap \mathrm{new}(T'_{p'}) = \varnothing$. Assume the first case. Then $W_{p'} = \{\overline{m}m.\{\overline{m}m.Z[x/n]\}\} \cup T'_{p'}[x/n]$, since $n \notin \mathrm{new}(S)$. Clearly $\mathrm{new}(V') \cap \mathrm{new}(T'_{p'}[x/n]) = \varnothing$. Moreover, since $n \notin \mathrm{new}(V')$, $V' = V'[x/n] \subseteq^{\mathrm{n}} Z[x/n]$ and so $V \prec W_{p'}$. In the second case it can be shown similarly that $V \prec V_{k'}$ and thus $V < W_{p'}$, since $V_{k'}$ is a connected component of $X_{p'}$.

To show the if-part, it suffices to show that $V \prec W[n/x]$ whenever $V \prec W_{p'}$ for some $p' \in \{1, \dots, s\}$ with $x \notin \mathrm{fn}(V)$. Then $W_{p'} = \{\overline{m}m.\{\overline{m}m.Z\}\} \cup T$ where $V' \subseteq^{\mathrm{n}} Z$ for some copy V' of V with $\mathrm{new}(V') \cap \mathrm{new}(T) = \varnothing$. Hence $W[n/x] = \{\overline{m}m.\{\overline{m}m.Z[n/x]\}\} \cup T'$, where $T' = (T \cup \bigcup_{p \in \{1,\dots,s\}-\{p'\}}(W_p \cup \bigcup_{k \in K_p^x} V_k) \cup \bigcup_{k \in K_{p'}^x} V_k)[n/x]$. Clearly $V' = V'[n/x] \subseteq^{\mathrm{n}} Z[n/x]$, since $x \notin \mathrm{fn}(V')$ and $n \notin \mathrm{new}(V')$. Also $\mathrm{new}(V') \cap \mathrm{new}(T') = \varnothing$, since $\mathrm{new}(V') \cap \mathrm{new}(T) = \varnothing$, the $\mathrm{new}(W_p)$ and $\mathrm{new}(V_k)$ are all mutually disjoint, and $n \notin \mathrm{new}(V')$. Hence $V \prec W[n/x]$. □

We have not yet explained why, in Definition 5.1, we have defined $<$ to be the transitive closure of $\prec$ rather than $\prec$ itself. In fact, this is only for technical convenience. Let us say that a cell C is *pure* if for every nonempty connected solution V, $V < \mathrm{nuc}(C)$ iff $V \prec \mathrm{nuc}(C)$. In the next section we will show (in Lemma 6.3) that every solution that is the semantics (under $\Rightarrow_{\mathbf{F}}$) of a process term is a colony, and, in fact, a colony of pure cells. Here we note that it follows from the proof of Lemma 5.10 that if (in the statement of the lemma) S is a colony of pure cells C_p, then $S[n/x]$ is a pure cell. In fact, we have shown that if $V < W[n/x]$ then $x \notin \mathrm{fn}(V)$ and $V < W_{p'}$, and so $V \prec W_{p'}$ because $C_{p'}$ is pure. And we have shown that if $V \prec W_{p'}$ and $x \notin \mathrm{fn}(V)$ then $V \prec W[n/x]$.

6 Cell Normal Form of Processes

Although by the absence of structural laws (3.2)–(3.5) not much freedom remains to normalize processes that are replications (as compared to the subconnected normal form of [3] where the replication is 'moved inwards' as much as possible), we still have structural law (2.3) in $\equiv^{\mathbf{ptl}}$ which allows us to 'strive for' process terms of which restrictions cannot be moved inwards any further. As a consequence of this absence, we miss the interplay between restriction and replication that, in the normal form of [3], guarantees that a restriction is connected (i.e., yields a connected solution by $\Rightarrow$) and that only connected process terms are replicated. The normal form we propose maps a restriction, a replication, and a guarded process term to a cell (by $\Rightarrow_{\mathbf{F}}$), and, in general, maps a process term to a colony.

We will write $P_1 \mid P_2 \mid \dots \mid P_k$ for any process term that is obtained from the process term $(\dots((P_1 \mid P_2) \mid P_3) \mid \dots \mid P_{k-1}) \mid P_k$ by structural law (1.3), i.e., by associativity of parallel composition. Moreover, for $k = 0$ we assume that this process term equals the inactive process $\mathbf{0}$. For $k \geq 0$, we let

$$P^k = \overbrace{P \mid \dots \mid P}^{k},$$

the parallel composition of k copies of P.

Definition 6.1 A *process cell* is a process term defined inductively as follows:

if $P_1, \ldots, P_k, P_{k+1}$, $k \geq 0$, are process cells, then

(i) $g.(P_1 \mid \ldots \mid P_k)$ is a process cell,
(ii) $!\,(P_1 \mid \ldots \mid P_k)$ is a process cell, and
(iii) $(\nu x)(P_1 \mid \ldots \mid P_k \mid P_{k+1})$ is a process cell, provided $x \in \mathrm{fn}(P_i)$ for all $i \in \{1, \ldots, k+1\}$.

A process term P is in *cell normal form (cnf)*, if $P = P_1 \mid \ldots \mid P_k$, $k \geq 0$, where P_i is a process cell for $i \in \{1, \ldots, k\}$.

Note that in Definition 6.1 the basis of the induction is formed by $k = 0$, since then $P_1 \mid \ldots \mid P_k = \mathbf{0}$ as assumed. In the remainder, if there is no danger of confusion, we abbreviate 'process cell' also to 'cell'.

The next two lemmas show that cnf is indeed a normal form of processes (even for $\equiv^{\mathbf{std}}$) and that the term 'process cell' in Definition 6.1 is not chosen arbitrarily.

Lemma 6.2 *For every process term P, there exists a process term P' in cnf such that $P' \equiv^{\mathbf{std}} P$.*

Proof The proof is by induction on the structure of P. It is trivial for $P = \mathbf{0}$, by taking $k = 0$ in Definition 6.1. The cases $P = g.P_1$ and $P = \,!\,P_1$ are easy and are treated in one stroke; by induction $P_1 \equiv^{\mathbf{std}} P_1'$ where P_1' is in cnf. By Definition 6.1(i) and (ii), respectively, $g.P_1'$ and $!\,P_1'$ are cells (and hence are in cnf), and obviously $g.P_1 \equiv^{\mathbf{std}} g.P_1'$ and $!\,P_1 \equiv^{\mathbf{std}} \,!\,P_1'$. The case $P = P_1 \mid P_2$ is obvious and left to the reader. Finally, let $P = (\nu x)P_1$. By induction $P_1 \equiv^{\mathbf{std}} P_1'$ where P_1' is in cnf, so let $P_1' = P_{1,1} \mid \ldots \mid P_{1,k_1}$, $k_1 \geq 0$, where each $P_{1,j}$ is a cell. Clearly $(\nu x)P_1' \equiv^{\mathbf{std}} P$ by congruence. Let $P_{1,1}', \ldots, P_{1,k_1}'$ be a permutation of $P_{1,1}, \ldots, P_{1,k_1}$ such that for l_1 with $0 \leq l_1 \leq k_1$, we have $x \in \mathrm{fn}(P_{1,j}')$ for $1 \leq j \leq l_1$, and $x \notin \mathrm{fn}(P_{1,j}')$ for $l_1 < j \leq k_1$. Obviously, by application of structural law (1.2) only, $P_1' \equiv^{\mathbf{std}} P_{1,1}' \mid \ldots \mid P_{1,k_1}'$. We consider three cases. If $l_1 = 0$, then $x \notin \mathrm{fn}(P_1')$, and hence choosing $P' = P_1'$, $P' \equiv^{\mathbf{std}} (\nu x)P_1' \equiv^{\mathbf{std}} P$ by structural law (2.2). If $l_1 = k_1 > 0$, then $x \in \mathrm{fn}(P_{1,j})$ for every $j \in \{1, \ldots, k_1\}$ and thus, letting $P' = (\nu x)P_1'$, P' is a cell (and hence is in cnf), and $P' \equiv^{\mathbf{std}} P$ is immediate. Thirdly, if $0 < l_1 < k_1$, then let $P' = (\nu x)(P_{1,1}' | \ldots | P_{1,l_1}') | P_{1,l_1+1}' | \ldots | P_{1,k_1}'$. Since the $P_{1,j}'$ are cells for $1 \leq j \leq k_1$, $(\nu x)(P_{1,1}' \mid \ldots \mid P_{1,l_1}')$ is a cell because $l_1 > 0$, and thus P' is in cnf. Moreover, $P' \equiv^{\mathbf{std}} (\nu x)(P_{1,1}' | \ldots | P_{1,k_1}') \equiv^{\mathbf{std}} (\nu x)P_1' \equiv^{\mathbf{std}} P$ by structural law (2.3). $\square$

Lemma 6.3 *For every cell P, if $P \Rightarrow_{\mathbf{F}} S$, then S is a cell. In general, for every process term P, if $P \Rightarrow_{\mathbf{F}} S$, then S is a colony.*

Proof By Lemma 6.2 and Corollary 3.2, it suffices to show the second part of the lemma for process terms P in cnf only. This is obvious from (SF0), (SF1), and the first part of the lemma. The proof of the first part is by induction on the definition of process cell, cf. Definition 6.1.

Let $P = g.P'$. By (SF3), $P \Rightarrow_{\mathbf{F}} \{g.T\}$ for certain solutions T. Obviously, $\{g.T\}$ is a cell, since it consists of a single connected component with an empty initial segment (because $g \neq \overline{m}m$ for $m \in \mathrm{New}$). Note that in this case the induction hypothesis is not needed.

