Lecture 3: Continuous Probability Spaces

- Continuous probability spaces definitions
- Density functions
- Cumulative distribution functions

Continuous Probability Spaces

We first consider (non-discrete) sample spaces $\Omega \subseteq \mathbb{R}$

Goal, again, is to be able to compute the probability of events $E \subseteq \Omega$

This time we do not specify the probability for individual outcomes $m(\omega)$, but a probability density function $f(\omega)$, or usually: $f(x)$

Definition:

Let X be a continuous real-valued random variable. A **density function** for X is a real-valued function f that satisfies

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx \qquad \text{, for all } a, b \in \mathbb{R}$$

Continuous Probability Spaces

Read $P(a \leq X \leq b)$ as $P(E)$, with $E = \{ \omega | \, \omega \in \Omega, a \leq \omega \leq b\}$

A density function is a density in the sense that it gives the probability **per unit sample space**

Analogy is mass density of a wire: Suppose we have a wire and its '*mass density along its length*' is given by $f(x)$.

Example 1: We have a wire of 2 meters long with a uniform density (*along its length*) of 10 kg/m^2. Draw a graph. Explain some masses.

Example 2: Now for a general density function $f(x)$. Now $M \approx \sum_{i=1}^{N} f(x_i)\Delta x$

We can get the exact mass by letting $\Delta x \to 0$: $M = \int_a^b f(x)\, dx$

Continuous Probability Spaces

Using the density function we can compute the probability of (almost) any (reasonable) event $E \subseteq \Omega$:

$$P(E) = \int_E f(x)\, dx$$

Note:

$$P(\Omega) = \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$P(\{x\}) = \int_x^x f(t)\, dt = 0$$

Two examples of density functions

1. Uniform distribution on an interval [a, b]:

$$f(x) = \begin{cases} \frac{1}{b-a} & if \quad a \le x \le b \\ 0 & elsewhere \end{cases}$$

2. Exponential distribution: often a good model for times between occurrences

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & if \quad t \ge 0 \\ 0 & if \quad t < 0 \end{cases}$$

Uniform density for 2 variables

Now consider $\Omega \subseteq \mathbb{R}^2$

For a uniform density $f(x,y) = \frac{1}{area(\Omega)}$

Probability of an event $E \subseteq \Omega$:

$$P(E) = \iint_E f(x)\, dx\, dy = \frac{area(E)}{area(\Omega)}$$

Dart example:

Compute the probability that the dart lands in a certain region (for example the first quadrant, half slice near rim 3/16)

Cumulative distribution functions

Let X be a continuous real-valued random variable with density function f(x). The cumulative distribution function F(x) is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

F(x) is a specific primitive of f(x):

$$\frac{d}{dx} F(x) = f(x)$$

Cumulative distribution functions

Uses of the cumulative distribution function F(x), which is defined by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

- Sometimes easier to determine than the density function
- It's already integrated out making it easier to use to compute probabilities:

$$P(X \leq a) = F(a)$$

$$P(X > a) = 1 - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a)$$

e.g. $P(a \leq X \leq b) = \int_{a}^{b} f(x)dx = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx$

Cumulative distribution functions

The uniform density

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} \frac{1}{b-1}\, dt = \frac{x-a}{b-a}$$

The exponential density

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} \lambda e^{-\lambda t}\, dt = [\, -e^{-\lambda t}\,]_{-\infty}^{x} = 1 - e^{-\lambda x}$$

Assignment

The density of a continuous random variable X is given by

$$f(x) = \begin{cases} x & if \quad 0 < x < 1 \\ \frac{1}{2} & if \quad 1 < x < 2 \\ 0 & \quad elsewhere \end{cases}$$

(a) Compute the cumulative distribution function F(x)

(b) Compute $P(X > \frac{3}{2})$

(c) Compute $P(\frac{1}{2} < X < \frac{3}{2})$