Lecture 2: Discrete Probability

- Sample space
- Distribution Function
- Events
- Probability of Events

Probability: Basic Definitions

In probability theory we consider experiments whose outcome depends on chance or are uncertain.

- How do we model an experiment?
- The outcome of the experiment is represented by a random variable, e.g. X
- The sample space Ω of the experiment is the set of all possible outcomes
- Every experiment has exactly one outcome!

Example:

We roll a die once. X denotes the outcome of the experiment.

The sample space of the experiment is

$$\Omega = \{ 1, 2, 3, 4, 5, 6 \}$$

Example:

We toss a coin twice. The sample space of the experiment:

$$\Omega = \{ HH, HT, TH, TT \}$$

Probability: Basic Definitions

Example as a Venn diagram:

We roll a die once. X denotes the outcome of the experiment.

The sample space of the experiment is

Probability: Basic Definitions

If we can represent the sample space as a finite set

$$\Omega = \{ \omega_1, \omega_2, . . . , \omega_n \}$$

or as a countable infinite set

$$\Omega = \{ \omega_1, \omega_2, . . . \}$$

then the sample space is called discrete.

Example of a countable infinite sample space:

Take the set of outcomes for a random variable X that gives the number of the first toss that a head comes up. Sample space: $\Omega = \{ 1, 2, 3, . . . \}$

Example of a non-discrete/continuous sample space:

The interval of all real numbers $0 \leq \omega \leq 1$: $\Omega = [0, 1]$

Probability: Basic Definitions

We assign probabilities to the sample points of a sample space Ω according to two rules:

1. All sample point probabilities must lie between 0 and 1.
2. The probabilities of all the sample points within a sample space must add up to 1.

This is formalized through a distribution function m for X. This is a function m that satisfies:

1. $m(\omega) \geq 0$, for all $\omega \in \Omega$
2. $\sum_{\omega \in \Omega} m(\omega) = 1$

Examples:

- we roll a die once: $m(1) = \frac{1}{6}, \ldots, m(6) = \frac{1}{6}$
- The uniform distribution on a sample space Ω containing n elements is the function m defined by: $m(\omega) = \frac{1}{n}$
- we roll a "loaded" die once, e.g.:

$$m(1) = \frac{1}{12},\ m(2) = \frac{1}{12},\ m(3) = \frac{1}{12},\ m(4) = \frac{1}{12},\ m(5) = \frac{1}{3},\ m(6) = \frac{1}{3}$$

Probability: Basic Ideas

- An event E is a subset of sample space Ω: $E \subseteq \Omega$
- An event E "is realized" if the outcome $X = \omega$ of the experiment is in the event E: $\omega \in E$
- The probability P(E) of an event E is given by:

$$P(E) = \sum_{\omega \in E} m(\omega)$$

- The distribution function m defines the probability of simple events $E = \{\omega\}$:

$$P(\{\omega\}) = m(\omega)$$

- After the experiment, the event E has either taken place or not!

Probability: Example

Example:

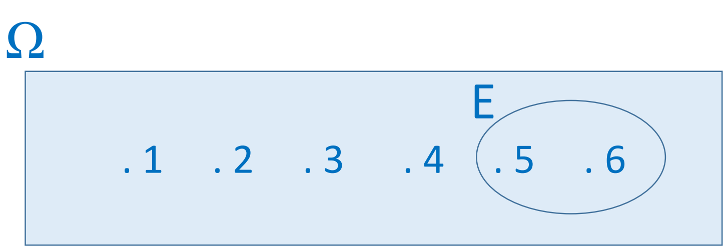
Roll a dice once. What is the probability that we throw more than 4?

Define the event: $E = \{ 5, 6 \}$

Calculate the probability:

$$P(E) = P(\{ 5, 6 \}) = m(5) + m(6) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Probability: Example in a Venn Diagram



Probability: Basic Ideas

Steps for calculating the probability of an event:

1. Define the sample space of the experiment: list all possible outcomes (sample points)
2. Define the distribution function: assign probabilities to the sample points
3. Determine the sample points contained in the event of interest
4. Sum the sample point probabilities to get the event probability

Probability of Events: Properties

- Probabilities are always between zero and one:
  - $P(\emptyset) = 0$
  - $P(\Omega) = 1$
  - If $E \subset F \subset \Omega$, then $0 \leq P(\emptyset) \leq P(E) \leq P(F) \leq P(\Omega) \leq 1$
- If A and B are events, then:
  - $A \cap B$ is the event that *both* event A and B occur
  - $A \cup B$ is the event that *either* A or B occurs, or both
- If A and B are disjoint events (or mutually exclusive; $A \cap B = \emptyset$) only one event can occur at the same time; then

$$P(A \cup B) = P(A) + P(B)$$

This is of course also the case for more than one subset (Proof).

Probability of Events: Properties

- If A and B are not disjoint, i.e. $A \cap B \neq \emptyset$, then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(explain with a Venn diagram)

- Probability that an event *A* does not happen:

$$P(\tilde{A}) = 1 - P(A)$$

$$P(A) + P(\tilde{A}) = P(\Omega) = 1$$

(draw a Venn diagram)

Probability of Events: Properties

Let $A_1, \dots, A_n$ be pairwise disjoint events with $\Omega = A_1 \cup \dots \cup A_n$ and let E be any event, then

$$P(E) = \sum_{i=1}^{n} P(E \cap A_i)$$

Assignments

We have a bowl with one hundred balls, 60 are white and 40 are black. We mix the balls well. We pick a ball from the bowl and write down its color. Next, we put it back, and mix the balls again. Then we take another ball, and again write down its color.

- What is the probability that we picked at least one white ball?
- Work out with a tree diagram, and use the complement insight.