Let $P = \,!\,(P_1 \,|\, \ldots \,|\, P_k)$. Assume by induction that $P_j \Rightarrow_{\mathbf{F}} C_j, C_{i,j}$, $1 \leq j \leq k$ and $i \in \mathbb{N}$, where the C_j and $C_{i,j}$ are cells with mutually disjoint $\mathrm{new}(C_j)$ and with mutually disjoint $\mathrm{new}(C_{i,j})$. Let $S' = \bigcup_{j=1}^{k} C_j$ and assume that $\mathrm{new}(C_{i,j})$ is disjoint with $\mathrm{new}(S')$. Let $W = \{\overline{m}m.\{\overline{m}m.S'\}\}$ and let $X = \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{k} C_{i,j}$, where $m \in \mathrm{New}$ with $m \notin \mathrm{new}(S')$ and $m \notin \mathrm{new}(C_{i,j})$. Then $P \Rightarrow_{\mathbf{F}} W \cup X$, by (SF1) and (SF4), respectively. We show that X is an initial segment of W. Since for fixed j, $C_{i,j}$ is a copy of $C_{i',j}$ for all $i, i' \in \mathbb{N}$ (Lemma 2.4), we have $\mathrm{mult}(V, X) \in \{0, \omega\}$ for all connected V. Assume $V < W$. Then $V \prec W$ or $V < V_1$ with $V_1 \prec W$. Assume the first case. Then $\mathrm{mult}(V, S') \geq 1$ and hence $\mathrm{mult}(V, \bigcup_{j=1}^{k} C_{i,j}) \geq 1$ for every $i \in \mathbb{N}$, since $\bigcup_{j=1}^{k} C_{i,j}$ is a copy of S' by (Pb). Thus $\mathrm{mult}(V, X) = \omega$ by (Pc). Next, assume the second case. By the same argument, $\mathrm{mult}(V_1, \bigcup_{j=1}^{k} C_{i,j}) \geq 1$ for every $i \in \mathbb{N}$. Thus $\mathrm{mult}(V_1, C_{i,j}) \geq 1$ for some i, j with $i \in \mathbb{N}$ and $1 \leq j \leq k$. Since $C_{i,j}$ is a cell, we have $\mathrm{mult}(V, C_{i,j}) = \omega$ and hence $\mathrm{mult}(V, X) = \omega$, by (Pc). To show the converse, note that $\mathrm{mult}(V, X) = \omega$ implies that $\mathrm{mult}(V, C_{i,j}) \geq 1$ for some i, j with $i \in \mathbb{N}$ and $1 \leq j \leq k$, and hence $\mathrm{mult}(V, S') \geq 1$, since S' is a copy of $\bigcup_{j=1}^{k} C_{i,j}$. Thus $V \prec W$ and hence $V < W$.

Finally, let $P = (\nu x)P'$ where $P' = P_1 \mid \ldots \mid P_k \mid P_{k+1}$ with $x \in \mathrm{fn}(P_i)$ for all $1 \leq i \leq k+1$. Assume by induction that $P_i \Rightarrow_{\mathbf{F}} C_i$, where C_i is a cell, and assume that the $\mathrm{new}(C_i)$ are mutually disjoint. Then $P' \Rightarrow_{\mathbf{F}} S' = \bigcup_{i=1}^{k+1} C_i$ by (SF1), and $x \in \mathrm{fn}(C_i)$ for all i. Recall from Lemma 4.3 that S' has the first BC property. By (SF2) we have $P \Rightarrow_{\mathbf{F}} S'[n/x]$ where $n \notin \mathrm{new}(S')$. By Lemma 5.10, $S'[n/x]$ is a cell. □

It follows from this proof that for every process term P, if $P \Rightarrow_{\mathbf{F}} S$ then S is in fact a colony of *pure* cells, cf. the discussion following Lemma 5.10 (note in particular that in the replication case we have shown that $\mathrm{mult}(V, X) = \omega$ implies $V \prec W$).

For a cell $P = \,!\,(P_1 \mid \ldots \mid P_k)$, by structural law (3.6), P can produce any process term of the form $P \mid P_i$, $1 \leq i \leq k$; in fact by multiple applications of (3.6), $P \equiv^{\mathbf{ptl}} P \,|\, (P_{f(1)} \,|\, \ldots \,|\, P_{f(p)})$, for every mapping $f : \{1, \ldots, p\} \to \{1, \ldots, k\}$. If any of the P_i is itself a replication, still other combinations are possible. We want to give a full characterization of cells Q that P can produce in parallel with itself, for arbitrary cells P; that is, we do not restrict ourselves to cells that are replications: also cells of the form $(\nu x)(P_1 \mid \ldots \mid P_k)$ can 'behave' in this manner. To see this, let, for instance, $P' = \,!\,(R_1 \mid R_2)$, where $R_1 = \overline{x}z.\mathbf{0}$ and $R_2 = x(y).\mathbf{0}$, and let $P = (\nu z)P'$. Now by structural laws (3.6) and (2.3) respectively, $P \equiv^{\mathbf{ptl}} (\nu z)(P' \mid R_2) \equiv^{\mathbf{ptl}} P \mid R_2$, but P cannot produce R_1.

Definition 6.4 For a cell P, the *offspring of* P is the finite set $\mathcal{O}(P)$ of subterms of P defined inductively by

(i) $\mathcal{O}(g.(P_1 \mid \ldots \mid P_k)) = \varnothing$,

(ii) $\mathcal{O}(\,!\,(P_1 \mid \ldots \mid P_k)) = \bigcup_{i=1}^{k}(\mathcal{O}(P_i) \cup \{P_i\})$, and

(iii) $\mathcal{O}((\nu x)(P_1 \mid \ldots \mid P_k \mid P_{k+1})) = \{Q \in \mathcal{O}(P_i) \mid 1 \leq i \leq k+1 \text{ and } x \notin \mathrm{fn}(Q)\}$,

where P_i is a cell for $i \in \{1, \ldots, k, k+1\}$ with $k \geq 0$.

Observe that every process term in $\mathcal{O}(P)$ is a cell. We now prove the first half of the characterization mentioned earlier; the second half will be shown in Theorem 6.6 below.

Lemma 6.5 *For every cell P and every $Q \in \mathcal{O}(P)$, $P \mid Q \equiv^{\mathbf{ptl}} P$.*

Proof The proof is by induction on the structure of P, according to Definition 6.1. It is obvious for the case $P = g.P'$ with P' in cnf, since then $\mathcal{O}(P) = \varnothing$. Next, let $P = \,!\,(P_1 \mid \ldots \mid P_k)$, $k \geq 0$, where P_i is a cell for all $i \in \{1, \ldots, k\}$. Let $Q \in \mathcal{O}(P)$. Then, for some $j \in \{1, \ldots, k\}$, $Q = P_j$ or $Q \in \mathcal{O}(P_j)$. Assume the first case. Then, using structural laws (1.2) and (3.6),

$$\begin{aligned} P \mid Q \quad &\equiv^{\mathbf{ptl}} \quad !\,(P_j \mid (P_1 \mid \ldots \mid P_{j-1} \mid P_{j+1} \mid \ldots \mid P_k)) \mid P_j \\ &\equiv^{\mathbf{ptl}} \quad !\,(P_j \mid (P_1 \mid \ldots \mid P_{j-1} \mid P_{j+1} \mid \ldots \mid P_k)) \\ &\equiv^{\mathbf{ptl}} \quad P. \end{aligned}$$

Assume the second case. By induction, $P_j \mid Q \equiv^{\mathbf{ptl}} P_j$. By the previous case, $P \equiv^{\mathbf{ptl}} P \mid P_j$. Hence $P \mid Q \equiv^{\mathbf{ptl}} P \mid P_j \mid Q \equiv^{\mathbf{ptl}} P \mid P_j \equiv^{\mathbf{ptl}} P$. Finally, let $P = (\nu x)(P_1 \mid \ldots \mid P_k \mid P_{k+1})$, $k \geq 0$, where P_i is a cell and $x \in \mathrm{fn}(P_i)$ for all $i \in \{1, \ldots, k, k+1\}$. Then, for some $j \in \{1, \ldots, k, k+1\}$, $Q \in \mathcal{O}(P_j)$ and $x \notin \mathrm{fn}(Q)$. By induction we have $P_j \mid Q \equiv^{\mathbf{ptl}} P_j$. Hence, using structural laws (1.2) and (2.3),

$$\begin{aligned} P \mid Q \quad &= \quad (\nu x)(P_1 \mid \ldots \mid P_k \mid P_{k+1}) \mid Q \\ &\equiv^{\mathbf{ptl}} \quad (\nu x)(P_1 \mid \ldots \mid P_j \mid Q \mid P_{j+1} \mid \ldots \mid P_k \mid P_{k+1}) \\ &\equiv^{\mathbf{ptl}} \quad (\nu x)(P_1 \mid \ldots \mid P_j \mid P_{j+1} \mid \ldots \mid P_k \mid P_{k+1}) = P. \end{aligned}$$

□

By multiple applications of Lemma 6.5, clearly we have $P \mid Q_1 \mid \ldots \mid Q_p \equiv^{\mathbf{ptl}} P$, $p \geq 1$, where $Q_i \in \mathcal{O}(P)$; by structural law (1.1) this even holds for $p \geq 0$.

We now turn to the main result of this section, showing the connection between the offspring $\mathcal{O}(P)$ of a cell P and the 'offspring' of a cell C discussed informally in the beginning of Section 5 and just before Lemma 5.7 (in which C is divided into two colonies S_1 and S_2). Note that this gives an answer to the question put in the beginning of Section 5.

Theorem 6.6 *For every cell P, if $P \Rightarrow_{\mathbf{F}} S_1 \cup S_2$, where S_1, S_2 are colonies with* $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$*, then there exist $Q_1, \ldots, Q_s \in \mathcal{O}(P)$ with $s \geq 0$, such that either $P \Rightarrow_{\mathbf{F}} S_1$ and $Q_1 \mid \ldots \mid Q_s \Rightarrow_{\mathbf{F}} S_2$, or conversely, $P \Rightarrow_{\mathbf{F}} S_2$ and $Q_1 \mid \ldots \mid Q_s \Rightarrow_{\mathbf{F}} S_1$.*

Proof The proof is by induction on the structure of P, according to Definition 6.1.

Let $P = g.P'$ where P' is in cnf. Then $P \Rightarrow_{\mathbf{F}} \{g.T\}$. Assume $S_1 \cup S_2 = \{g.T\}$. Then $S_1 = \varnothing$ and $S_2 = \{g.T\}$, or $S_2 = \varnothing$ and $S_1 = \{g.T\}$. In both cases the statement holds if $s = 0$ is chosen.

