From: S. Lang, Linear Algebra, 2^{nd} Ed.
Addison-Wesley Publ. Comp., Reading, 1970.

Fields

$K \subseteq \mathbb{C}$ is a field if it satisfies:

a) If $x, y \in K$, then $x+y \in K$ and $xy \in K$

b) If $x \in K$, then $-x \in K$ and if $x \neq 0$ also, then $x^{-1} \in K$

c) $0 \in K$ and $1 \in K$ (additive and multiplicative nullelements, resp.)

Examples of Fields:

- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$
- Note, $\mathbb{Z}$ is not a field as b) does not hold.

Vector Spaces

V is a vector space over K if the following is true:

- if $u, v \in V$, then $u+v \in V$
- if $u \in V$ and $\lambda \in K$, then $\lambda u \in V$

1) $u, v, w \in V$, then $(u+v)+w = u+(v+w)$

2) $\exists\, \mathbf{0} \in V$ such that $\mathbf{0} + u = u + \mathbf{0} = u \quad \forall u \in V$

3) given $u \in V$, $\exists -u \in V$ such that $u + (-u) = \mathbf{0}$

4) $\forall u, v \in V$: $u + v = v + u$

5) $\forall c \in K$: $c(u+v) = cu + cv$ for all $u, v \in V$

6) $\forall a, b \in K$: $(a+b)v = av + bv$ for all $v \in V$

7) $\forall a, b \in K$: $(ab)v = a(bv)$ for all $v \in V$

8) $\forall u \in V$ $1 \cdot u = u$ (where $1 \in K$)

Examples of Vector Spaces

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, ..., $\mathbb{C}^1$, $\mathbb{C}^2$, $\mathbb{C}^3$, ...

Examples of Vector Spaces (Cont'd)

Function Spaces

Let S be a set and K a field and $f : S \to K$ a K-valued function, i.e., a rule that associates to each element of S a unique element of K.

Let V be the set of all functions of S into K.

A] If $f, g \in V$ we define $f+g$ as the function whose value at $x \in S$ is the value $f(x)+g(x)$ (again $\in K$ as K is a field).

B] If $c \in K$, we define cf to be the function whose value at $x \in S$ is equal to $c\,f(x)$ (again $\in K$ as K is a field).

- Then it is easy to verify that V is a Vector Space over K.

[$f_0 : S \to K$ where $f(x)=0$ for all $x \in S$ is the 0 element]

Other Examples of Function Spaces which are Vector Spaces:

- V the set of all functions of $\mathbb{R}$ into $\mathbb{R}$
- V the set of all continuous functions of $\mathbb{R}$ into $\mathbb{R}$
- V the set of all differentiable functions of $\mathbb{R}$ into $\mathbb{R}$
- V the subspace generated by the functions $f(t)=e^t$ and $g(t)=e^{2t}$ (for all $t \in \mathbb{R}$.)

[Just check that A] and B] hold. As $\mathbb{R}$ is a field the claim that V is Vector Space follows.].

Linearly Dependence

Let V be a vector space over the field K.

Let $v_1, \dots, v_n \in V$. $v_1, \dots, v_n$ are linearly dependent over K if $\exists a_1, \dots, a_n \in K$ not all equal 0 such that $a_1v_1 + \dots + a_nv_n = 0$.

- If there do not exist such numbers, i.e., if $a_1, \dots, a_n \in K$ such that $a_1v_1 + \dots + a_nv_n = 0$, then $a_i = 0$ $\forall i = 1, \dots, n$. then $v_1, \dots v_n$ are linearly independent.

Example: - Let $V = \mathbb{R}^n$, then $E_1 = (1, 0, \dots, 0)$ $\vdots$ $E_n = (0, 0, \dots, 1)$ are linearly independent.

- also e^t, e^{2t} are linearly independent.

Basis

If elements $v_1, \dots, v_n \in V$ generate V, and $v_1, \dots, v_n$ are linearly independent. $\{v_1, \dots, v_n\}$ is called a basis of V.

- $v_1, \dots, v_n \in V$ generate V, that is every element of V can be expressed as a linear combination of $v_1, \dots, v_n$.
- and indeed if $x_1v_1 + \dots + x_nv_n = x = y_1v_1 + \dots + y_nv_n$ with $x_1, \dots, x_n, y_1, \dots, y_n \in K$ and for a $x \in V$,

  then $(x_1 - y_1)v_1 + \dots + (x_n - y_n)v_n = 0$

  thus $x_1 = y_1, \dots, x_n = y_n$.

Scalar Products

Let V a vector space over a field K. (real)

A <u>scalar product</u> on V is an association which to any pair $v, w \in V$ associates a scalar $\langle v, w \rangle$ (also $v \cdot w$) satisfying:

1] $\forall v, w \in V \quad \langle v, w \rangle = \langle w, v \rangle$

2] Let $u, v, w \in V$, then $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

3] Let $c \in K$, then $\langle cu, v \rangle = c \langle u, v \rangle$

and $\langle u, cv \rangle = c \langle u, v \rangle$

A scalar product is <u>non-degenerate</u>, if also:

4] If $v \in V$ and $\langle v, w \rangle = 0$ for all $w \in V$, then $v = 0$.

Examples of Scalar Products:

- $V = K^n \quad \langle x, y \rangle : \ X, Y \to X \cdot Y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ is a scalar product. [this is the 'standard' dot-product.]
- Let V be the space of continuous real-valued functions on the interval $[0, 1]$. If $f, g \in V$,

  $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$.

  Then $\langle f, g \rangle$ is a scalar product.

Orthogonality

$v, w \in V$ are orthogonal: $v \perp w$ if $\langle v, w \rangle = 0$.

