Expected Value and Variance

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Introduction

- This class we will, finally, discuss expectation and variance.
- Often used concepts to summarize probability distributions: what to expect and how much does it vary around the expectation.
- As usual we first look at the discrete case, then at the continuous. For the discrete case we only look at variables with numerical values (for categorical ones, the expectation usually doesn't make sense).

Expected Value

- Let X be a numerically valued discrete rv with sample space Ω and distribution function m(x). The expected value E(X) is defined by

$$E(X) = \sum_{x \in \Omega} x m(x)$$

[provided that this sum converges; the expectation is often referred to as the mean, and denoted by mu; mean and mu on blackboard]

- [Expectation can be interpreted as the average outcome: Law of Large numbers; next time]

Examples

- Example 1: Throw a die: what is the expected outcome?
- Example 2: Toss a coin three times, what is the expected number of heads that appear
- $0 * 1/8 + 1 * 3/8 + 2 * 3/8 + 3 * 1/8 = 3/2$

Expectation of the Geometric Distribution

- T: first success in a Bernoulli trials process with parameters n and p ; sample space: $\Omega = \{1, 2, 3, \ldots\}$
- Distribution function: $m(j) = P(T = j) = q^{j-1}p$
- Expectation:

$$E(T) = 1 \cdot p + 2 \cdot qp + 3 \cdot q^2 p + \ldots$$
$$= p(1 + 2q + 3q^2 + \ldots)$$

Differentiate the geometric series:

$$1 + s + s^2 + s^3 + \ldots = \frac{1}{1-s} = (1-s)^{-1}$$
$$1 + 2s + 3s^2 + \ldots = \frac{1}{(1-s)^2}$$

So

$$E(T) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Expectation of a Function of a Random Variable

X a random variable with sample space Ω and distribution function m(x); $\phi : \Omega \to \mathbb{R}$ a function. Then

$$E(\phi(X)) = \sum_{x \in \Omega} \phi(x) m(x)$$

Example: O = {-1, 1, 3} with m(-1)=1/8, m(1) = ¼, m(3)=5/8

What is the expectation of phi(x) = x^2

Method 1: first find the distribution of the new variable

x^2 has possible outcomes 1 and 9 with probs. 3/8 and 5/8. So E = 3/8 + 9 * 5/8 =48/8=6

Method 2: use the law (of the unconscious statistician)

E = (-1)^2 * 1/8 + 1^2 * 3/8 + 9 * 5/8

More Properties

- Sum of random variables (do not have to be independent!):

$$E(X+Y) = E(X) + E(Y)$$

- Scalar multiplication:

$$E(cX) = cE(X)$$

Expectation of Binomial Distribution

- Counts the number of successes B in a Bernoulli trials process with parameters n and p

- Sample space: $\Omega = \{1, 2, 3, \ldots, n\}$

$$B = \sum_{i=1}^{N} X_i$$

$$E(X_i) = 0 \cdot q + 1 \cdot p = p$$

$$\mathrm{E(B)} = \sum_{i=1}^{n} E(X_i) = np$$

$$b(p, n, k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k}$$

Expectation of the Poisson Distribution

- The Poisson distribution approximates the binomial distribution for large n and small p with $\lambda = np$
- So the expectation of a Poisson variable with parameter λ is λ

Independence and Expectation

- If X and Y are independent then E(XY) = E(X)E(Y) [for variables that are not independent this does not have to be true]
- Very easy to proof using P(X,Y)=P(X)P(Y)

Conditional Expectation

- Let X be a random variable with sample space Ω and F an event then:

$$E(X|F) = \sum_j x_j P(X = x_j|F)$$

- [Probably skip: Often used as:

$$\Omega = \cup_j F_j,\ F_i \cap F_j = \varnothing$$

$$E(X) = \sum_j E(X|F_j)P(F_j)$$

Variance of Discrete Random Variables

- [The expectation tells you what to expect, the variance is a measure from how much the actual is expected to deviate]
- Let X be a numerically valued rv with distribution function m(x) and expected value mu=E(X). Then the variance V(X) of X is:

$$V(X) = E((X-\mu)^2) = \sum_x (x-\mu)^2 m(x)$$

- The standard deviation of X is the square root of the variance and is usually denoted by σ :

$$V(X) = \sigma^2$$

Example

- Compute the variance of throwing a die; $E(X) = 7/2$
- $V(X) = 1/6\ (25/4 + 9/4 + ¼ + ¼ + 9/4 + 25/4) = 35/12$

Alternative Method of Computation

$$V(X) = E(X^2) - \mu^2$$

- Proof it… (when there's time)
- Example: Throwing a die, again
- E(X^2) = 1/6(1 + 4 + 9 + 16 + 25 + 36) = 91/6

Properties

- Variance is not linear like the expectation

$$V(cX) = c^2 V(X)$$

$$V(X + c) = V(X)$$

- If X and Y are independent:

$$V(X + Y) = V(X) + V(Y)$$

- Point out theorem 6.9 as a direct consequence

Bernoulli, Binomial, Geometric, Poisson variables

- X: succes with probability p; failure with q
- Variance p – p^2 = p (1 –p)=pq
- Binomial: npq

- Geometric: no time to show it, but E(X)=1/p; Var(X) = q/(p^2); see book example 6.19

- Poisson: Easier npq with limit lam x q = lambda (p to zero)

Continuous Random Variables

- Expected Value:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

- Same properties apply (also E(XY)=E(X)E(Y) for indep's)
- Example: uniform distribution: ½
- Expectation of a function of a random variable:

$$E(\phi(X)) = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx$$

Variance

- Is still:

$$V(X) = E((X-\mu)^2)$$

$$V(X) = \sigma^2 = \int\limits_{-\infty}^{\infty} (x-\mu)^2 \, dx$$

- Properties stay the same.

Assignment

- 1. (a), (b)