Let $P = \,!(P_1 \mid \ldots \mid P_k)$, $k \geq 0$, where P_j is a cell. Let $P_j \Rightarrow_{\mathbf{F}} T_{i,j}, T'_j$, $1 \leq j \leq k$ and $i \in \mathbb{N}$, such that the $\mathrm{new}(T_{i,j})$ are mutually disjoint, the $\mathrm{new}(T'_j)$ are mutually disjoint, and $\mathrm{new}(\bigcup_{j=1}^{k} \bigcup_{i \in \mathbb{N}} T_{i,j}) \cap \mathrm{new}(\bigcup_{j=1}^{k} T'_j) = \varnothing$. By Lemma 6.3, $T_{i,j}$ and T'_j are cells. Let $T' = \bigcup_{j=1}^{k} T'_j$, let $X = \bigcup_{j=1}^{k} \bigcup_{i \in \mathbb{N}} T_{i,j}$, and let $m \in \mathrm{New}$ with $m \notin \mathrm{new}(T')$ and $m \notin \mathrm{new}(X)$. Let $W = \{\overline{m}m.\{\overline{m}m.T'\}\}$ and assume $P \Rightarrow_{\mathbf{F}} W \cup X = S_1 \cup S_2$. Observe that by the proof of Lemma 6.3, X is an initial segment of W. By Lemma 5.7, there exists a copy X' of X such that either $S_1 = W \cup X'$ and $X = X' \cup S_2$, or the converse, $S_2 = W \cup X'$ and $X = X' \cup S_1$. Assume the first case; the proof of the second is the same. Clearly, by (Pb), S_1 is a copy of $S_1 \cup S_2$ and so $P \Rightarrow_{\mathbf{F}} S_1$ by Lemma 2.4. By Lemma 5.9 (applied to $X = X' \cup S_2 = \bigcup_{j=1}^{k} \bigcup_{i \in \mathbb{N}} T_{i,j}$), there exist colonies $U_{i,j,h}$ with mutually disjoint $\mathrm{new}(U_{i,j,h})$ such that $U_{i,j,1} \cup U_{i,j,2}$ is a copy of $T_{i,j}$ and $\bigcup_{j=1}^{k} \bigcup_{i \in \mathbb{N}} U_{i,j,2}$ is a copy of S_2. Thus for all $j \in \{1, \ldots, k\}$ and $i \in \mathbb{N}$, we have by Lemma 2.4 that $P_j \Rightarrow_{\mathbf{F}} U_{i,j,1} \cup U_{i,j,2}$. By induction there exist $Q_{i,j,1}, \ldots, Q_{i,j,s_{i,j}} \in \mathcal{O}(P_j)$ such that either $P_j \Rightarrow_{\mathbf{F}} U_{i,j,1}$ and $Q_{i,j,1} \mid \ldots \mid Q_{i,j,s_{i,j}} \Rightarrow_{\mathbf{F}} U_{i,j,2}$, or $P_j \Rightarrow_{\mathbf{F}} U_{i,j,2}$ and $Q_{i,j,1} \mid \ldots \mid Q_{i,j,s_{i,j}} \Rightarrow_{\mathbf{F}} U_{i,j,1}$. Observe that $Q_{i,j,p} \in \mathcal{O}(P)$ and $P_j \in \mathcal{O}(P)$ for all i, j, p. This implies that there exist $Q_a \in \mathcal{O}(P)$, $a \in A$, such that S_2 is a copy of $\bigcup_{a \in A} D_a$ where $Q_a \Rightarrow_{\mathbf{F}} D_a$ and the $\mathrm{new}(D_a)$ are mutually disjoint. Note that, by Lemma 6.3, D_a is a cell. For $Q \in \mathcal{O}(P)$, let $r(Q) = \#\{a \mid Q_a = Q\}$, let $A_{\mathrm{fin}} = \{a \mid 0 < r(Q_a) < \omega\}$, and $A_{\mathrm{infin}} = \{a \mid r(Q_a) = \omega\}$. Let A_l, $l \in L$, be the equivalence classes of A_{infin} under the equivalence relation $\{(a, b) \mid Q_a = Q_b\}$. Since $\mathcal{O}(P)$ is finite, L is finite, and clearly for all $l \in L$ and $a, b \in A_l$, we have by Lemma 2.4 that D_a is a copy of D_b. Hence by $\#L$ applications of Lemma 5.6, $\bigcup_{a \in A} D_a$ is a copy of $\bigcup_{a \in A_{\mathrm{fin}}} D_a$. Now let $Q_{\mathrm{fin}} = \{Q_a \in \mathcal{O}(P) \mid a \in A_{\mathrm{fin}}\}$. Since $\mathcal{O}(P)$ is finite, Q_{fin} is finite, say $Q_{\mathrm{fin}} = \{R_1, \ldots, R_t\}$, and hence A_{fin} is finite. Clearly, $R_1^{r(R_1)} | \ldots | R_t^{r(R_t)} \Rightarrow_{\mathbf{F}} \bigcup_{a \in A_{\mathrm{fin}}} D_a$, and hence $R_1^{r(R_1)} | \ldots | R_t^{r(R_t)} \Rightarrow_{\mathbf{F}} S_2$.

Finally, let $P = (\nu x)(P_1 \mid \ldots \mid P_k \mid P_{k+1})$ where P_i is a cell with $x \in \mathrm{fn}(P_i)$ for all $i \in \{1, \ldots, k+1\}$. Let $P_i \Rightarrow_{\mathbf{F}} T_i$, where the $\mathrm{new}(T_i)$ are mutually disjoint, and let $T = T_1 \cup \ldots \cup T_k \cup T_{k+1}$. Assume $P \Rightarrow_{\mathbf{F}} T[n/x] = S_1 \cup S_2$, where $n \in \mathrm{New} - \mathrm{new}(T)$. By Lemma 5.10, $T[n/x]$ is a cell (cf. the proof of Lemma 6.3); let $T[n/x] = W \cup X$, where $W = \mathrm{nuc}(T[n/x])$ and X is an initial segment of W. Note that $n \in \mathrm{new}(W)$ by Lemma 5.10. By Lemma 5.7, there exists a copy X' of X such that either $S_1 = W \cup X'$ and $X = X' \cup S_2$, or the converse, $S_2 = W \cup X'$

and $X = X' \cup S_2$. Assume the first case. Then S_1 is a copy of $T[n/x]$, so $P \Rightarrow_{\mathbf{F}} S_1$. Moreover, since $T[n/x] = W \cup X' \cup S_2$ where $n \in \mathrm{new}(W)$ and the $\mathrm{new}(W)$, $\mathrm{new}(X')$, and $\mathrm{new}(S_2)$ are mutually disjoint, $T = W[x/n] \cup X' \cup S_2$. Now let $S = W[x/n] \cup X'$, so $T = S \cup S_2$ with $\mathrm{new}(S) \cap \mathrm{new}(S_2) = \varnothing$. By Lemma 5.9, there exist colonies $U_{i,j}$, $1 \leq i \leq k+1$ and $j \in \{1, 2\}$, with mutually disjoint $\mathrm{new}(U_{i,j})$ such that $U_{i,1} \cup U_{i,2}$ is a copy of T_i and $\bigcup_{i=1}^{k+1} U_{i,2}$ is a copy of S_2. Hence $P_i \Rightarrow_{\mathbf{F}} U_{i,1} \cup U_{i,2}$. By induction, there exist $s_i \geq 0$ and $Q_{i,p} \in \mathcal{O}(P_i)$, $p \in \{1, \ldots, s_i\}$, such that $P_i \Rightarrow_{\mathbf{F}} U_{i,1}$ and $Q_{i,1} \mid \ldots \mid Q_{i,s_i} \Rightarrow_{\mathbf{F}} U_{i,2}$. Note that the converse, $P_i \Rightarrow_{\mathbf{F}} U_{i,2}$ and $Q_{i,1} \mid \ldots \mid Q_{i,s_i} \Rightarrow_{\mathbf{F}} U_{i,1}$, is impossible since $x \notin \mathrm{fn}(S_2)$ and hence $x \notin \mathrm{fn}(U_{i,2})$. Thus $x \notin \mathrm{fn}(Q_{i,p})$ and consequently $Q_{i,p} \in \mathcal{O}(P)$. Moreover,

$$Q_{1,1} \mid \ldots \mid Q_{1,s_1} \mid \ldots \mid Q_{k+1,1} \mid \ldots \mid Q_{k+1,s_{k+1}} \Rightarrow_{\mathbf{F}} S_2,$$

since $\bigcup_{i=1}^{k+1} U_{i,2}$ is a copy of S_2. □

7 The Completeness of Potential Structural Congruence

As explained in Section 2, the proof of the decidability of potential structural congruence is, more or less, immediate from its completeness, i.e., the converse of Corollary 3.2. We use a method of proof similar to the one in [3], where it was used to show that $\equiv$ and $\equiv_m$ are identical, and the one in [4], where it was used in the proof of completeness of structural inclusion. Apart from technical details, $P \equiv_m^{\mathbf{F}} Q$ implies $P \equiv^{\mathbf{ptl}} Q$ is shown by induction on the structure of Q; the first lemma forms the induction basis (which is the case $Q = \mathbf{0}$), and the next four the induction steps depending on the form of Q, i.e., whether Q is a guarded process term, a parallel composition, a restriction, or a replication, respectively.

Lemma 7.1 *For all process terms P, $P \equiv_m^{\mathbf{F}} \mathbf{0}$ if and only if $P \equiv^{\mathbf{ptl}} \mathbf{0}$.*

Proof The if-direction is by Corollary 3.2. The only if-direction is shown by induction on the structure of P. Assume $P \Rightarrow_{\mathbf{F}} \varnothing$ in each case. The proof is trivial for $P = \mathbf{0}$. For the cases $P = \,!P'$ and $P = g.P'$ the statement vacuously holds, since then $P \Rightarrow_{\mathbf{F}} \varnothing$ is impossible. Next, let $P = P_1 \mid P_2$. Then $P_1 \Rightarrow_{\mathbf{F}} \varnothing$ and $P_2 \Rightarrow_{\mathbf{F}} \varnothing$, and so $P_1 \equiv_m^{\mathbf{F}} \mathbf{0}$ and $P_2 \equiv_m^{\mathbf{F}} \mathbf{0}$. Hence, by induction, $P_1 \equiv^{\mathbf{ptl}} \mathbf{0}$ and $P_2 \equiv^{\mathbf{ptl}} \mathbf{0}$, and thus $P_1 \mid P_2 \equiv^{\mathbf{ptl}} \mathbf{0} \mid \mathbf{0} \equiv^{\mathbf{ptl}} \mathbf{0}$, by structural law (1.1). Finally, let $P = (\nu x)P'$. Then $P' \Rightarrow_{\mathbf{F}} \varnothing$, and so $P' \equiv_m^{\mathbf{F}} \mathbf{0}$. By induction $P' \equiv^{\mathbf{ptl}} \mathbf{0}$, and hence $(\nu x)P' \equiv^{\mathbf{ptl}} (\nu x)\mathbf{0} \equiv^{\mathbf{ptl}} \mathbf{0}$, by structural law (2.2). □

In the next lemma, we treat the case that Q is a guarded process term $g.Q'$. In its proof we exploit the first BC property in the case that P is a restriction $(\nu x)P'$ in the following way: $(\nu x)P' \equiv_m^{\mathbf{F}} g.Q'$ implies that $(\nu x)P' \Rightarrow_{\mathbf{F}} \{g.T\}$

where $T = T_0 \cup \{\overline{m}m.\{\overline{m'}m'.T_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$ (with the restrictions of (SF3)) and $Q' \Rightarrow_{\mathbf{F}} T_0, T_1$. Then, cf. (SF2), $S'[n/x] = \{g.T\}$ where $P' \Rightarrow_{\mathbf{F}} S'$ and $n \in \mathrm{New} - \mathrm{new}(S')$. Since g is a guard over $\mathbf{N}$, the n must occur in T (assuming it occurs at all in $g.T$) which implies exactly one of the following cases: (1) $m = n$, (2) $m' = n$, (3) n occurs in T_0, or (4) n occurs in T_1. The first BC property of $S' = \{g.T[x/n]\}$ forbids each of them: if $m = n$ or $m' = n$, then S' would consist of a molecule that is not a cherry anymore (neither a berry). If n occurred in T_0 or in T_1, then (since $\mathrm{new}(T_0) \cap \mathrm{new}(T_1) = \varnothing$) $\mathrm{fn}(T_0[x/n])$ would contain x, but $\mathrm{fn}(T_1[x/n])$ would not, or vice versa, and thus, again, $g.T[x/n]$ would not be a cherry anymore. Hence the only case in which $(\nu x)P' \equiv^{\mathbf{F}}_m g.Q'$ is when $x \notin \mathrm{fn}(P')$. Note that, consequently, structural law (2.4) is not needed.