Norm

- The norm of $v \in W$ can be defined by $\|v\| = \sqrt{\langle v, v \rangle}$

  It is clear that: $\|cv\| = |c| \, \|v\|$
- $v \in V$ is a unit vector if $\|v\| = 1$

  ($v / \|v\|$ is always a unit vector, if $v \neq 0$)

Some Theorems (easy)

We have the following theorems:

Th. If $v, w \in V$ and $v \perp w$ (i.e. $\langle v, w \rangle = 0$), then (Pythagoras)

$\|v + w\|^2 = \|v\|^2 + \|w\|^2$

Proof: $\|v + w\|^2 = \langle v + w, v + w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle$

$\underset{v \perp w}{=} \|v\|^2 + \|w^2\|$ (as $\langle v, w \rangle = 0$). ☒

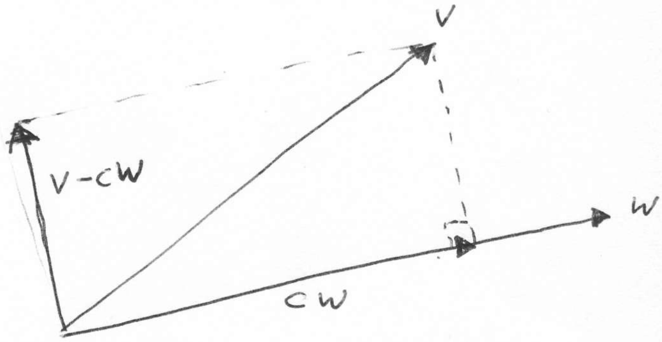
Parallelogram law:

$\forall\, v, w \in V$ we have $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

Fourier Coefficients

Observation:

Let $w \in V$ such that $\|w\| \neq 0$.
For any $v \in V$ there exists a unique $c \in K$ such that $v - cw$ is perpendicular to w.



Now $v - cw$ perpendicular to w means that $\langle v - cw, w \rangle = 0$.

$\Rightarrow \langle v, w \rangle - c\langle w, w \rangle = 0 \quad \Rightarrow \quad c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}$

Conversely, if $c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}$ then $c\langle w, w \rangle = \langle v, w \rangle \Rightarrow$

$\langle v, w \rangle - c\langle w, w \rangle = 0 \Rightarrow$

$\langle v - cw, w \rangle = 0$

hence $v - cw$ perpendicular to w.

We call c the component of v along w, or
the Fourier coefficient of v with respect to w.

We call cw the projection of v along w.

- Note if w is ~~the~~ unit vector, i.e. $\|w\| = 1$, then $\langle w, w \rangle = 1$, and hence c is simply $\langle v, w \rangle$

Example Fourier Coefficients.

Let V be the space of continuous functions on $[-\pi, \pi]$.

Let $f : x \to \sin kx$, where $k \in \mathbb{Z}_{>0}$.

Then $\|f\| = \sqrt{\langle f,f\rangle} = \left(\int_{-\pi}^{\pi} \sin^2 kx \, dx\right)^{1/2} = \sqrt{\pi}$

In this case, if g is <u>any</u> continuous function on $[-\pi, \pi]$, then the <u>Fourier coefficient</u> of g with respect to f is

$$\frac{\langle g,f\rangle}{\langle f,f\rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin kx \, dx$$

The Complex ($\mathbb{C}$) Case

Let V be a vector space over the complex numbers.
A <u>hermitian product</u> on V is a rule $\langle v,w\rangle$ satisfying.

1) $\langle v,w\rangle = \overline{\langle w,v\rangle}$ for all $v,w \in V$
2) $u,v,w \in V$, then $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$
3) if $\alpha \in \mathbb{C}$, then $\langle \alpha u, v\rangle = \alpha\langle u,v\rangle$
$\langle u, \alpha v\rangle = \bar{\alpha}\langle u,v\rangle$.

$\langle \cdot,\cdot\rangle$ is <u>positive definite</u>
if $\langle v,v\rangle \geq 0$ for all $v \in V$ and
$\langle v,v\rangle > 0$ if $v \neq 0$

<u>Note</u> <u>Orthogonal</u>, <u>perpendicular</u>, <u>orthogonal basis</u>, <u>orthogonal complement</u>, as before!

Also the <u>Fourier coefficient</u> and the <u>projection of</u> v along w are as before.

Example

Let V be the space of continuous complex-valued functions on the interval $[-\pi, \pi]$.

- If $f, g \in V$, we define $\langle \cdot, \cdot \rangle$ as follows:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\, \overline{g(t)}\, dt$$

This can be shown, using standard properties of the integral, to be a positive definite hermitean product.

- Let $f_n(t) = e^{int}$

A] if $n \neq m$, then $\langle f_n, f_m \rangle = \int_{-\pi} e^{int}\, \overline{e^{imt}}\, dt = \int_{-\pi} e^{ikt}\, dt = 0$

if $n = m$, then we have $\langle f_n, f_n \rangle = \int_{-\pi} e^{int}\, e^{-int}\, dt = \int_{-\pi} 1\, dt = 2\pi$

- If $f \in V$, then its Fourier coefficient with respect to f_n is equal to:

$$\frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt.$$

Note: A] shows that f_n and f_m with $n \neq m$ are orthogonal.

Furthermore it can be shown that $\{f_n\}, n \in \mathbb{N}^+$ constitutes a basis for V.

Hence $\{e^{it}, e^{2it}, e^{3it}, \ldots\}$ is an orthogonal basis of V the vector space of continuous complex-valued functions on the interval $[-\pi, \pi]$. (Note, by dividing through $\langle f_n, f_n \rangle$ you get normalized basis.)