Lecture 4 (Enumerative) Combinatorics

- Principles of counting
- Counting possibilities
- Examples
  - In how many ways can we
    - Pick 6 numbers from 1 to 15 so that no two are consecutive?
    - Can 7 balls be placed in 4 boxes, if no box is to be left empty?

Fundamental Principle of Counting

With m elements $a_1, .. ,a_m$ and n elements $b_1, \dots b_n$ it is possible to form $m \times n$ (ordered) pairs (a_i, b_j) containing one element from each group

If we have n_r successive stages/selections/decisions with exactly n_k choices possible at the k-th step, then we can have a total of

$N = n_1 \, n_2 \dots n_r$ different results

Example:

3 starters, 4 mains, 2 dessert gives 24 different menus.

4 methods to generate collections (of size k) from a set of n elements:

1. Ordered sampling from the set with replacement
2. Ordered sampling from the set without replacement
3. Taking a combination from the set without replacement
4. Taking a combination from the set with replacement

Ordered sampling with replacement

"order matters, and elements can be drawn more than once"

Draw *k* elements from the set of n elements by selecting elements one by one:

- This gives an ordered sample of length *k*.
- If each time the selection is taken from the entire set (i.e. the same element can be drawn more than once), then there exist $N = n^k$ different samples.

Example:

A={a,b,c}. How many words of size 4 can we form?

Ordered sampling without replacement

"Order matters, elements can be drawn only once", i.e.

- Draw k elements from the set by selecting elements one by one and remove them from the set once they are chosen.
- There exist a total of $N = (n)_k$ different samples without repetitions, where $(n)_k = n(n-1) \ldots (n-k+1)$

Additional terminology:

- If we select n elements like this we get a permutation of the set.
- A permutation of a set A is an ordered listing of the elements of the set. We represent it as $a_1 \, a_2 \ldots a_n$.
- A permutation can be seen as a one-to-one mapping of A onto itself

Example:

A = {1, 2, 3} has six permutations 123, 132, 213, 231, 312, 321

Ordered sampling without replacement

- The total number of different orderings (= permutations) of n elements is: n! (n factorial); Note: 0! =1
- A k-permutation of a set A is an ordered listing of a k-subset of A
- We represent it as $a_1 \, a_2 \ldots a_k$, e.g. there are 6 2-permutations of A={1,2,3}: 12, 13, 21, 23, 31, 32
- As we have seen, a set of n elements has $(n)_k = n!/(n-k)!$ k-permutations
- Give the probability of sampling without any repetitions: $(n)_k/n^k$

Combinations without replacement

"order does not matter, elements can be used only once"

- Take k elements without minding their order. Without replacement this corresponds to taking a subset of the set.

Example:
A={1,2,3}. There are 3 such combinations: {1,2}, {2,3}, {1,3}

Define the binomial coefficient (pronounced as "n choose k") as $\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$

A set of n elements has $\binom{n}{k}$ subsets (combinations without repetition) of size $k \leq n$

Note: $\binom{n}{0} = 1$

Combinations with replacement

Order does not matter, and elements can be chosen more than once.

Example: A={a, b, c, d}. Pick 4 elements, e.g. a, a, b, d.

Encode a, a, b, d as 1, 1, /, 1, /, /, 1 (Go through A, pick one from A (code '1'), or go to next element from A (code '/').)
Encode b, c, c, d as /, 1, /, 1, 1, /, 1.

Every selection corresponds to such strings. Strings always have length n+k-1; there are always (n-1) next-symbols, and k 1's.

The total number of combinations with repetition (also knows as bags or multisets) of size k from a set of n elements is:

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Example: 3 types of pies. In how many ways can we pick 10 pies. (10 + 3 − 1)! / 3! (10 − 1)! = 220

Bernoulli trials

Bernoulli trials process consisting of n experiments:

- Each experiment has two possible outcomes, e.g. 0 or 1, success or failure: $\Omega_i = \{0, 1\}, i = 0, \ldots, n$
- The probability p of success (1) is the same for each experiment; the probability q of failure(0) is 1 – p:

$m_i(1) = m_i(S) = p$

$m_i(0) = m_i(F) = q = 1 - p$

Tree:

probabilities of outcomes of entire experiment (see figure page 96)

Binomial distribution

Given *n* Bernoulli trials with probability *p* of success on each experiment, the probability of exactly *k* successes is:

$$b(n, p, k) = \binom{n}{k} p^k q^{n-k}$$

Explanation:

P(E) with E = {ω | ω has k successes }

$P(E) = \sum_{\omega \in E} m(\omega)$

Using the tree: every path with *k* successes and *n-k* failures:

m({ k successes, n-k failures }) = $p^k q^{n-k}$

How many such paths are there? n possible trials, k should be successes: $\binom{n}{k}$

If B is a random variable counting the number of successes in a Bernoulli trials process with parameters n and p. Then the distribution m(k) = b(n, p, k) is called the Binomial distribution.

Binomial distribution

A dice is rolled four times.

What is the probability that exactly one 6 turns up:

$$b(4, \tfrac{1}{6}, 1) = \binom{4}{1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^3$$