Lemma 7.2 *For every process term P, if $P \equiv^{\mathbf{F}}_m g.Q'$ where g is a guard over $\mathbf{N}$, then there exists a process term R such that $P \equiv^{\mathbf{ptl}} g.R$ and $R \equiv^{\mathbf{F}}_m Q'$.*

Proof We first observe that it suffices to prove the statement of the lemma for the case that x does not occur bound in P if x occurs bound in g. In fact, using α-conversion (i.e., structural law (α)) to rename all bound occurrences of x in P, a process term $\overline{P}$ can be constructed such that $\overline{P} \equiv^{\mathbf{ptl}} P$. The statement can then be proved for $\overline{P}$ (cf. Corollary 3.2).

Throughout the proof, assume that $P \Rightarrow_{\mathbf{F}} \{g.T\}$ with

$$T = T_0 \cup \{\overline{m}m.\{\overline{m'}m'.T_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$$

such that $m, m' \in \mathrm{New} - \mathrm{new}(T_0 \cup T_1)$, $m \neq m'$, $\mathrm{new}(T_0) \cap \mathrm{new}(T_1) = \varnothing$, and $Q' \Rightarrow_{\mathbf{F}} T_0, T_1$. We proceed by induction on the structure of P. The cases $P = \mathbf{0}$ and $P = \,!P'$ are trivial, since then $P \Rightarrow_{\mathbf{F}} \{g.T\}$ is impossible.

Let $P = h.P'$ where h is a guard over $\mathbf{N}$. Then $\{g.T\} = \{h.S\}$ with $S = S_0 \cup \{\overline{n}n.\{\overline{n'}n'.S_1\}, \overline{n}n.\{\overline{n'}n'.\varnothing\}\}$ and $P' \Rightarrow_{\mathbf{F}} S_0, S_1$, such that $\mathrm{new}(S_0) \cap \mathrm{new}(S_1) = \varnothing$ and $n, n' \in \mathrm{New} - \mathrm{new}(S_0 \cup S_1)$ with $n \neq n'$. Observe that $h.S$ is cherry, cf. (the proof of) Lemma 4.3. First consider the case that g is an output guard. Then $h = g$ and $S = T$. Since all molecules of S_0 of the form $g_1.\{g_2.S'\}$ are berries and since evidently both $\overline{m}m.\{\overline{m'}m'.T_1\}$ and $\overline{m}m.\{\overline{m'}m'.\varnothing\}$ are not because $m \neq m'$, we have $S_i = T_i$, $i \in \{0,1\}$. Hence, letting $R = P'$, we immediately obtain $P \equiv^{\mathbf{ptl}} g.R$ and $R \equiv^{\mathbf{F}}_m Q'$. Next, consider the case that g is an input guard; assume $g = x(y)$ and $h = x(v)$. Recall that $\{x(v).S\} = \{x(y).T\}$ abbreviates $\{x(-).\mathrm{inc}(S)[1/v]\} = \{x(-).\mathrm{inc}(T)[1/y]\}$. By the same argument as above, it follows that $\mathrm{inc}(S_i)[1/v] = \mathrm{inc}(T_i)[1/y]$, $i \in \{0,1\}$. Note that if $y \neq v$, then $y \notin \mathrm{fn}(S_i)$ and hence $y \notin \mathrm{fn}(P')$. Now $\mathrm{inc}(T_i) = \mathrm{inc}(S_i)[1/v][y/1] = \mathrm{inc}(S_i)[y/v] = \mathrm{inc}(S_i[y/v])$, so $T_i = S_i[y/v]$. Now let $R = P'[y/v]$. By Lemma 2.5(1), $R \Rightarrow_{\mathbf{F}} T_i$ and so $R \equiv^{\mathbf{F}}_m Q'$. Moreover, $P = x(v).P' \equiv^{\mathbf{ptl}} x(y).R$ by structural law (α), since $y \notin \mathrm{fn}(P')$ if $y \neq v$.

Let $P = (\nu x)P'$. Then $\{g.T\} = S'[n/x]$ where $P' \Rightarrow_{\mathbf{F}} S'$ and $n \in \mathrm{New} - \mathrm{new}(S')$. Note that $x \notin \mathrm{fn}(g.T)$ and hence, by the assumption in the beginning of the proof, x does not occur in g. Then $S' = \{g.T[x/n]\}$, since g is a guard over $\mathbf{N}$, i.e., n does not occur in g. Moreover, since $g.T[x/n]$ is evidently not a berry, by the first BC property of S' (Lemma 4.3) it must be a cherry, which implies that $m \neq n$ and $m' \neq n$. Thus $T[x/n] = T_0[x/n] \cup \{\overline{m}m.\{\overline{m'}m'.T_1[x/n]\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$ with $z \in \mathrm{fn}(T_0[x/n])$ iff $z \in \mathrm{fn}(T_1[x/n])$ for all $z \in \mathbf{N}$ that do not occur bound in g. Hence if $x \in \mathrm{fn}(T_0[x/n])$, then $x \in \mathrm{fn}(T_1[x/n])$, and so $n \in \mathrm{new}(T_0) \cap \mathrm{new}(T_1)$ which contradicts $\mathrm{new}(T_0) \cap \mathrm{new}(T_1) = \varnothing$. For the same reason, $x \notin \mathrm{fn}(T_1[x/n])$. Consequently, $x \notin \mathrm{fn}(S')$, i.e., $S'[n/x] = S'$, and so $P' \equiv^{\mathbf{F}}_m g.Q'$. By induction, there exists R with $P' \equiv^{\mathbf{ptl}} g.R$ and $R \equiv^{\mathbf{F}}_m Q'$. Hence $P \equiv^{\mathbf{ptl}} g.R$, since $x \notin \mathrm{fn}(P')$ and thus $(\nu x)P' \equiv^{\mathbf{ptl}} P'$ by structural law (2.2).

Finally, let $P = P_1 \mid P_2$. Then $\{g.T\} = S_1 \cup S_2$ where $P_j \Rightarrow_{\mathbf{F}} S_j$, $j \in \{1,2\}$, with $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$. Thus either $S_1 = \{g.T\}$ and $S_2 = \varnothing$, or $S_1 = \varnothing$ and $S_2 = \{g.T\}$. Assume the first case, the proof of the second is the same. Then $P_1 \equiv^{\mathbf{F}}_m g.Q'$ and $P_2 \equiv^{\mathbf{F}}_m \mathbf{0}$. By induction there exists R with $P_1 \equiv^{\mathbf{ptl}} g.R$ and $R \equiv^{\mathbf{F}}_m Q'$, and by Lemma 7.1 we obtain $P_2 \equiv^{\mathbf{ptl}} \mathbf{0}$. Hence $P_1 \mid P_2 \equiv^{\mathbf{ptl}} g.R \mid \mathbf{0} \equiv^{\mathbf{ptl}} g.R$ by structural law (1.1). $\square$

To treat the case that Q is a parallel composition $Q_1 \mid Q_2$, what we want to prove (using induction on the structure of P) is

> if $P \equiv^{\mathbf{F}}_m Q_1 \mid Q_2$, then there exist R_1 and R_2 such that $P \equiv^{\mathbf{ptl}} R_1 \mid R_2$, $R_1 \equiv^{\mathbf{F}}_m Q_1$, and $R_2 \equiv^{\mathbf{F}}_m Q_2$;

this however, we cannot, because in the case that P is a parallel composition $P_1 \mid P_2$, it is impossible to find the process terms Q_1 and Q_2 needed in the induction step. Instead we prove the lemma below (of which the above statement is a consequence). We use Theorem 6.6 in its proof; this allows us to treat the three cases in which P can appear as a cell in one stroke.

Lemma 7.3 *For every process term P, if $P \Rightarrow_{\mathbf{F}} T_1 \cup T_2$ where T_1, T_2 are colonies with $\mathrm{new}(T_1) \cap \mathrm{new}(T_2) = \varnothing$, then there exist process terms R_1 and R_2 such that $R_1 \Rightarrow_{\mathbf{F}} T_1$, $R_2 \Rightarrow_{\mathbf{F}} T_2$, and $P \equiv^{\mathbf{ptl}} R_1 \mid R_2$.*

Proof The proof is by induction on the structure of P. Observe that by Lemma 6.2 and Corollary 3.2, it suffices to prove the statement of the lemma for the case that P is in cnf.

Let $P = \mathbf{0}$. Then $T_1 \cup T_2 = \varnothing$ and hence $T_1 = T_2 = \varnothing$. Now let $R_1 = R_2 = \mathbf{0}$, since then by structural law (1.1), $P \equiv^{\mathbf{ptl}} \mathbf{0} \mid \mathbf{0}$.

Let P be a cell. Then, by Theorem 6.6, either $P \Rightarrow_{\mathbf{F}} T_1$ and, for $s \geq 0$, $Q_1 \mid \ldots \mid Q_s \Rightarrow_{\mathbf{F}} T_2$ with $Q_i \in \mathcal{O}(P)$, or the converse, $Q_1 \mid \ldots \mid Q_s \Rightarrow_{\mathbf{F}} T_1$ and $P \Rightarrow_{\mathbf{F}} T_2$. Assume the first case; the second is the same. Let $R_1 = P$ and $R_2 = Q_1 \mid \ldots \mid Q_s$. By s applications of Lemma 6.5, $P = R_1 \equiv^{\mathbf{ptl}} R_1 \mid R_2$.

Let $P = P_1 \mid P_2$ with $P_i \Rightarrow_{\mathbf{F}} S_i$ and $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$. Suppose $S_1 \cup S_2 = T_1 \cup T_2$. By Lemma 6.3, S_1 and S_2 are colonies. Hence by Lemma 5.9, there exist colonies $U_{i,j}$, $i,j \in \{1,2\}$, with mutually disjoint $\mathrm{new}(U_{i,j})$, such that S_i is a copy of $U_{i,1} \cup U_{i,2}$ and T_j is a copy of $U_{1,j} \cup U_{2,j}$. Consequently,

$P_i \Rightarrow_{\mathbf{F}} U_{i,1} \cup U_{i,2}$. By induction, there exist $R_{i,j}$, $i, j \in \{1, 2\}$, with $R_{i,j} \Rightarrow_{\mathbf{F}} U_{i,j}$ and $P_i \equiv^{\mathbf{ptl}} R_{i,1} \mid R_{i,2}$. Now let $R_1 = R_{1,1} \mid R_{2,1}$ and $R_2 = R_{1,2} \mid R_{2,2}$, since then $R_j \Rightarrow_{\mathbf{F}} U_{1,j} \cup U_{2,j}$ by (SF1), and hence $R_j \Rightarrow_{\mathbf{F}} T_j$ since T_j is a copy of $U_{1,j} \cup U_{2,j}$. Moreover, $P = P_1 \mid P_2 \equiv^{\mathbf{ptl}} (R_{1,1} \mid R_{1,2}) \mid (R_{2,1} \mid R_{2,2}) \equiv^{\mathbf{ptl}} R_1 \mid R_2$ by structural laws (1.2) and (1.3). $\square$

Similar to the previous case, we are forced to include solutions (instead of process terms) in the statement of the lemma below to show the case $Q = (\nu x)Q'$; we cannot prove

> if $P \equiv^{\mathbf{F}}_m (\nu x)Q'$, then there exists R such that $P \equiv^{\mathbf{ptl}} (\nu x)R$ and $R \equiv^{\mathbf{F}}_m Q'$

immediately (again, in the case that P is a parallel composition $P_1 \mid P_2$, it is impossible to deduce the existence of a process term Q' needed in the induction step). Whereas in the previous case we required the solutions to be colonies, here we require them to have the two BC properties. The reason why the first BC property is included in the statement is much the same as why it was needed in the dual case $P = (\nu x)P'$ in Lemma 7.2. The second BC property of T is needed in the case that P is a replication $!\,P'$: by (SF4) we infer that $P \Rightarrow_{\mathbf{F}} \bigcup_{i \in \mathbb{N}} S_i \cup \{\overline{m}m.\{\overline{m}m.S\}\} = T[n/x]$ where the S_i, S are new-disjoint. Now if n appeared in S, i.e., $T = \bigcup_{i \in \mathbb{N}} S_i \cup \{\overline{m}m.\{\overline{m}m.S[x/n]\}\}$, then T would not have the second BC property because $x \in \mathrm{fn}(T)$, but x does not appear in any cherry of T. Thus, n does not appear in S and, similarly, $n \neq m$ (cf. the discussion in Section 4).

Lemma 7.4 *For every process term P, if $P \Rightarrow_{\mathbf{F}} T[n/x]$ where $x \in \mathbf{N}$, $n \in$ New $-$ new(T), and T has the first and second BC property, then there exists a process term R such that $P \equiv^{\mathbf{ptl}} (\nu x)R$ and $R \Rightarrow_{\mathbf{F}} T$.*

Proof The proof is by induction on the structure of P. Note that x does not occur free in P. In fact, by an argument similar to the one in the proof of Lemma 7.2, we may assume that x does not occur at all (i.e., neither free nor bound) in P.

Let $P = \mathbf{0}$ and assume $P \Rightarrow_{\mathbf{F}} T[n/x]$. Then $T = \varnothing$ and hence, letting $R = \mathbf{0}$, $R \Rightarrow_{\mathbf{F}} T$ and $P \equiv^{\mathbf{ptl}} (\nu x)R$ by structural law (2.2).

Let $P = g.P'$ and let $P' \Rightarrow_{\mathbf{F}} S_0, S_1$ with $\mathrm{new}(S_0) \cap \mathrm{new}(S_1) = \varnothing$. Then $P \Rightarrow_{\mathbf{F}} \{g.S\}$, where $S = S_0 \cup \{\overline{m}m.\{\overline{m'}m'.S_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$ with $m, m' \in$ New $-$ new$(S_0 \cup S_1)$ and $m \neq m'$. Assume $\{g.S\} = T[n/x]$; observe that $x \notin \mathrm{fn}(g.S)$, and that x does not occur in g. Also note that since g is a guard over $\mathbf{N}$, n does not occur in g, so $T = \{g.S[x/n]\}$. By the same argument as in the proof of Lemma 7.2 (case $P = (\nu x)P'$) using the first BC property of T, $m \neq n$ and $m' \neq n$, and thus $S[x/n] = S_0[x/n] \cup \{\overline{m}m.\{\overline{m'}m'.S_1[x/n]\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$. Tracing this argument even further, using that $\mathrm{new}(S_0) \cap \mathrm{new}(S_1) = \varnothing$, $x \notin \mathrm{fn}(T)$ and so $T = T[n/x]$. Hence, letting $R = P$, we have $R \Rightarrow_{\mathbf{F}} T$ and $P \equiv^{\mathbf{ptl}} (\nu x)R$ by structural law (2.2).

Let $P = \,!P'$ where $P' \Rightarrow_{\mathbf{F}} S, S_i$, $i \in \mathbb{N}$, such that the new(S_i) are mutually disjoint and disjoint with new(S). Let $m \in$ New with $m \notin$ new(S) and $m \notin$ new(S_i). Then $P \Rightarrow_{\mathbf{F}} (\bigcup_{i\in\mathbb{N}} S_i) \cup \{\overline{m}m.\{\overline{m}m.S\}\} = T[n/x]$. Now, $m \neq n$ and $n \notin$ new(S) by the second BC property of T and since the new(S_i) and new(S) are mutually disjoint. Thus $T = (\bigcup_{i\in\mathbb{N}-\{j\}} S_i) \cup S_j[x/n] \cup \{\overline{m}m.\{\overline{m}m.S\}\}$ for exactly one $j \in \mathbb{N}$. To use induction (applied to $P' \Rightarrow_{\mathbf{F}} S_j[x/n][n/x] = S_j$), we show that $S_j[x/n]$ has both the BC properties. Clearly, $S_j[x/n]$ has the first BC property, since $S_j[x/n] \subseteq T$ and T has the first BC property. To show that $S_j[x/n]$ has the second BC property, suppose that $y \in$ fn$(S_j[x/n])$. If $x \neq y$, then $y \in$ fn(S_j). Note that by Lemma 4.3, S_j has the second BC property. Thus there exists a cherry $g.T' \in D_{S_j}$ with $y \in$ fn$(g.T')$. Hence $y \in$ fn$((g.T')[x/n])$ and $(g.T')[x/n] \in D_{S_j[x/n]}$. Thus, $(g.T')[x/n] \in D_T$, and since $(g.T')[x/n]$ cannot be a berry, by the first BC property of T it must be a cherry. If $x = y$, then, by the second BC property of T, there exists a cherry $g.T' \in D_T$ with $y \in$ fn$(g.T')$. Hence $g.T' \in D_{S_j[x/n]}$, since $x \notin$ fn(S_i), $i \in \mathbb{N}$, and $x \notin$ fn$(\overline{m}m.\{\overline{m}m.S\})$. Consequently $S_j[x/n]$ has the second BC property. By induction there exists R' with $R' \Rightarrow_{\mathbf{F}} S_j[x/n]$ and $P' \equiv^{\mathbf{ptl}} (\nu x)R'$. Now take $R = P\,|\,R'$ since then clearly $R \Rightarrow_{\mathbf{F}} T$ and $P \equiv^{\mathbf{ptl}} P\,|\,P' \equiv^{\mathbf{ptl}} P\,|\,(\nu x)R' \equiv^{\mathbf{ptl}} (\nu x)R$, by structural laws (3.1) and (2.3), respectively, and the fact that x does not occur in P.

Let $P = P_1 \mid P_2$ and let $P_i \Rightarrow_{\mathbf{F}} S_i$ with new$(S_1) \cap$ new$(S_2) = \varnothing$. Then $P \Rightarrow_{\mathbf{F}} S_1 \cup S_2$. Assume $S_1 \cup S_2 = T[n/x]$. Since the new(S_i) are mutually disjoint, either $T = S_1[x/n] \cup S_2$, or $T = S_1 \cup S_2[x/n]$. Assume the first case. Note that $S_1[x/n]$ has the first BC property, since $S_1[x/n] \subseteq T$ and T has the first BC property. The second BC property of $S_1[x/n]$ follows from the second BC property of S_1, the first and the second BC property of T, and from $x \notin$ fn(S_2), by an argument similar to the previous case. By induction there exists R_1 with $R_1 \Rightarrow_{\mathbf{F}} S_1[x/n]$ and $P_1 \equiv^{\mathbf{ptl}} (\nu x)R_1$. Consequently, letting $R = R_1\,|\,P_2$, $R \Rightarrow_{\mathbf{F}} T$ and $P \equiv^{\mathbf{ptl}} (\nu x)R_1\,|\,P_2 \equiv^{\mathbf{ptl}} (\nu x)R$ by structural law (2.3).

Let $P = (\nu y)P'$ where $P' \Rightarrow_{\mathbf{F}} S$ with $m \in$ New $-$ new(S). Then $P \Rightarrow_{\mathbf{F}} S[m/y]$. Assume $S[m/y] = T[n/x]$. Note that since x does not occur in P, $y \neq x$. First consider the case $m = n$. Then $T = S[m/y][x/n] = S[x/y]$. Now take $R = P'[x/y]$, since then $P \equiv^{\mathbf{ptl}} (\nu x)R$ by structural law (α), and furthermore $R \Rightarrow_{\mathbf{F}} T$ by Lemma 2.5(1). Next, consider the case $m \neq n$. Then $S = T[n/x][y/m] = T[y/m][n/x]$. Since by Lemma 4.3, S has the first and second BC property, and since by assumption $T = T[y/m][m/y]$ has the first and second BC property, $T[y/m]$ has the first and second BC property by Lemma 4.4. By induction, applied to $P' \Rightarrow_{\mathbf{F}} S = T[y/m][n/x]$, there exists R' with $R' \Rightarrow_{\mathbf{F}} T[y/m]$ and $P' \equiv^{\mathbf{ptl}} (\nu x)R'$. Hence, letting $R = (\nu y)R'$, we have $R \Rightarrow_{\mathbf{F}} T[y/m][m/y] = T$ and $P \equiv^{\mathbf{ptl}} (\nu y)(\nu x)R' \equiv^{\mathbf{ptl}} (\nu x)R$ by structural law (2.1). $\square$

The final lemma treats the case that Q is a replication $!Q'$. As in Lemma 7.2 (for $Q = g.Q'$), the assertion is made directly on the process term $!Q'$. Also note that the cases in which P is the null process, a guarded process, or a restriction are excluded in its proof. The technical assumptions on the offspring $\mathcal{O}(\,!\,Q')$ are needed in the case that P is a parallel composition — the only nontrivial case remaining — which is shown by an application of Lemma 7.3.

Lemma 7.5 *Let $!\,Q'$ be a cell such that for every process term R', $R' \equiv^{\mathbf{F}}_m Q'$ implies $!\,R' \mid Q_2 \equiv^{\mathbf{ptl}} \,!\,R'$ whenever $Q_2 \equiv^{\mathbf{F}}_m Q_1 \in \mathcal{O}(\,!\,Q')$. For every process term P, if $P \equiv^{\mathbf{F}}_m \,!\,Q'$, then there exists a process term R with $P \equiv^{\mathbf{ptl}} \,!\,R$ and $R \equiv^{\mathbf{F}}_m Q'$.*

Proof The proof is by induction on the structure of P. By an argument similar to the one in the proof of Lemma 7.3, it suffices to prove the statement in the lemma for the case that P is in cnf. The proof is trivial for the cases $P = \mathbf{0}$ and $P = g.P'$; it is easily checked that then $P \equiv^{\mathbf{F}}_m \,!\,Q'$ is impossible. We can also exclude the case $P = (\nu x)P'$ as the following argument shows: assume $P' \Rightarrow_{\mathbf{F}} S$ with $n \in \mathrm{New} - \mathrm{new}(S)$. Observe that S is a colony by Lemma 6.3. Then $P \Rightarrow_{\mathbf{F}} S[n/x]$. Since P is a cell, $n \in \mathrm{new}(\mathrm{nuc}(S[n/x]))$ by Lemma 5.10. Now let $Q' \Rightarrow_{\mathbf{F}} T, T_i$, $i \in \mathbb{N}$, such that the $\mathrm{new}(T_i)$ are mutually disjoint and disjoint with $\mathrm{new}(T)$. Assume $!\,Q' \Rightarrow_{\mathbf{F}} (\bigcup_{i\in\mathbb{N}} T_i) \cup \{\overline{m}m.\{\overline{m}m.T\}\} = S[n/x]$. Since by Lemma 4.3 S has the first BC property, $n \neq m$, and so $n \in \mathrm{new}(T)$ since $\mathrm{nuc}(S[n/x]) = \{\overline{m}m.\{\overline{m}m.T\}\}$, cf. the proof of Lemma 6.3. Hence $S = (\bigcup_{i\in\mathbb{N}} T_i) \cup \{\overline{m}m.\{\overline{m}m.T[x/n]\}\}$. Now let V be a connected component of $T[x/n]$ containing x. Clearly $V < \{\overline{m}m.\{\overline{m}m.T[x/n]\}\}$. Since S is a colony, by its colony property we have that $\mathrm{mult}(V, S) = \mathrm{mult}(V, \bigcup_{i\in\mathbb{N}} T_i) = \omega$. However, this is impossible since $x \notin \mathrm{fn}(T_i)$.

Let $P = P_1 \mid P_2$ where $P_i \Rightarrow_{\mathbf{F}} S_i$, $i \in \{1,2\}$, with $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$. Assume $!\,Q' \Rightarrow_{\mathbf{F}} S_1 \cup S_2$. By Theorem 6.6, there exist $R_1, \ldots, R_s \in \mathcal{O}(\,!\,Q')$ such that either $P_1 \equiv^{\mathbf{F}}_m \,!\,Q'$ and $P_2 \equiv^{\mathbf{F}}_m R_1 | \ldots | R_s$, or the converse, $P_1 \equiv^{\mathbf{F}}_m R_1 | \ldots | R_s$ and $P_2 \equiv^{\mathbf{F}}_m \,!\,Q'$. Assume the first case; the second has the same proof. By induction, there exists R with $P_1 \equiv^{\mathbf{ptl}} \,!\,R$ and $R \equiv^{\mathbf{F}}_m Q'$. Moreover, by Lemma 7.3, there exist $R'_1, \ldots, R'_s$ such that $P_2 \equiv^{\mathbf{ptl}} R'_1 \mid \ldots \mid R'_s$ and $R'_p \equiv^{\mathbf{F}}_m R_p$, $1 \leq p \leq s$. Hence $P_1 \mid P_2 \equiv^{\mathbf{ptl}} \,!\,R \mid R'_1 \mid \ldots \mid R'_s \equiv^{\mathbf{ptl}} \,!\,R$ by the assumption in the statement of the lemma.

Let $P = \,!\,P'$ and assume $!\,P' \equiv^{\mathbf{F}}_m \,!\,Q'$. Since both $!\,P'$ and $!\,Q'$ are cells, there exists a cell C such that $!\,P' \Rightarrow_{\mathbf{F}} C$ and $!\,Q' \Rightarrow_{\mathbf{F}} C$. Hence (cf. the proof of Lemma 6.3) $\mathrm{nuc}(C) = \{\overline{m}m.\{\overline{m}m.S\}\}$ for some $m \in \mathrm{New}$, with $P' \Rightarrow_{\mathbf{F}} S$ and $Q' \Rightarrow_{\mathbf{F}} S$. Thus, choosing $R = P'$, we have $P = \,!\,R$ and $R \equiv^{\mathbf{F}}_m Q'$. $\square$

We are now able to show the completeness of potential structural congruence.

Lemma 7.6 *For all process terms P and Q, if $P \equiv^{\mathbf{F}}_m Q$, then $P \equiv^{\mathbf{ptl}} Q$.*

Proof The proof is by induction on the structure of Q. By Lemma 6.2 and Corollary 3.2 we may restrict ourselves to process terms Q in cnf; observe that if Q is in cnf then every subterm of Q is in cnf. If $Q = \mathbf{0}$, then $P \equiv^{\mathbf{ptl}} Q$ by Lemma 7.1.

Let $Q = g.Q'$. Then by Lemma 7.2, $P \equiv^{\mathbf{ptl}} g.R$ for some process term R with $R \equiv_m^{\mathbf{F}} Q'$. By induction, $R \equiv^{\mathbf{ptl}} Q'$ and hence $P \equiv^{\mathbf{ptl}} g.Q'$ by congruence of $\equiv^{\mathbf{ptl}}$.

Let $Q = Q_1 \mid Q_2$. Then $P \Rightarrow_{\mathbf{F}} T_1 \cup T_2$ with $Q_1 \Rightarrow_{\mathbf{F}} T_1$, $Q_2 \Rightarrow_{\mathbf{F}} T_2$, and $\mathrm{new}(T_1) \cap \mathrm{new}(T_2) = \varnothing$. By Lemma 6.3, T_1 and T_2 are colonies. By Lemma 7.3, there exist process terms R_1 and R_2 with $P \equiv^{\mathbf{ptl}} R_1 | R_2$, $R_1 \equiv_m^{\mathbf{F}} Q_1$, and $R_2 \equiv_m^{\mathbf{F}} Q_2$. By induction, $R_1 \equiv^{\mathbf{ptl}} Q_1$ and $R_2 \equiv^{\mathbf{ptl}} Q_2$. Hence $P \equiv^{\mathbf{ptl}} Q_1 \mid Q_2$.

Let $Q = (\nu x)Q'$ with $Q' \Rightarrow_{\mathbf{F}} T$ and $n \in \mathrm{New} - \mathrm{new}(T)$. Then $P \Rightarrow_{\mathbf{F}} T[n/x]$, and by Lemma 4.3, T has the first and second BC property. By Lemma 7.4, there exists a process term R with $P \equiv^{\mathbf{ptl}} (\nu x)R$ and $R \equiv_m^{\mathbf{F}} Q'$. By induction, $R \equiv^{\mathbf{ptl}} Q'$ and so $P \equiv^{\mathbf{ptl}} (\nu x)Q'$.

Let $Q = \,!Q'$. By assumption, $!Q'$ is a cell. By induction, $R' \equiv^{\mathbf{ptl}} Q'$ for every process term R' with $R' \equiv_m^{\mathbf{F}} Q'$, which implies $!R' \equiv^{\mathbf{ptl}} \,!Q'$. Also by induction, since $\mathcal{O}(!Q')$ consists only of subterms of Q', we have $Q_2 \equiv^{\mathbf{ptl}} Q_1$ for all process terms Q_1 and Q_2 with $Q_1 \in \mathcal{O}(!Q')$ and $Q_2 \equiv_m^{\mathbf{F}} Q_1$. Hence $!R' \mid Q_2 \equiv^{\mathbf{ptl}} \,!Q' \mid Q_1 \equiv^{\mathbf{ptl}} \,!Q' \equiv^{\mathbf{ptl}} \,!R'$, by Lemma 6.5. Thus, since $!Q'$ satisfies all the requirements in Lemma 7.5, there exists a process term R with $P \equiv^{\mathbf{ptl}} \,!R$ and $R \equiv_m^{\mathbf{F}} Q'$. By induction, $R \equiv^{\mathbf{ptl}} Q'$ and so $P \equiv^{\mathbf{ptl}} \,!Q'$. $\square$

This last result proves the first main result of this paper: potential structural congruence and multiset congruence by **F** are the same.

Theorem 7.7 *For all process terms P and Q, $P \equiv^{\mathbf{ptl}} Q$ if and only if $P \equiv_m^{\mathbf{F}} Q$.*

Proof Immediate by Corollary 3.2 and Lemma 7.6. $\square$

As we claimed earlier, the decidability of potential structural congruence is a consequence of the previous result. Thus we obtain the second main result of this paper.

Theorem 7.8 *It is decidable, for process terms P and Q, whether or not $P \equiv^{\mathbf{ptl}} Q$.*

Proof By Theorem 7.7 (and Theorem 33 of [3]), $P \equiv^{\mathbf{ptl}} Q$ if and only if $\mathbf{F}(P) \equiv \mathbf{F}(Q)$. Since by Theorem 34 of [3], $\equiv$ is decidable and since the reduction **F** is effective, $P \equiv^{\mathbf{ptl}} Q$ is decidable. $\square$

8 Behavioural Invariance of Process Transformation

We indicated in Section 2 that the mapping **F** does not change the behaviour of processes; since only inactive agents are added, one might expect then that **F** is a strong bisimulation on the transition system of the π-calculus, but, unfortunately, it is not, as the following trivial counterexample shows. Let $P = x(y).\mathbf{0} \mid \overline{x}z.\mathbf{0}$. Clearly $P \to \mathbf{0}|\mathbf{0}$, and so $P \to \mathbf{0}$. Let $R = (\nu v)(\nu w)(\overline{v}v.\overline{w}w.\mathbf{0}|\overline{v}v.\overline{w}w.\mathbf{0})$. It can be verified easily that $\mathbf{F}(P) = P_1 \,|\, P_2$, where $P_1 \equiv^{\mathbf{std}} x(y).R$ and $P_2 \equiv^{\mathbf{std}} \overline{x}z.R$. Clearly, whenever $\mathbf{F}(P) \to P'$, then $P' \equiv^{\mathbf{std}} R \mid R$, but $\mathbf{F}(\mathbf{0}) \equiv^{\mathbf{std}} R \mid R$ does not hold. Instead, we show that $\Rightarrow \circ \mathbf{F} \circ \Rightarrow^{-1}$ is *contained* in a strong bisimulation on the transition system of Mπ, which shows (since, by Theorem A of [2], $\Rightarrow$ is a strong bisimulation between the transition systems of the π-calculus and Mπ) that P and $\mathbf{F}(P)$ are strongly bisimilar. In fact, we will use a multiset transition system which is obtained from Mπ by adding *communication labels* to the transitions, and we show that the semantics of P and $\mathbf{F}(P)$ are strongly bisimilar even in that system; this is an immediate consequence of Theorem 8.7 and Theorem 8.8 that are proven at the end of this section.

Definition 8.1 The *Multiset π-Calculus with communication labels* Mcπ is the multiset transition system (Mol, T) where T consists of all the basic transitions

$$\{x(-).S, \overline{x}z.S'\} \overset{x(-),\overline{x}z}{\longrightarrow} \mathrm{dec}(S[z/1]) \cup S'$$

where $x, z \in \mathbf{N} \cup \mathrm{New}$, S and S' are solutions, and $\mathrm{dec}(S[z/1])$ decreases every number that occurs in $S[z/1]$ by one.

We refer to Sections 3 and 4 of [2] for a detailed discussion on multiset transition systems. In particular we note that by the (labeled analogue of the) chemical law, the transition relation of Mcπ consists of all transitions

$$\{x(-).S, \overline{x}z.S'\} \cup S'' \overset{x(-),\overline{x}z}{\longrightarrow} \mathrm{dec}(S[z/1]) \cup S' \cup S''.$$

Clearly, if $S \overset{x(-),\overline{x}z}{\longrightarrow} S'$ in Mcπ, then $S \to S'$ in Mπ. Conversely, if $S \to S'$ in Mπ, then there exist $x, z \in \mathbf{N} \cup \mathrm{New}$ such that $S \overset{x(-),\overline{x}z}{\longrightarrow} S'$ in Mcπ. However, solutions that are bisimilar in Mπ need not be bisimilar in Mcπ (take, e.g., $\{x(-).\varnothing, \overline{x}z.\varnothing\}$ and $\{y(-).\varnothing, \overline{y}z'.\varnothing\}$). Thus bisimilarity in Mcπ is stricter than bisimilarity in Mπ.

In the remainder of this section we will show that $\Rightarrow \circ \mathbf{F} \circ \Rightarrow^{-1}$ is an instance of expanding a solution by adding dummy molecules to its subsolutions. Moreover, we prove that this expansion does not change the solution's behaviour. Of course, this depends on what exactly we mean by dummy molecules. To be on the safe side, to a solution S we only add output molecules $\overline{n}m.U$ with $n, m \in \mathrm{New} - \mathrm{new}(S)$. This guarantees that whenever a communication takes place in the expansion, the molecule $\overline{n}m.U$ has no part in it, since it needs the input molecule $n(-).U'$ to communicate with (that is evidently not in the expansion of S). The new names in the dummy molecules we take from a predetermined set $D \subseteq \mathrm{New}$.

Definition 8.2 Let $D \subseteq \mathrm{New}$. *D-Expansion*, denoted $\lhd_D$, is the smallest binary relation on Sol such that

if $S_i \lhd_D T_i$ for all $i \in I$,
then $\bigcup_{i \in I}\{g_i.S_i\} \lhd_D \bigcup_{i \in I}\{g_i.T_i\} \cup \bigcup_{k \in K}\{\overline{n_k}m_k.U_k\}$,

where the g_i are schematic guards, and $n_k, m_k \in D$. If $S \lhd_D T$, then T is called a D-expansion of S. *D-Exclusive expansion*, denoted $\lhd^{\mathrm{e}}_D$, is the relation

$$\{(S,T) \mid S \lhd_D T \text{ and } \mathrm{new}(S) \cap D = \varnothing\}.$$

If $S \lhd^{\mathrm{e}}_D T$, then T is called a D-exclusive expansion of S.

Evidently, $S \lhd^{\mathrm{e}}_D T \implies S \lhd_D T \implies S \subseteq^{\mathrm{g}} T$ for all $D \subseteq \mathrm{New}$ and all solutions S and T, where $\subseteq^{\mathrm{g}}$ is the nested containment relation of [4]. Hence by Lemma 4.11 of [4], $\mathrm{new}(S) \subseteq \mathrm{new}(T)$ whenever $S \lhd^{\mathrm{e}}_D T$. It is easy to show that $S \lhd^{\mathrm{e}}_D T$ implies $S \lhd^{\mathrm{e}}_D T \cup \bigcup_{k\in K}\{\overline{n_k}m_k.U_k\}$ with $n_k, m_k \in D$. Below we collect some other easy to prove properties of expansion.

Lemma 8.3 *Let $D \subseteq \mathrm{New}$. For all solutions S, T, S_i and T_i, $i \in I$,*

(1) if $S_i \lhd^{\mathrm{e}}_D T_i$ for every $i \in I$, then $\bigcup_{i\in I} S_i \lhd^{\mathrm{e}}_D \bigcup_{i\in I} T_i$,

(2) if $S \lhd^{\mathrm{e}}_D T$, then for every mapping

$$f : \mathbf{N} \cup (\mathrm{New} - D) \cup \mathbb{N}_+ \to \mathbf{N} \cup (\mathrm{New} - D) \cup \mathbb{N}_+,$$

$f(S) \lhd^{\mathrm{e}}_D f(T)$, and

(3) if $S \lhd^{\mathrm{e}}_D T$, then for every guard g over $\mathbf{N}$, $\{g.S\} \lhd^{\mathrm{e}}_D \{g.T\}$.

Proof The proof of (1) is obvious, cf. the proof of Lemma 4.3(1) of [4]. We show (2) by induction on the definition of $\lhd_D$ (cf. the notion of induction on the definition of $\subseteq^{\mathrm{g}}$ in [4]). Let $S = \bigcup_{i\in I}\{g_i.S_i\}$ and $T = \bigcup_{i\in I}\{g_i.T_i\} \cup \bigcup_{k\in K}\{\overline{n_k}m_k.U_k\}$ with $S_i \lhd_D T_i$ and $n_k, m_k \in D$, such that $\mathrm{new}(S) \cap D = \varnothing$. By induction, $f(S_i) \lhd_D f(T_i)$. Now $f(S) = \bigcup_{i\in I}\{f(g_i).f(S_i)\} \lhd_D \bigcup_{i\in I}\{f(g_i).f(T_i)\} \cup \bigcup_{k\in K}\{\overline{n_k}m_k.f(U_k)\} = f(T)$, and hence $f(S) \lhd^{\mathrm{e}}_D f(T)$, since $\mathrm{new}(f(S)) \cap D = f(\mathrm{new}(S)) \cap D = \varnothing$. The proof of (3) is similar to the proof of Lemma 4.3(3) of [4] (and uses (2) of the current lemma). $\square$

If T is a D-exclusive expansion of S, then it is, of course, also an expansion with respect to the subset of D consisting of new names that actually *occur* in T. Or we can add to D any name from New, as long as it does not appear in S; also with respect to this set T is an expansion of S.

Lemma 8.4 *Let $D \subseteq \mathrm{New}$. If $S \lhd^{\mathrm{e}}_D T$, then $S \lhd^{\mathrm{e}}_{D'} T$ for all D' with $\mathrm{new}(T) \cap D \subseteq D' \subseteq \mathrm{New} - \mathrm{new}(S)$.*

Proof The proof is by induction on the definition of $\lhd_D$. Let $S \lhd_D T$ with $\mathrm{new}(S) \cap D = \varnothing$, and let $\mathrm{new}(T) \cap D \subseteq D' \subseteq \mathrm{New} - \mathrm{new}(S)$. It follows that $S = \bigcup_{i\in I}\{g_i.S_i\}$ and $T = \bigcup_{i\in I}\{g_i.T_i\} \cup \bigcup_{k\in K}\{\overline{n_k}m_k.U_k\}$ with $S_i \lhd_D T_i$ for all $i \in I$. By induction, $S_i \lhd_{D'_i} T_i$ for all D'_i with $\mathrm{new}(T_i) \cap D \subseteq D'_i \subseteq \mathrm{New} - \mathrm{new}(S_i)$. Since in particular $\mathrm{new}(T_i) \cap D \subseteq \mathrm{new}(T) \cap D \subseteq D' \subseteq \mathrm{New} - \mathrm{new}(S) \subseteq \mathrm{New} - \mathrm{new}(S_i)$, we have $S_i \lhd_{D'} T_i$ for all $i \in I$. Hence $S \lhd^{\mathrm{e}}_{D'} T$, since obviously $n_k, m_k \in D'$ and $\mathrm{new}(S) \cap D' = \varnothing$. $\square$

We additionally need a result similar to Lemma 8.3(1), only for different sets D_i.

Lemma 8.5 *Let $D_i \subseteq \mathrm{New}$, $i \in I$. If $S_i \lhd^{\mathrm{e}}_{D_i} T_i$ for all $i \in I$, and the* $\mathrm{new}(T_i)$ *are mutually disjoint, then $\bigcup_{i\in I} S_i \lhd^{\mathrm{e}}_D \bigcup_{i\in I} T_i$ with $D = \bigcup_{i\in I}(\mathrm{new}(T_i) \cap D_i)$.*

Proof We show that $D \cap \mathrm{new}(\bigcup_{i\in I} S_i) = \varnothing$. Since the $\mathrm{new}(T_i)$ are mutually disjoint and $\mathrm{new}(S_i) \subseteq \mathrm{new}(T_i)$ for all $i \in I$, $\mathrm{new}(T_i) \cap \mathrm{new}(S_j) = \varnothing$ if $i \neq j$. Moreover, since $D_i \cap \mathrm{new}(S_i) = \varnothing$, we have $D \cap \mathrm{new}(\bigcup_{i\in I} S_i) = \varnothing$. Thus $\mathrm{new}(T_i) \cap D_i \subseteq D \subseteq \mathrm{New} - \mathrm{new}(\bigcup_{j\in I} S_j) \subseteq \mathrm{New} - \mathrm{new}(S_i)$, for all $i \in I$. Consequently, by Lemma 8.4, $S_i \lhd^{\mathrm{e}}_D T_i$ for all $i \in I$, and so $S \lhd^{\mathrm{e}}_D T$ by Lemma 8.3(1). $\square$

The result in Lemma 8.4 induces a relation that is independent of the set D with which solutions are expanded.

Definition 8.6 For solutions S and T, if $S \lhd_{\mathrm{New}-\mathrm{new}(S)} T$, then T is an *expansion of* S, denoted $S \lhd T$.

Before we show that expansion is a strong bisimulation on the transition system Mcπ, we show that a solution corresponding to a process term P can be expanded to one corresponding to $\mathbf{F}(P)$. Note that $S \lhd_{\mathrm{New}-\mathrm{new}(S)} T$ if and only if $S \lhd^{\mathrm{e}}_{\mathrm{New}-\mathrm{new}(S)} T$; this is implicitly used in the proof of the theorem below. Note also that, by Lemma 8.4, $S \lhd^{\mathrm{e}}_D T$ implies $S \lhd T$.

Theorem 8.7 *For every process term P, there exist solutions S and T such that $P \Rightarrow S$, $P \Rightarrow_{\mathbf{F}} T$, and $S \lhd T$.*

Proof The statement in the lemma clearly is a consequence of the following asymmetrical statement (using Lemma 4 of [2]):

if $P \Rightarrow_{\mathbf{F}} T$, then there exists S with $P \Rightarrow S$ and $S \lhd T$,

that we will show by induction on the structure of P. It is obvious for $P = \mathbf{0}$, since $\varnothing \lhd \varnothing$.

Let $P = P_1 \mid P_2$. Then $P \Rightarrow_{\mathbf{F}} T_1 \cup T_2 = T$ with $\mathrm{new}(T_1) \cap \mathrm{new}(T_2) = \varnothing$ and $P_i \Rightarrow_{\mathbf{F}} T_i$, $i \in \{1,2\}$. By induction there exist S_i with $P_i \Rightarrow S_i$ and $S_i \lhd T_i$. Since $\mathrm{new}(S_i) \subseteq \mathrm{new}(T_i)$ we have that $\mathrm{new}(S_1) \cap \mathrm{new}(S_2) = \varnothing$, and so $P \Rightarrow S_1 \cup S_2 = S$. By Lemma 8.5, $S_1 \cup S_2 \lhd^{\mathrm{e}}_D T_1 \cup T_2$ with $D = \mathrm{new}(T_1 \cup T_2) - \mathrm{new}(S_1 \cup S_2)$. Hence $S \lhd T$ by Lemma 8.4.

Let $P = g.P'$. Then, for $T' = T_0 \cup \{\overline{m}m.\{\overline{m'}m'.T_1\}, \overline{m}m.\{\overline{m'}m'.\varnothing\}\}$ with $m, m' \notin \mathrm{new}(T_0 \cup T_1)$ and $P' \Rightarrow_{\mathbf{F}} T_0$, we have $P \Rightarrow_{\mathbf{F}} \{g.T'\} = T$. By induction, there exists S' with $P' \Rightarrow S'$ and $S' \lhd T_0$. Hence $P \Rightarrow \{g.S'\} = S$. Note that $S' \lhd^{\mathrm{e}}_{\mathrm{New}-\mathrm{new}(S)} T_0$ because $\mathrm{new}(S) = \mathrm{new}(S')$. Now since $m \notin \mathrm{new}(T_0)$, $m \notin \mathrm{new}(S)$, and thus $S' \lhd^{\mathrm{e}}_{\mathrm{New}-\mathrm{new}(S)} T'$ by Definition 8.2. Consequently $S \lhd T$ by Lemma 8.3(3).

Let $P = (\nu x)P'$. Then $P \Rightarrow_{\mathbf{F}} T'[n/x] = T$ with $n \in \text{New} - \text{new}(T')$ and $P' \Rightarrow_{\mathbf{F}} T'$. By induction, $P' \Rightarrow S'$ with $S' \lhd T'$. Since $n \notin \text{new}(S')$, $P \Rightarrow S'[n/x] = S$. Now since $\text{new}(S) \subseteq \text{new}(S') \cup \{n\}$ and $n \notin \text{new}(T')$, we obtain $\text{new}(T') \cap (\text{New} - \text{new}(S')) \subseteq \text{New} - \text{new}(S)$. Consequently, $S' \lhd^{\mathrm{e}}_{\text{New}-\text{new}(S)} T'$ by Lemma 8.4, and so $S \lhd T$ by Lemma 8.3(2).

Let $P = \,!P'$. Then $P \Rightarrow_{\mathbf{F}} \bigcup_{i \in \mathbb{N}} T_i \cup \{\overline{m}m.\{\overline{m}m.T'\}\} = T$ with $P' \Rightarrow_{\mathbf{F}} T_i$, $m \notin \text{new}(T_i)$, and the $\text{new}(T_i)$ are mutually disjoint. By induction, there exist S_i, $i \in \mathbb{N}$, with $P' \Rightarrow S_i$ and $S_i \lhd T_i$. Thus the $\text{new}(S_i)$ are mutually disjoint and so $P \Rightarrow \bigcup_{i \in \mathbb{N}} S_i = S$. By Lemma 8.5, we have $S \lhd^{\mathrm{e}}_D \bigcup_{i \in \mathbb{N}} T_i$ with $D = \bigcup_{i \in \mathbb{N}} \text{new}(T_i) - \bigcup_{i \in \mathbb{N}} \text{new}(S_i)$. Hence also $S \lhd^{\mathrm{e}}_{\text{New}-\text{new}(S)} \bigcup_{i \in \mathbb{N}} T_i$ by Lemma 8.4. Thus, by Definition 8.2, $S \lhd T$ since $m \in \text{New} - \text{new}(S)$. $\square$

Theorem 8.8 *Expansion is a strong bisimulation on the multiset transition system $Mc\pi$.*

Proof We prove that for every set $D \subseteq \text{New}$ and all solutions S and T with $S \lhd^{\mathrm{e}}_D T$,

(1) if $S \xrightarrow{x(-),\overline{x}z} S'$, then there exists T' with $T \xrightarrow{x(-),\overline{x}z} T'$ and $S' \lhd^{\mathrm{e}}_D T'$, and

(2) if $T \xrightarrow{x(-),\overline{x}z} T'$, then there exists S' with $S \xrightarrow{x(-),\overline{x}z} S'$ and $S' \lhd^{\mathrm{e}}_D T'$.

Thus in particular, choosing $D = \text{New} - \text{new}(S)$, it follows from Lemma 8.4 that $S \lhd T$ implies

(1) if $S \xrightarrow{x(-),\overline{x}z} S'$, then there exists T' with $T \xrightarrow{x(-),\overline{x}z} T'$ and $S' \lhd T'$, and

(2) if $T \xrightarrow{x(-),\overline{x}z} T'$, then there exists S' with $S \xrightarrow{x(-),\overline{x}z} S'$ and $S' \lhd T'$.

Let $S = \bigcup_{i \in I} \{g_i.S_i\}$ and let $T = \bigcup_{i \in I} \{g_i.T_i\} \cup U$ with $U = \bigcup_{k \in K} \{g_k.U_k\}$, $g_k = \overline{n_k}m_k$, $S_i \lhd^{\mathrm{e}}_D T_i$, $n_k, m_k \in D$, and $K \cap I = \varnothing$.

To show (1), $S \xrightarrow{x(-),\overline{x}z} S'$ means that there exist $i_1, i_2 \in I$, such that $g_{i_1} = x(-)$, $g_{i_2} = \overline{x}z$, and $S' = \text{dec}(S_{i_1}[z/1]) \cup S_{i_2} \cup S_3$, with $S_3 = \bigcup_{i \in I - \{i_1, i_2\}} \{g_i.S_i\}$. Hence $T = \{x(-).T_{i_1}, \overline{x}z.T_{i_2}\} \cup \bigcup_{i \in I - \{i_1, i_2\}} \{g_i.T_i\} \cup U$ and so $T \xrightarrow{x(-),\overline{x}z} T' = \text{dec}(T_{i_1}[z/1]) \cup T_{i_2} \cup T_3 \cup U$, with $T_3 = \bigcup_{i \in I - \{i_1, i_2\}} \{g_i.T_i\}$. It suffices to show that $S' \lhd^{\mathrm{e}}_D \text{dec}(T_{i_1}[z/1]) \cup T_{i_2} \cup T_3$. Since $\text{new}(S) \cap D = \varnothing$, $z \notin D$, and so we have $\text{dec}(S_{i_1}[z/1]) \lhd^{\mathrm{e}}_D \text{dec}(T_{i_1}[z/1])$ by Lemma 8.3(2). Obviously, $S_3 \lhd^{\mathrm{e}}_D T_3$, and thus $S' \lhd^{\mathrm{e}}_D T'$, by Lemma 8.3(1).

To show (2), $T \xrightarrow{x(-),\overline{x}z} T'$ implies that there exist $i_1, i_2 \in I \cup K$, such that $g_{i_1} = x(-)$ and $g_{i_2} = \overline{x}z$. Evidently $i_1 \notin K$, so $x \in \text{fn}(S)$. Since $\text{new}(S) \cap D = \varnothing$, $x \notin D$, which implies that $i_2 \notin K$. The proof now proceeds as in (1). $\square$

Conclusion

We presented a multiset semantics for π-calculus process terms such that potential structural congruence is sound and complete with respect to that semantics. It is still open whether a similar result holds for the standard structural congruence, but we believe that it cannot be done in the same manner. Presently, there is some confidence that (at least for process terms without restriction) the decidability of standard structural congruence can be shown using linear equations of *finite* multisets.

Since the process transformation $\mathbf{F}$ in Section 2 only adds dummy agents to its argument, it should be clear that $\mathbf{F}$ also preserves the *concurrent* behaviour of a process. To formalize this, we need a proper notion of bisimulation of Petri net processes (which express the concurrent behaviour of a Petri net). This notion however, we have not developed yet.

The process transformation $\mathbf{F}$ bears some other reductions that might be useful; for instance, we claim that if we modify the definition of $\mathbf{F}(\,!P)$ such that it respects replication — i.e., change it into $\mathbf{F}(\,!P) = \,!\mathbf{F}(P)$ — then it induces a model for extended structural congruence in which only structural law (2.4) is no longer valid. To model the dual congruence — the one obtained by adding (2.4) to potential structural congruence — we claim that $\mathbf{F}$ with the following modification

$$\mathbf{F}(g.P) = g.((\nu v)(\nu w)(\overline{v}v.\overline{w}w.\mathbf{F}(P) \mid \overline{v}v.\overline{w}w.\mathbf{0}))$$

will do the job.

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